UNIVERSITY OF CAMBRIDGE

MAT1

MATHEMATICAL TRIPOS Part IB

Thursday, 12 June, 2025 1:30pm to 4:30pm

PAPER 3

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most four** questions from Section I and **at most six** questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Groups, Rings and Modules

Let G be a finite group. Show that there exists subgroups

$$G = H_1 \rhd H_2 \rhd \cdots \rhd H_n = \{1\}$$

such that H_{i+1} is a normal subgroup of H_i, and the quotient H_i/H_{i+1} is simple.

Write such a series for $G = S_4$, the symmetric group on 4 letters, and determine whether each H_i in your series is normal in G.

2E Geometry

Define what it means for an element $g \in SL_2(\mathbb{R})$, $g \neq \pm I$ to be *elliptic, parabolic* or *hyperbolic* in terms of the action of g on the hyperbolic plane $\mathfrak{h}$ and its boundary.

Give an example of each.

Prove that every element $g \neq \pm I$ in $SL_2(\mathbb{R})$ is precisely one of the three.

Let $g \in SL_2(\mathbb{R})$ be an element with $g^n = I$, $g \neq \pm I$, $n > 1$. Determine when g is elliptic, parabolic, or hyperbolic.

3A Complex Methods

(a) Consider the complex function f analytic in the open disc D centred at zero, except with a singularity at $z = 0$, and its Laurent series around zero,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n \, z^n = \cdots + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \cdots .$$

Show that if f is even, that is $f(z) = f(-z)$ for $z \in D \setminus 0$, then $a_n = 0$ when n is odd.

(b) Evaluate the integrals

$$I_n = \oint_{C_n} \frac{1}{z^3 \sin(z)} dz, \quad n = 0, 1, 2, \ldots ,$$

where C_n is the circle $\{z \in \mathbb{C} : |z| = (n + 1/2)\pi\}$, with the counterclockwise orientation.

4C Variational Principles

Consider the functional

$$S[x,y] = \frac{1}{2}\int_{t_0}^{t_1}\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + 2\sin\omega t\,\frac{dx}{dt} - y^2\right)dt$$

defined on smooth curves $t \mapsto (x(t), y(t))$ in the plane. Assume $\omega \in \mathbb{R}$ is constant.

Write out the Euler-Lagrange equations for S and find the general solution.

What symmetries does the system have?

Find all the first integrals (conserved quantities) of the system.

[*You may either use the Noether theorem or work with the Euler-Lagrange equations. Consider all values of* $\omega \in \mathbb{R}$.]

5D Methods

The Fourier transform $\tilde{f}(k)$ is defined in this question by $\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$.

Prove the convolution theorem in the form

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(k)\tilde{g}(k)e^{ikx}\,dk = \int_{-\infty}^{\infty} f(u)g(x-u)\,du$$

for suitably integrable functions $f(x)$, $g(x)$.

Determine the Fourier transform of the function

$$f(x) = \begin{cases} 1 \text{ for } -1 < x < 1, \\ 0 \text{ otherwise.} \end{cases}$$

Hence calculate $\displaystyle\int_{-\infty}^{\infty}\frac{\sin^2 k}{k^2}\,dk$.

6C Quantum Mechanics

Let $\psi(x,t)$ solve the time-dependent Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}kx^2\psi\,, \qquad -\infty < x < +\infty\,,$$

for a particle of mass m moving in a potential $V(x) = \frac{1}{2}kx^2$. If $\mathcal{O}$ is an operator representing an observable, its expectation value at time t in a normalized state ψ is

$$\langle\mathcal{O}\rangle(t) = \int \psi(x,t)^*\mathcal{O}\psi(x,t)dx\,.$$

Write down operators Q and P representing, respectively, the position and the momentum of the particle. Calculate the time derivative of $\langle P\rangle(t)$ as a function of $\langle Q\rangle(t)$ and interpret the answer.

[*Hint: You may assume ψ and its derivatives are smooth and decrease to zero at infinity as needed.*]

7D Fluid Dynamics

A layer of viscous fluid of density ρ, dynamic viscosity μ, and uniform thickness h flows down a rigid, vertical wall, adjacent to stationary, inviscid, ambient fluid of density ρ_a. The ambient fluid exerts no shear stress on the viscous fluid layer.

What is the hydrostatic pressure in the ambient fluid? Write down the dynamic boundary conditions at the interface between the two fluids and the boundary condition to be applied at the vertical wall.

Write down the equations governing steady, parallel flow of the viscous fluid, and solve them to determine its pressure and velocity fields.

8H Markov Chains

(a) What does it mean to say a Markov chain is *recurrent*?

(b) Let $X_0 = 0$ and $X_n = Z_1 + \ldots + Z_n$ for $n \geqslant 1$, where $Z_1, Z_2, \ldots$ are independent and $\mathbb{P}(Z_n = +1) = p = 1 - \mathbb{P}(Z_n = -1)$ for each n. Prove that the Markov chain $(X_n)_{n\geqslant 0}$ is recurrent if and only if $p = 1/2$.

[*You may use the fact that there is a constant $A > 0$ such that $k!\ (e/k)^k k^{-1/2} \to A$ as $k \to \infty$.*]

SECTION II

9F Linear Algebra

Throughout this question V is an n-dimensional complex vector space and $\varphi : V \to V$ is a linear endomorphism of V.

(a) Define the *minimal polynomial* m_φ of φ and explain why m_φ is uniquely defined. State the Cayley–Hamilton theorem. What can we deduce about the relationship between m_φ and the characteristic polynomial of φ?

(b) Now suppose that V has a basis $(a_1, \ldots, a_n)$ such that $\varphi(a_k) = a_{k+1}$, for each $k = 1, 2, \ldots, n-1$. Let θ be a map assigning to each complex polynomial p the vector $p(\varphi)(a_1) \in V$. By considering θ, or otherwise, show that the minimal polynomial of φ is

$$m_\varphi(x) = x^n - \sum_{k=0}^{n-1} c_{k+1} x^k$$

where the coefficients c_j are determined by $\varphi(a_n) = \sum_{j=1}^{n} c_j a_j$. [*Hint: You do not need to determine the eigenvalues of* φ.]

(c) Show that if the minimal polynomial of a linear endomorphism $\varphi : V \to V$ is of the form $m_\varphi(x) = (x - \alpha)^n$ for some constant α (where as above $n = \dim V$), then V cannot be written as a direct sum $V_1 \oplus V_2$, where V_1, V_2 are non-zero, proper subspaces of V such that $\varphi(V_i) \subset V_i$, $i = 1, 2$. Show also that V has a basis $\mathcal{B} = (b_1, \ldots, b_n)$ such that $\varphi(b_k) = b_{k+1}$ for $k = 1, 2, \ldots, n-1$ and compute the matrix of φ with respect to the basis $\mathcal{B}$.

10E Groups, Rings and Modules

(a) Let R be a unique factorisation domain, and $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x]$.

Define the content $c(f)$ of f, and what it means for f to be *primitive*.

(i) Prove that if $f, g \in R[x]$ are primitive, then so is fg. Deduce that $c(fg) = c(f)c(g)u$, for some unit $u \in R^*$.

(ii) Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$ be a monic polynomial with integer coefficients. Suppose $f(\lambda) = 0$ for some $\lambda \in \mathbb{Q}$. Deduce from (i) that $\lambda \in \mathbb{Z}$.

(b) Show $x^3y^3 + x^3y + x^2y^2 + 1 - x - y - y^2$ is irreducible in $\mathbb{C}[x, y]$.

11G Analysis and Topology

Let X, Y be topological spaces.

Define what it means for X to be *connected*. Show that if X is connected and $f : X \to Y$ is continuous then $f(X)$ is connected, where $f(X)$ inherits the subspace topology from Y.

Show that X is connected if and only if every continuous map $g : X \to \{0,1\}$ is constant, where $\{0,1\}$ carries the discrete topology.

Show that $\mathbb{R}$ with the topology induced by the Euclidean metric is connected. You may assume the intermediate value theorem.

Let $\sim$ be the equivalence relation on $\mathbb{R}$ given by $x \sim y$ if $x - y \in \mathbb{Q}$. Is $\mathbb{R}/\sim$ connected in the quotient topology? Justify your answer.

For $A \subset X$ define

$$\mathrm{Cl}(A) = \bigcap_{E \text{ closed}; A \subset E} E.$$

Suppose A is connected and $A \subset B \subset \mathrm{Cl}(A)$. Show that B is connected.

12F Geometry

What is the *Euler characteristic* of a closed topological surface? State the Gauss–Bonnet theorem for geodesic polygons and for closed smooth surfaces.

Let $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that $f(x,y) = 0$ when $x^2 + y^2 > 1$ and let $S \subset \mathbb{R}^3$ be a non-compact embedded surface parametrized by $\varphi(x,y) = (x, y, f(x,y))$ for $(x,y) \in \mathbb{R}^2$. Prove that if the Gaussian curvature K of S is everywhere non-negative, then K is everywhere zero.

Let $T \subset \mathbb{R}^3$ be the torus given by

$$\sigma(u,v) = \big((2+\cos u)\cos v,\ (2+\cos u)\sin v,\ \sin u\big), \quad 0 \leqslant u, v < 2\pi.$$

Determine the regions T_+ and T_- of T where the Gaussian curvature is positive and negative, respectively. Show, by considering an appropriate closed surface but without explicitly integrating K, that

$$\int_{T_+} K dA = -\int_{T_-} K dA = 4\pi.$$

[*You may assume that the Gauss–Bonnet theorem holds when a closed topological surface in $\mathbb{R}^3$ is a union of finitely many smooth surfaces joined at their boundaries.*]

13E Complex Analysis

State and prove Liouville's theorem. You may assume Cauchy's integral formula provided it is clearly stated.

For $R > 0$ let $A = \{z \in \mathbb{C} : |z| > R\}$. Suppose $f : A \to \mathbb{C}$ is holomorphic and satisfies $f(z) \to a$ as $|z| \to \infty$ for some $a \in \mathbb{C}$. Show that $z^2 f'(z) \to b$ as $|z| \to \infty$ for some $b \in \mathbb{C}$.

Let g be holomorphic on $\mathbb{C}$ except at $z \in \{p_1, \ldots, p_n\}$ where g has a simple pole. Assume that g has simple zeros at $z \in \{q_1, \ldots, q_m\}$ and no other zeros, and that $g(z) \to 1$ as $z \to \infty$. Show that $m = n$ and hence determine g.

14D Methods

Prove that, for scalar fields $\phi(\mathbf{x})$ and $\psi(\mathbf{x})$ in a three-dimensional domain $\mathcal{D}$ with boundary $\partial\mathcal{D}$,

$$\int_{\mathcal{D}} (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV \equiv \int_{\partial\mathcal{D}} \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) dS,$$

where $\mathbf{n}$ is the outward unit normal to $\partial\mathcal{D}$.

Let $\psi(\mathbf{x})$ satisfy

$$\nabla^2\psi = \delta(\mathbf{x} - \mathbf{x}_0) \quad \text{in} \quad z > 0$$

with

$$\psi = 0 \quad \text{on} \quad z = 0, \qquad \psi \to 0 \text{ as } |\mathbf{x}| \to \infty$$

in Cartesian coordinates $\mathbf{x} = (x, y, z)$. Use the method of images to determine $\psi(x, y, z)$.

Now use the identity above to solve the equation

$$\nabla^2\phi = 0 \quad \text{in} \quad z > 0, \qquad \phi \to 0 \text{ as } |\mathbf{x}| \to \infty$$

with $\phi = 1$ on $z = 0$, $x^2 + y^2 < 1$, while $\phi = 0$ on $z = 0$, $x^2 + y^2 > 1$ in terms of a surface integral. Find the closed-form solution for $\phi(0, 0, z)$.

15B Electromagnetism

Give the electromagnetic tensor $F_{\mu\nu}$ explicitly in terms of the components of the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$. Show that the dual electromagnetic tensor defined as $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}$, is given by

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix}.$$

Calculate the Lorentz scalar $F^{\mu\nu}F_{\mu\nu}$ and express in terms of the fields $\mathbf{E}$ and $\mathbf{B}$. State what the remaining Lorentz scalars $F^{\mu\nu}\tilde{F}_{\mu\nu}$ and $\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu}$ are in terms of $\mathbf{E}$ and $\mathbf{B}$.

In a particular reference frame S, the components of uniform electric and magnetic fields are restricted to the y-z plane, taking the form $\mathbf{E} = (0, E_y, E_z)$ and $\mathbf{B} = (0, B_y, B_z)$. Consider another inertial frame S' related by the Lorentz transformation,

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where v is the velocity of S' in S along the x-axis with $\gamma = (1 - v^2/c^2)^{-1/2}$. Determine the components of the fields $\mathbf{E}'$ and $\mathbf{B}'$ in the new frame S'.

Now suppose that $\mathbf{E} = E_0(0, 1, 0)$ lies parallel to the y-axis, while $\mathbf{B}$ has magnitude $B_0 = E_0/c$ and lies in the y-z plane at an angle θ to the y-axis with $0 \leqslant \theta \leqslant \pi/2$. Determine the velocity of a reference frame S' in which the two fields align to become parallel. Briefly discuss the two limits when $\theta \ll 1$ and when $\theta \to \pi/2$.

16D Fluid Dynamics

Incompressible fluid is contained between rigid plates at $\theta = \pm\alpha$ hinged together at $r = 0$ in plane polar coordinates (r, θ). The fluid was initially at rest but is set into motion by rotating the plates towards each other, each with angular speed Ω. The subsequent instantaneous fluid flow $\mathbf{u}(r, \theta)$ can be treated as being inviscid.

Explain why the flow can be written in terms of a velocity potential ϕ that satisfies Laplace's equation $\nabla^2\phi = 0$. What boundary conditions are satisfied by ϕ?

Find the velocity potential in the form $\phi = r^2 f(\theta)$, determining the function $f(\theta)$ explicitly.

Determine the velocity field and thence determine a streamfunction for the flow. Describe and sketch the streamlines.

Calculate the flux of fluid across the radial arc $r = R$, $-\alpha < \theta < \alpha$.

17B Numerical Analysis

Consider $C[-1,1]$ equipped with the inner product $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)w(x)\,dx$, where $w(x)>0$ for $x\in(-1,1)$. Moreover, for $n\in\mathbb{N}$, let

$$A_n = \begin{bmatrix} \alpha_1 & \sqrt{\beta_2} & 0 & \cdots & 0 \\ \sqrt{\beta_2} & \alpha_2 & \sqrt{\beta_3} & \ddots & \vdots \\ 0 & \sqrt{\beta_3} & \alpha_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \sqrt{\beta_n} \\ 0 & \cdots & 0 & \sqrt{\beta_n} & \alpha_n \end{bmatrix},$$

where $\alpha_n\in\mathbb{R}$ and $\beta_n>0$.

(a) Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of monic polynomials of degree n orthogonal with respect to the above inner product. Prove that for $n\geqslant 1$ each p_n has n distinct zeros in the interval $(-1,1)$.

(b) Let $P_0(x)=1$, $P_1(x)=x-\alpha_1$, and let P_n satisfy the following recurrence relation:

$$P_n(x) = (x-\alpha_n)P_{n-1}(x) - \beta_n P_{n-2}(x), \quad n\geqslant 2.$$

Prove that for $n>1$ we have $P_n(x)=\det(xI-A_n)$.

(c) Prove that if $p_0(x)=1$ and

$$\alpha_n = \frac{\langle p_n, xp_n\rangle}{\langle p_n,p_n\rangle}, \qquad \beta_n = \frac{\langle p_n,p_n\rangle}{\langle p_{n-1},p_{n-1}\rangle}$$

then all the eigenvalues of A_n are distinct and reside in $(-1,1)$.

[*Hint: You may quote the three-term recurrence relation theorem from the class notes.*]

18H Statistics

Let $X=(X_1,\ldots,X_n)$ be a discrete random vector with probability mass function $f(x;\theta)$, where θ is an unknown parameter.

(a) In this context, what is a *sufficient statistic* for θ? State and prove the factorisation criterion for sufficiency.

(b) State and prove the Rao–Blackwell theorem.

(c) Let $X_1,\ldots,X_n$ be independent and identically distributed Poisson random variables with mean $\sqrt{q}$ where q is unknown and $n\geqslant 2$ is given. Find a one-dimensional sufficient statistic T for q. Show that $\widetilde{q}=X_1^2-X_1$ is an unbiased estimator of q. Find another unbiased estimator of q that is a function of T and that has strictly smaller variance than $\widetilde{q}$.

19H Optimisation

Let A be the $m \times n$ payoff matrix of a two-person, zero-sum game.

(a) Write down the necessary and sufficient conditions that a vector $p \in \mathbb{R}^m$ is an optimal mixed strategy for Player I in terms of the optimal mixed strategy $q \in \mathbb{R}^n$ for Player II and the value v of the game.

(b) In the anti-symmetric case where $m = n$ and $A = -A^\top$, show that the value of the game is zero.

(c) Suppose there are rows i_0 and i_1 such that $A_{i_0 j} \leqslant A_{i_1 j}$ for all $1 \leqslant j \leqslant n$. Show that there is an optimal strategy $p \in \mathbb{R}^m$ for Player I such that $p_{i_0} = 0$.

(d) Find the optimal strategies for both players for the game with payoff matrix

$$A = \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 2 \\ 1 & -1 & 0 & 1 \\ 0 & -2 & -1 & 0 \end{pmatrix}$$

END OF PAPER