UNIVERSITY OF CAMBRIDGE

MAT1

MATHEMATICAL TRIPOS Part IB

Wednesday, 11 June, 2025 9:00am to 12:00pm

PAPER 2

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most four** questions from Section I and **at most six** questions from Section II.

If you attempt the joint Complex Analysis and Complex Methods question, you may submit an answer to at most one of the two subquestions.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Groups, Rings and Modules

(a) Let R be a ring. Define what it means for R to be i) an *integral domain*, ii) *Noetherian*, and iii) a *Principal Ideal Domain (PID)*.

(b) Let $R = \{(f(x), g(y)) \in \mathbb{C}[x] \times \mathbb{C}[y] \mid f(0) = g(0)\}$. Verify that this is a subring of $\mathbb{C}[x] \times \mathbb{C}[y]$.

Let $p : R \to \mathbb{C}[x], (f, g) \mapsto f$. Determine the kernel of p.

Is R an integral domain? A PID?

2G Analysis and Topology

For each of the following sequences of functions $f_n : \mathbb{R} \to \mathbb{R}$, determine whether $(f_n)_{n=1}^{\infty}$ converges uniformly, justifying your answer:

i) $f_n(x) = \tanh(nx)$,

ii) $f_n(x) = \sin\left(x + \frac{1}{n}\right)$,

iii) $f_n(x) = \dfrac{\exp\left(x + \frac{1}{n}\right)}{\cosh(ax)}$ (your answer may depend on the value of $a \in \mathbb{R}$),

iv) $f_n(x) = 2n\left[h\left(x + \dfrac{1}{n}\right) - h\left(x - \dfrac{1}{n}\right)\right]$,
where $h : \mathbb{R} \to \mathbb{R}$ is differentiable with h' uniformly continuous.

3B Methods

Consider the initial value problem for a second-order differential operator with constant coefficients and a forcing term:

$$\mathcal{L}y(t) \equiv \alpha y'' + \beta y' + \gamma y = f(t), \quad t > a, \quad y(a) = y'(a) = 0\,,$$

with $\alpha \neq 0$. Write down the Green's function $G(t, \tau)$ constructed to satisfy $\mathcal{L}G = \delta(t - \tau)$.

Use the Green's function approach to determine an explicit solution for the forced oscillator problem

$$y'' + \omega^2 y = \sin(\lambda t), \quad t > 0\,, \quad y(0) = y'(0) = 0\,.$$

4B Electromagnetism

State Maxwell's equations in the presence of charge density ρ and current density $\mathbf{J}$. Derive the continuity equation that ensures the conservation of charge,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0\,.$$

Suppose that all non-zero ρ and $\mathbf{J}$ are confined to a finite time-independent volume D (vanishing on the boundary ∂D). Show that the total charge Q in the region D remains constant. In addition, prove the following relation:

$$\frac{d}{dt}\int_D \mathbf{x}\,\rho\, d^3x = \int_D \mathbf{J}\, d^3x\,.$$

5D Fluid Dynamics

Write down the Euler equations governing the inviscid flow $\mathbf{u}$ of an incompressible fluid with no body force. Derive the corresponding vorticity equation.

At some initial time, the velocity

$$\mathbf{u} = (A\sin z, B\sin x + A\cos z, B\cos x)$$

in Cartesian coordinates (x, y, z), where A and B are constants. Show that the vorticity is parallel to $\mathbf{u}$. Hence show that the vorticity is constant, independent of time.

Use the Euler equation to show that $H \equiv \frac{1}{2}\rho|\mathbf{u}|^2 + p$ is uniform in space, where p is the fluid pressure and ρ is its density.

[*Hint: You may use the vector identities* $\mathbf{u}\times(\nabla\times\mathbf{u}) = \nabla(\frac{1}{2}|\mathbf{u}|^2) - \mathbf{u}\cdot\nabla\mathbf{u}$ *and* $\nabla\times(\mathbf{a}\times\mathbf{b}) = (\nabla\cdot\mathbf{b})\mathbf{a} + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\nabla\cdot\mathbf{a})\mathbf{b} - (\mathbf{a}\cdot\nabla)\mathbf{b}$.]

6H Statistics

Let X be a random variable with the $\mathrm{Exp}(\theta)$ distribution. Suppose the prior distribution of θ is $\Gamma(m, \lambda)$ for known parameters m and λ; that is, the prior density is $p(\theta) = C_{m,\lambda}\theta^{m-1}e^{-\lambda\theta}$ where $C_{m,\lambda} = \lambda^m/\Gamma(m)$.

(a) Find the posterior distribution of θ.

(b) Show that the Bayesian estimator of θ for the loss function $L(\theta, a) = (\theta - a)^2$ is given by $\hat{\theta}_{\mathrm{Bayes}} = (m+1)/(\lambda + X)$.

(c) What is the Bayesian estimator of θ for the loss function $L(\theta, a) = \cosh\big(r(\theta - a)\big)$ for a given positive constant $r < \lambda$. [*Recall that* $\cosh u = \frac{1}{2}(e^u + e^{-u})$.]

7H Optimisation

(a) What does it mean to say a set $X \subseteq \mathbb{R}^m$ is *convex*? Assuming X is convex, what does it mean to say a function $f : X \to \mathbb{R}$ is *convex*?

(b) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is convex. Let $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ be defined by $g(t, x) = tf\left(\frac{x}{t}\right)$, where $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$. Show that g is convex.

(c) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ has the property that there is a function $\lambda : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f(x) - f(y) \leqslant \lambda(x)^\top (x - y)$$

for all $x, y \in \mathbb{R}^n$. Prove that f is convex.

SECTION II

8F Linear Algebra

Let m be a positive integer and $\alpha \in \mathbb{C}$. What is a *Jordan block* $J_m(\alpha)$?

Let $p(x)$ be a polynomial with complex coefficients. Show that

$$p(J_2(\alpha)) = \begin{pmatrix} p(\alpha) & p'(\alpha) \\ 0 & p(\alpha) \end{pmatrix}.$$

Let A be an $n \times n$ complex matrix. Define the *Jordan normal form* of A.

Show that the Jordan normal form of A is a diagonal matrix if and only if $\operatorname{Ker}(A - \lambda I_n)^2 = \operatorname{Ker}(A - \lambda I_n)$ for all $\lambda \in \mathbb{C}$, where I_n is the identity matrix of size n.

Let $A \in M_n(\mathbb{C})$ be an $n \times n$ complex matrix and let B be the Jordan normal form of A. Show that B is also the Jordan normal form of the transpose matrix A^T.

Show that A can be factorised as $A = CD$ where the matrices C and D are symmetric and C is non-singular.

9E Groups, Rings and Modules

Define what it means for a ring to be a *Euclidean domain*.

Let R be a Euclidean domain.

(a) Prove that R is a principal ideal domain (PID).

(b) Let p be a prime in R. Fix $e \geqslant 1$, and let $\varphi : R/(p^e) \to R/(p)$ be the natural ring homomorphism. Let $a \in R/(p^e)$.

Describe in terms of $\varphi(a)$ (i) when a is a unit, and (ii) when a is nilpotent. You must justify your answer.

(c) Let $I = (p_1^{e_1} p_2^{e_2} \ldots p_n^{e_n})$ be an ideal in R, where $p_1, \ldots, p_n$ are irreducible elements, pairwise non-associated, and $e_1, \ldots, e_n \geqslant 1$ are integers. Describe the units in R/I.

(d) Let $d = p_1 p_2 \cdots p_n$, where $p_1, \ldots, p_n$ are as defined in (c). Prove that the homomorphism $R/I \to R/(d)$ induces a surjective map on units, $(R/I)^* \to (R/(d))^*$. Here we write S^* for the set of invertible elements in a ring S.

10G Analysis and Topology

Let X, Y be topological spaces. Briefly describe the product topology on the space $X \times Y$ and show that the maps

$$\begin{array}{rcl} \Pi_X : X \times Y & \to & X \\ (x,y) & \mapsto & x \end{array}, \quad \text{and} \quad \begin{array}{rcl} \Pi_Y : X \times Y & \to & Y \\ (x,y) & \mapsto & y \end{array}$$

are continuous.

Let $\Delta = \{(y_1, y_2) \in Y \times Y : y_1 = y_2\}$ be the diagonal in $Y \times Y$. Show that Δ is closed if and only if Y is Hausdorff.

Suppose that $f : X \to Y$ is continuous and Y is Hausdorff. Show that the graph of f

$$\Gamma_f = \{(x,y) \in X \times Y : y = f(x)\}$$

is closed.

Let $f : X \to Y$ where X and Y are both compact Hausdorff spaces and suppose that Γ_f is closed. By considering $\Pi_X(\Gamma_f \cap (X \times C))$ for C a closed set in Y, or otherwise, show that f is continuous.

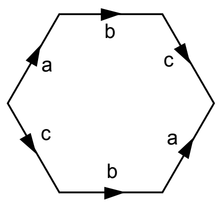
Give an example of a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ whose graph is closed to show that the previous result need not hold if X and Y are only assumed to be Hausdorff.

[*You may use results from lectures provided they are clearly stated.*]

11F Geometry

(a) What is a *topological surface*?

Show that a regular hexagon Σ with opposite sides identified as shown is a compact topological surface.



Explain briefly why Σ is homeomorphic to a torus.

Show that a cone $C = \{(x,y,z) : x^2 + y^2 - z^2 = 0\}$ in $\mathbb{R}^3$ is not a topological surface but a half-cone $C_+ = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 0,\ z > 0\}$ is.

(b) By considering parametrizations, construct a map $\pi : \mathbb{R}^2 \setminus \{(0,0)\} \to C_+$ which is a local isometry. Show that if f is an isometry of C_+ onto itself and $f(p) = p$ for some point $p \in C_+$, then f is either a restriction to C_+ of a reflection in a plane in $\mathbb{R}^3$ or the identity map.

[*The expression for the first fundamental form on $\mathbb{R}^2$ in polar coordinates can be assumed without proof. You may assume that local isometries map geodesics to geodesics.*]

12 Complex Analysis OR Complex Methods

This is the joint question for Complex Analysis/Complex Methods. Attempt only ONE of the sub-questions. On your answer sheet, specify the question number as either "12.1G" or "12.2A".

(12.1G) Complex Analysis

Use the residue theorem to give a proof of Cauchy's derivative formula: if f is holomorphic on $D(a,R) = \{z \in \mathbb{C} : |z-a| < R\}$ and $|w-a| < r < R$ then

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_{|z-a|=r} \frac{f(z)}{(z-w)^{n+1}} dz.$$

Let (g_k) be a sequence of holomorphic functions $g_k : D(a,R) \to \mathbb{C}$ which converges locally uniformly to a holomorphic function g.

Show that $\left(g_k^{(n)}\right)$ converges locally uniformly to $g^{(n)}$ for all $n = 0, 1, \ldots$.

Suppose further that g has a zero of order $m \geqslant 1$ at a and vanishes nowhere else in $D(a,R)$. Show that for any $0 < \epsilon < R$ there exists $K \in \mathbb{N}$ such that for all $k \geqslant K$, g_k has exactly m zeros in $D(a,\epsilon)$, counting with multiplicity.

(12.2A) Complex Methods

(a) Let f be a holomorphic function in the complex plane, except for potentially $n \in \mathbb{N}$ points that are poles. Suppose also that $\int_\gamma p(z)^2 f(z)\, dz = 0$ for all complex polynomials p and every closed contour γ avoiding the potential poles of f. Show that f is entire.

(b) Suppose that h is entire, of the form $h(z) = h(x+iy) = u(x,y) + iv(x,y)$, is real on the real axis, and has positive imaginary part in the upper half-plane (that is $v(x,y) > 0$ when $y > 0$).

  (i) Show that $h'(x) \geqslant 0$ when x is real.

  (ii) Show that if $h(0) = 0$, then $h'(0) \neq 0$.

13C Variational Principles

This question concerns the movement of a particle in space $\mathbb{R}^3$. Introduce cylindrical coordinates (ρ, ϕ, z) and assume that the trajectory of the particle can be parameterized as a curve

$$z \mapsto (\rho(z), \phi(z))$$

going from $A = (\rho(z_0), \phi(z_0), z_0)$ to $B = (\rho(z_1), \phi(z_1), z_1)$, and is such as to make the following functional stationary:

$$F[\rho, \phi] = \int_{z_0}^{z_1} n(\rho, \phi, z) \sqrt{1 + \left(\frac{\partial \rho}{\partial z}\right)^2 + \rho^2 \left(\frac{\partial \phi}{\partial z}\right)^2} dz\,, \quad \text{where } z_1 > z_0\,,$$

where the function $n = n(\rho, \phi, z)$ is positive and smooth. Write down the Euler-Lagrange equations for this functional.

In the case that $n = n(\rho)$ depends only on ρ, show that there are special solutions to the Euler-Lagrange equations of the form

$$\rho(z) = R, \qquad \phi(z) = \phi_0 + \omega(R) z,$$

where R and ϕ_0 are constants, and $\omega = \omega(R)$ solves an equation

$$n(R) R \omega^2 + a(1 + R^2 \omega^2) n'(R) = 0 \qquad (*)$$

for some constant a which you should find. [*You may assume (∗) has two solutions* $\pm\omega$, *with* $\omega > 0$.]

Find a condition on a positive number L which implies that the points having cylindrical coordinates $(R, \phi_0, 0)$ and (R, ϕ_0, L) can be joined by means of these special solutions and sketch two of them.

14B Methods

Consider a string of uniform mass density ρ that is stretched under tension τ along the x-axis. The string undergoes small transverse oscillations in the (x, y) plane, with displacement represented by $y(x,t)$. Derive the equation of motion governing $y(x,t)$, identifying the wave speed c in terms of ρ and τ (neglecting gravity).

The string is fixed at both ends, $x = 0$ and $x = L$. Determine the general solution for the oscillatory motion of the string using the method of separation of variables.

Assume the string is at rest for $t < 0$. At time $t = 0$, the string is struck by a hammer within the interval $[l - \epsilon/2, l + \epsilon/2]$, where $x = l$ represents the position along the string. The hammer's impact imparts a constant velocity $v/\sqrt{\epsilon}$ to the section of the string within this interval, while the rest of the string remains unaffected. Calculate the total energy imparted to the string by this blow. Determine the eigenmode coefficients for the resulting string solution and the energy excited in each mode relative to the total energy.

In musical terms, the $n = 7$ eigenmode is generally regarded as dissonant. Where can you strike the string in order to minimise the vibration of this mode? Briefly comment on the power law fall-off of the energy in each mode as the hammer head narrows, $\epsilon \to 0$?

15C Quantum Mechanics

This question concerns one dimensional quantum mechanics on the real line, with the momentum operator given by $p = -i\hbar\frac{d}{dx}$ as usual. A pair of distinct one dimensional Hamiltonians

$$H_+ = \frac{p^2}{2m} + V_+(x) \;\text{ and }\; H_- = \frac{p^2}{2m} + V_-(x)\,, \qquad -\infty < x < +\infty,$$

are said to be *partners* if there exists a function $f = f(x)$ such that

$$H_\pm = \frac{1}{2m}(p \pm if)(p \mp if)\,.$$

Show that $[f(x), p] = i\hbar f'(x)$. Taking the upper sign show that

$$V_+(x) = \frac{1}{2m}\left(f(x)^2 - \hbar f'(x)\right)$$

and find V_-.

Choosing f appropriately, find a partner Hamiltonian $H_+ = \frac{p^2}{2m} + V_+(x)$ to

$$H_- = \frac{p^2}{2m} + 2m$$

giving $V_+(x)$ explicitly in as simple a form as possible. [*Hint:* $\operatorname{sech}^2 z + \tanh^2 z = 1$.] Show that $\lim_{x\to\pm\infty} V_+(x) = 2m$.

By considering the solutions $e_k(x) = e^{ikx}$ to $H_- e_k = \left(\frac{\hbar^2k^2}{2m} + 2m\right) e_k$ and applying the operator $p + if$, show that it is possible to generate a corresponding solution to the partner Hamiltonian H_+. Hence compute the reflection and transmission coefficients for the Hamiltonian

$$\frac{p^2}{2m} - 4m\,\operatorname{sech}^2\left(\frac{2mx}{\hbar}\right)\,.$$

16B Electromagnetism

Consider two different vector potential fields $\mathbf{A}_1(x,y,z) = b_1\,(-y,\ x,\ 0)$ and $\mathbf{A}_2(x,y,z) = b_2\,(-y\,z,\ x\,z,\ 0)$, where b_1, b_2 are constants and the vertical direction is aligned with the z-axis. Calculate the associated static magnetic fields $\mathbf{B}_1(x,y,z)$ and $\mathbf{B}_2(x,y,z)$ and show they satisfy the vacuum Maxwell equations.

Consider a circular conducting loop of radius r and resistance R that is constrained to lie horizontally and centred along the z-axis. For the two magnetic fields $\mathbf{B}_1$ and $\mathbf{B}_2$, determine the respective magnetic fluxes through the loop at a vertical position z. Suppose a current I flows around the loop in a clockwise direction and calculate the force $\mathbf{F}$ from the magnetic field acting on the loop in each case.

Suppose the loop has a mass m and is allowed to fall from rest at position z_0 under the influence of gravity (with no initial current). What is the induced current that results from its motion $\dot{z}$ for the two different magnetic fields $\mathbf{B}_1$ and $\mathbf{B}_2$? Hence calculate any resistive forces that emerge. Write down an equation of motion for the vertical position in each case and identify the asymptotic behaviour of the trajectories. In one case, the trajectory approaches a terminal velocity at which you should compare the power loss from the current with the change in the gravitational potential energy. Briefly comment on the behaviour of a superconducting loop with $R = 0$ if it is dropped in the same manner.

17A Numerical Analysis

Consider the scalar autonomous ODE of the form

$$y' = f(y), \quad y(0) = y_0 \in \mathbb{R}, \tag{$*$}$$

where $y(t)$ exists and is unique for $t \in [0,T]$ and $T > 0$. Consider also the following two Runge-Kutta methods:

$$k_1 = f(y_n), \quad k_2 = f\big(y_n + \frac{h}{2}k_1 + \frac{h}{2}k_2\big), \quad y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2), \tag{$\dagger$}$$

$$\text{same as } (\dagger) \text{ except } k_2 = f\big(y_n + \frac{h}{4}k_1 + \frac{3h}{4}k_2\big), \tag{$\ddagger$}$$

both producing a sequence $\{y_n\}_{n\leqslant N}$, where $N = \lfloor \frac{T}{h} \rfloor$ and $h > 0$ is the step-size.

(a) Do the above Runge-Kutta methods have the same order? If so, determine the order. If not, determine which method has the highest order.
[*Hint: Think about how both methods can be written in terms of a single parameter.*]

(b) For a numerical method approximating the solution of ($*$), define the linear stability domain. What does it mean for such a numerical method to be A-stable?

(c) Are any of the Runge-Kutta methods ($\dagger$) and ($\ddagger$) A-stable? If so, determine the linear stability domain for the method(s).

18H Markov Chains

Let $T_1, T_2, \ldots$ be independent and identically distributed random variables taking values in $\{1, \ldots, N\}$. Construct $(X_n)_{n \geqslant 0}$ as follows. First $X_0 = 0$. For $1 \leqslant n \leqslant T_1$, let $X_1 = T_1 - 1$ and $X_n = X_{n-1} - 1$. Note that $X_{T_1} = 0$. For $T_1 + 1 \leqslant n \leqslant T_1 + T_2$, let $X_{T_1+1} = T_2 - 1$ and $X_n = X_{n-1} - 1$ until $X_{T_1+T_2} = 0$. This pattern repeats forever with $X_{T_1+T_2+1} = T_3 - 1$ and so forth.

(a) Let $S_0 = 0$ and $S_k = T_1 + \ldots + T_k$ for $k \geqslant 1$. Show that

$$X_n = \max\{S_{k+1} : S_k < n\} - n$$

for $n \geqslant 1$.

(b) Find the transition probabilities of the Markov chain $(X_n)_{n \geqslant 0}$ in terms of the given constants $q_j = \mathbb{P}(T_1 = j)$ for $1 \leqslant j \leqslant N$.

(c) Show that there is a unique invariant distribution $(\pi_i)_{0 \leqslant i \leqslant N-1}$ for the Markov chain and compute it in terms of $(q_j)_{1 \leqslant j \leqslant N}$.

(d) Find an example of $(q_j)_{1 \leqslant j \leqslant N}$ such that $\mathbb{P}(X_n = 0)$ does not converge as $n \to \infty$.

Pick ε such that $0 < \varepsilon < 1$ and consider a Markov chain $(X_n^{(\varepsilon)})_{n \geqslant 0}$ on $\{0, \ldots, N-1\}$ with $X_0^{(\varepsilon)} = 0$ and transition matrix $P^{(\varepsilon)} = (p_{i,j}^{(\varepsilon)})_{i,j}$ given by

$$P^{(\varepsilon)} = (1 - \varepsilon)P + \varepsilon I$$

where $P = (p_{i,j})_{i,j}$ is the transition matrix for $(X_n)_{n \geqslant 0}$ found in part (b) and I is the $N \times N$ identity matrix.

(e) Show that $\mathbb{P}(X_n^{(\varepsilon)} = 0)$ converges as $n \to \infty$, and compute the limit in terms of $(q_j)_{1 \leqslant j \leqslant N}$ and ε.

END OF PAPER