MAT1

MATHEMATICAL TRIPOS Part IB

Tuesday, 10 June, 2025 9:00am to 12:00pm

PAPER 1

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most four** questions from Section I and **at most six** questions from Section II.

If you attempt any of the joint Complex Analysis and Complex Methods questions, you may submit an answer to at most one of the two subquestions.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Linear Algebra

Define the *determinant* of an $n \times n$ matrix A. Define the *adjugate matrix* $\operatorname{adj}(A)$. Express $\det A$ in terms of $\operatorname{adj}(A)$ and A.

For each $n \geqslant 2$ let A_n be the $n \times n$ matrix defined by

$$(A_n)_{ij} = \begin{cases} 2 & i = j, \\ -1 & |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

What is $\det A_n$? Justify your answer.

2F Geometry

Let $\sigma : U \to \Sigma \subset \mathbb{R}^3$ be a smooth parametrization of an embedded surface in $\mathbb{R}^3$ and let $\gamma : [a,b] \to \Sigma$ be a smooth curve on Σ. Define the *energy* of γ. Deduce from the Euler–Lagrange equations of a stationary curve for the energy function the ordinary differential equations on U defining the geodesics on $\sigma(U) \subset \Sigma$.

Suppose that a plane $P \subset \mathbb{R}^3$ contains the unit normal vector for Σ at each point of the intersection $\Sigma \cap P$. If a curve η is parametrized with constant speed and is contained in $\Sigma \cap P$, show that η is a geodesic on Σ.

3 Complex Analysis OR Complex Methods

This is the joint question for Complex Analysis/Complex Methods. Attempt only ONE of the sub-questions. On your answer sheet, specify the question number as either "3.1G" or "3.2A".

(3.1G) Complex Analysis

State and prove Jordan's Lemma.

Find

$$\int_{-\infty}^{\infty} \frac{x \sin x}{1+x^2} dx.$$

(3.2A) Complex Methods

(a) State the Residue Theorem.

(b) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos(nx)}{x^4+1} dx, \quad n \in \mathbb{N}.$$

4C Variational Principles

Given a real symmetric $n \times n$ matrix A, consider the quadratic function

$$Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}\,,$$

on the unit sphere $S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{x} = 1\}$. Assume that $\mathbf{x_0}$ is a unit vector such that

$$Q(\mathbf{x_0}) \geqslant Q(\mathbf{x})\,, \quad \forall\, \mathbf{x} \in S^{n-1}\,.$$

Show that $\mathbf{x_0}$ is an eigenvector of the matrix A and determine the corresponding eigenvalue E. How does this eigenvalue compare to the other eigenvalues of A?

For the case that

$$A = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix} \qquad -\infty < t < +\infty$$

calculate E, as a function of $t \in \mathbb{R}$, and draw a sketch to show that it is convex.

5A Numerical Analysis

Consider the ODE of the form

$$y'(t) = f(t, y(t)), \quad y(0) = y_0 \in \mathbb{R}, \qquad (*)$$

where $y(t)$ exists and is unique for $t \in [0, T]$ and $T > 0$.

(a) State the Dahlquist equivalence theorem regarding convergence of a multistep method.

(b) Consider the following multistep method for $(*)$ with a parameter $\alpha \in \mathbb{R}$:

$$y_{n+3} + (2\alpha - 3)(y_{n+2} - y_{n+1}) - y_n = h\alpha\big(f(t_{n+2}, y_{n+2}) - f(t_{n+1}, y_{n+1})\big),$$

producing a sequence $\{y_n\}_{n \leqslant N}$, where $N = \lfloor \frac{T}{h} \rfloor$ and $h > 0$ is the step-size. It is given that the method is of at least order 2 for any α and also of order 3 for $\alpha = 6$. Determine all values of α for which the method is convergent, and find the order of convergence.

6H Statistics

The distribution of a random variable X depends on an unknown parameter θ. Consider testing the null hypothesis $H_0 : \theta = \theta_0$ versus the alternative hypothesis $H_1 : \theta = \theta_1$. State and prove the Neyman–Pearson lemma in the case where X has a probability density function $p(x; \theta)$.

Let X have probability density function $p(x; \theta) = \frac{1}{2}\theta e^{-\theta|x|}$ for $x \in \mathbb{R}$ and $\theta > 0$. Find the critical region of the most powerful test of size α when $\theta_0 < \theta_1$.

7H Optimisation

(a) Derive the dual problem to

$$\text{maximise } c^\top x \text{ subject to } Ax \leqslant b, \ x \geqslant 0$$

where the vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and the $m \times n$ matrix A are given, and the inequalities are interpreted component-wise.

(b) Find the optimal solution to

$$\text{maximise } 2x_1 + 3x_2 + 4x_3 \quad \text{subject to} \quad \begin{aligned} &x_1 + 3x_2 + x_3 \leqslant 2, \\ &x_1 + x_2 + 4x_3 \leqslant 1, \\ &x_1, x_2, x_3 \geqslant 0. \end{aligned}$$

SECTION II

8F Linear Algebra

State the rank-nullity theorem, explaining all the quantities that appear in it.

Let U, V, W be vector spaces where U and V are finite-dimensional. If $\alpha : V \to W$, $\beta : U \to V$ are linear maps, prove that

$$\mathrm{rk}(\alpha \circ \beta) \geqslant \mathrm{rk}(\alpha) + \mathrm{rk}(\beta) - \dim V.$$

If X and Y are matrices representing the same linear map between two finite-dimensional vector spaces with respect to different bases, write down the relation satisfied by X and Y. [You should explain the terms appearing in this relation.]

Let $M_k(\mathbb{C})$ denote the vector space of all $k \times k$ complex matrices. Let A be a block matrix of the form $A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$, where $P \in M_n(\mathbb{C})$, $S \in M_m(\mathbb{C})$ and P is invertible. Show that

$$\mathrm{rk}(A) = \mathrm{rk}(P) + \mathrm{rk}(S - RP^{-1}Q).$$

Deduce that

$$\mathrm{rk}(I_n - QR) = \mathrm{rk}(I_m - RQ) + n - m$$

where I_r is the identity matrix of size r.

9E Groups, Rings and Modules

(a) Let H be a proper subgroup of a finite, non-abelian group G. Prove that if G is simple, $|G/H| \geqslant 5$.

(b) Let S be a Sylow p-subgroup of a finite group G. Suppose that $gSg^{-1} \cap S = \{1\}$ for all $g \in G \setminus N_G(S)$.

Show that the number of Sylow p-subgroups is congruent to 1 mod $|S|$.

(c) Let G be a simple group of order 168.

(i) Compute the number of Sylow 7-subgroups of G. Compute the number of elements of G of order exactly 7.

(ii) Let S be a Sylow 2-subgroup of G. Show that there exists a Sylow 2-subgroup S' of G, $S \neq S'$, with $S \cap S' \neq \{1\}$. Show $S \cap S'$ contains an element of order 2.

10G Analysis and Topology

Given a metric space (X, d) state what it means for a function $f : X \to \mathbb{R}$ to be uniformly continuous. Let $C_{b,u}(X)$ be the space of bounded, uniformly continuous, functions $f : X \to \mathbb{R}$ equipped with the metric

$$d'(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

Show that $(C_{b,u}(X), d')$ is complete.

Assume $\mathbb{R}^n$ carries the Euclidean metric and let $C_0(\mathbb{R}^n)$ be the space of continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ satisfying $f(x) \to 0$ as $|x| \to \infty$. Show that $C_0(\mathbb{R}^n)$ is a subset of $C_{b,u}(\mathbb{R}^n)$. Is $C_0(\mathbb{R}^n)$ a closed subset of $C_{b,u}(\mathbb{R}^n)$? Is it compact? Justify your answer in each case.

11F Geometry

Define an (allowable) parametrization of an embedded smooth surface $S \subset \mathbb{R}^3$.

Suppose that S is a surface of revolution, meaning for each real θ the rotation

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

defines a diffeomorphism of S onto itself. Stating any result(s) that you use, show that S admits, around each point which is not on the z-axis, a local parametrization of the form

$$\sigma(u, v) = (f(u)\cos v, f(u)\sin v, g(u)) \quad \text{with } (f'(u), g'(u)) \neq (0, 0),$$

where $|v| < \pi$, $|u| < \varepsilon$ for some $\varepsilon > 0$.

We say that an embedded smooth $\Sigma \subset \mathbb{R}^3$ is a *ruled surface* if Σ admits a parametrization of the form

$$\psi(s, t) = a(s) + tb(s), \qquad (s, t) \in I \times \mathbb{R},$$

where $I \subset \mathbb{R}$ is an interval, $a, b : I \to \mathbb{R}^3$ are embedded smooth curves and $b(s)$ is a unit vector for all $s \in I$. Explain why we must have $a'(s) \times b(s) + tb'(s) \times b(s) \neq 0$ for all s, t.

Suppose that a path-connected, ruled surface Σ is also a surface of revolution. Suppose also that for some s_0 the affine line $a(s_0) + tb(s_0)$, $t \in \mathbb{R}$, in $\mathbb{R}^3$ neither meets the z-axis, nor is parallel to the z-axis. Prove that then Σ is diffeomorphic to a one-sheet hyperboloid $x^2 + y^2 = 1 + z^2$.

[*You may assume if you wish that a hyperboloid $x^2 + y^2 = 1 + z^2$ is a complete surface, not contained as a proper subset in any embedded smooth connected surface.*]

12 Complex Analysis OR Complex Methods

This is the joint question for Complex Analysis/Complex Methods. Attempt only ONE of the sub-questions. On your answer sheet, specify the question number as either "12.1G" or "12.2A".

(12.1G) Complex Analysis

Let $A = \{z \in \mathbb{C} : r < |z| < R\}$ and suppose $f : A \to \mathbb{C}$ is holomorphic. Show that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

with the sum converging locally uniformly, where you should give an expression for the coefficients $a_n \in \mathbb{C}$ in terms of a contour integral involving f.

Let $D^*(R) = \{z \in \mathbb{C} : 0 < |z| < R\}$ and suppose $f : D^*(R) \to \mathbb{C}$ is holomorphic. What does it mean in terms of the a_n for f to have a (*i*) *removable singularity*; (*ii*) *pole of order* $k \geqslant 1$; (*iii*) *essential singularity* at $z = 0$.

For each of the following holomorphic functions $f_i : D^*(1) \to \mathbb{C}$, determine the type of the singularity at $z = 0$:

(i) $f_1(z) = \dfrac{1}{z^2} - \dfrac{1}{\sin^2 z}$;

(ii) $f_2(z) = \displaystyle\int_{-1}^{1} e^{-t^2/z^2} dt.$

(12.2A) Complex Methods

(a) Let f be an analytic function on an open disc D whose centre is the point $z_0 \in \mathbb{C}$. Assume that $|f'(z) - f'(z_0)| < |f'(z_0)|$ on D. Prove that f is one-to-one on D.

(b) What does it mean for a function $g : \mathbb{R}^2 \to \mathbb{R}$ to be harmonic?

  (i) Suppose that $\tilde{u}$ is a positive ($\tilde{u} \geqslant 0$) harmonic function on $\mathbb{R}^2$. Show that $\tilde{u}$ is constant.

  (ii) Let u be a real valued harmonic function in the complex plane (we identify $\mathbb{C}$ with $\mathbb{R}^2$) such that
  $$u(z) \leqslant a\big|\log(|z|)\big| + b$$
  for all $z \in \mathbb{C}$, where a and b are positive constants. Prove that u is constant.

13B Methods

(a) Legendre's differential equation on the domain $-1 < x < 1$ is given by

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda\, y = 0\,.$$

Put this equation in Sturm-Liouville form and show that the Sturm-Liouville operator is self-adjoint with respect to an inner product you should specify. Briefly state some key properties of the eigenvalues λ_k and eigenfunctions $y_k(x)$ of any Sturm-Liouville differential equation.

Consider a series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ of Legendre's equation and show that the coefficients a_n satisfy the recurrence relation

$$\frac{a_{n+2}}{a_n} = \frac{n(n+1)-\lambda}{(n+1)(n+2)}\,.$$

Hence, show that polynomial solutions $y(x) = P_\ell(x)$ of degree ℓ exist when $\lambda = \ell(\ell+1)$, where ℓ is a non-negative integer ($\ell \geqslant 0$). Find expressions for $P_1(x)$ and $P_3(x)$, adopting the convention that $P_n(1) = 1$.

(b) Laplace's equation in spherical polars for the axisymmetric case takes the form

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{2}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial x}\left(\left(1-x^2\right)\frac{\partial\Phi}{\partial x}\right) = 0\,,$$

where $x = \cos\theta$. State the general form of the solution $\Phi(r,x)$ obtained using the method of separation of variables (derivation not required).

Suppose that on the sphere at $r = R$, the boundary condition is $\Phi(R,x) = x(1-x^2)$. Find the regular solution in the interior of the sphere.

14C Quantum Mechanics

Consider the one-dimensional potential

$$V(x) = \begin{cases} 0 & \text{if } |x| > a\,, \\ -\frac{1}{2a} & \text{if } |x| \leqslant a\,, \end{cases}$$

where $a > 0$. Show that for positive a there exist *normalizable* and *even* solutions to the stationary Schrödinger equation

$$-\frac{\hbar^2}{2m}\psi'' + V(x)\psi = E\psi\,, \qquad -\infty < x < +\infty\,,$$

with energy $E = E(a) = -\frac{\hbar^2 \kappa(a)^2}{2m}$, where $\kappa(a) > 0$ satisfies an equation which you should give. Show that for small positive a the energy is unique and the solution is unique up to multiplication by a constant.

Now consider the limit $a \to 0+$. Calculate the limiting value $E_0 = \lim_{a\to 0+} E(a)$, and show that this is the energy of a normalizable and even solution ψ_0 to the stationary Schrödinger equation with a singular potential V_0; give V_0 and ψ_0 explicitly.

15B Electromagnetism

In a volume V, an electrostatic charge density $\rho(\mathbf{x})$ induces an electric field $\mathbf{E}(\mathbf{x})$ with electrostatic potential $\phi(\mathbf{x})$ which vanishes on the boundary. Use Maxwell's equations, to show that the electrostatic energy,

$$U = \frac{1}{2}\int_V d^3x\, \rho(\mathbf{x})\, \phi(\mathbf{x})\,,$$

can be expressed in terms of the electric field $\mathbf{E}(\mathbf{x})$.

Consider three concentric spherical shells with uniformly distributed surface charges $Q_1 = q$, $Q_2 = -2q$, $Q_3 = q$, placed around the origin at radii $r_1 = R$, $r_2 = 2R$, $r_3 = 3R$, respectively. Use Gauss's Law to find the electric field $\mathbf{E}(\mathbf{x})$ at all points in space. Likewise determine the potential $\phi(\mathbf{x})$ everywhere. Calculate the total electrostatic energy U, both by using the displayed equation above and using the electric-field formulation, and verify that they agree.

16D Fluid Dynamics

State Bernoulli's equation for steady flow.

Starting from Euler's equations governing steady, inviscid, flow $\mathbf{u}$ of an incompressible fluid of density ρ subject to a conservative body force $\mathbf{f} = -\nabla\chi$, derive the integral momentum equation

$$\int_{\partial V} (\rho \mathbf{u} \cdot \mathbf{n}\, \mathbf{u} + p\mathbf{n} + \chi\mathbf{n})\, dS = 0,$$

where p is the fluid pressure and $\mathbf{n}$ is the unit normal to the surface ∂V of a closed domain V.

A large circular blood vessel of cross-sectional area A bifurcates symmetrically with respect to its axis into two smaller circular blood vessels, each of cross-sectional area a and each inclined at angle α to the axis of the larger blood vessel. The constant volume flux through the system is q.

(i) Determine the pressure drop between a location in the larger vessel upstream of the junction and a location in one of the smaller blood vessels downstream of the junction.

(ii) Given that the pressure inside the smaller vessels far downstream of the junction is equal to the uniform pressure of the body tissue surrounding the vessels, determine the force on the junction in terms of the parameters given above.

17A Numerical Analysis

(a) Define *Householder reflections* and show that a real Householder reflection is a symmetric and orthogonal matrix.

(b) Let $H \in \mathbb{R}^{n\times n}$ be a Householder reflection. Determine the eigenvalues of H and their multiplicities.

(c) Show that for any $A \in \mathbb{R}^{n\times n}$ there exist Householder reflections $H_1, \ldots, H_n$ such that $H_n H_{n-1} \cdots H_1 A = R$, where R is upper triangular.

(d) Show that if A is symmetric there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that $C = QAQ^T \in \mathbb{R}^{n\times n}$ is symmetric and tridiagonal (that is, only the diagonal, super and subdiagonal have non-zero entries), and C can be computed in finitely many operations $(+, -, \times, \div, \sqrt{\ })$.

18H Statistics

A data set contains the ordered pairs of observations $(X_1, Y_1), \ldots, (X_n, Y_n)$. A statistician models these data as $Y_i = X_i\beta + \varepsilon_i$, where $X_1, \ldots, X_n$ are known real parameters, the noise $\varepsilon_i \sim N(0, \sigma^2)$ are independent and identically distributed, and the real parameters β and σ^2 are unknown.

(a) Find the maximum likelihood estimator $\widehat{\beta}$ for β and $\widehat{\sigma^2}$ for σ^2. Using standard properties of normal random variables, show that $\widehat{\beta}$ and $\widehat{\sigma^2}$ are independent.

(b) Find a $(1-\alpha)$-confidence interval for β. Express your answer in terms of the cumulative distribution function of the t_k distribution for an appropriately chosen k.

(c) Let $\widetilde{\beta} = \sum_{i=1}^n c_i Y_i$, where $c_1, \ldots, c_n$ are known constants. If $\widetilde{\beta}$ is an unbiased estimator of β, show that

$$\mathrm{Var}(\widetilde{\beta}) \geqslant \frac{\sigma^2}{\sum_{i=1}^n X_i^2}.$$

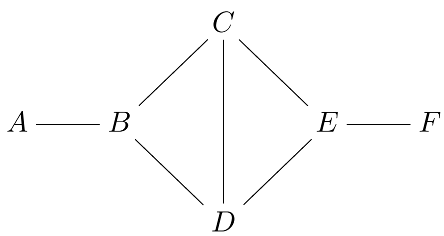
For which choice of constants $c_1, \ldots, c_n$ is there equality for all (β, σ^2)? [If you use the Gauss–Markov theorem, you must prove it.]

(d) Another statistician models the same data as $X_i = Y_i b + e_i$, where now it is assumed that $Y_1, \ldots, Y_n$ are known parameters, the noise $e_i \sim N(0, s^2)$ are i.i.d., and the real parameters b and s^2 are unknown. Let $\widehat{b}$ and $\widehat{s^2}$ be the maximum likelihood estimators of b and s^2 respectively. Show that $\hat{b}\hat{\beta} \leqslant 1$, with equality only if $\widehat{\sigma^2} = 0 = \widehat{s^2}$.

19H Markov Chains

(a) What does it mean to say a Markov chain is *reversible*? Show that a random walk on a finite connected graph is reversible.

Consider the random walk $(X_n)_{n\geqslant 0}$ on this graph, where $X_0 = A$.



(b) Find the expected number of steps until the walk first returns to A.

(c) Find the probability that the walk returns to A before hitting F.

(d) Given that the walk returns to A before hitting F, find the conditional expected number of steps until the walk first returns to A.

END OF PAPER