UNIVERSITY OF CAMBRIDGE

MAT0

MATHEMATICAL TRIPOS Part IA

Wednesday, 11 June, 2025 1:30pm to 4:30pm

PAPER 4

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **all four** questions from Section I and **at most five** questions from Section II. Of the Section II questions, no more than three may be on the same course.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Numbers and Sets

What does it mean to say that a series $\sum_{n=1}^{\infty} a_n$ *converges*? For $n \geqslant 1$, let

$$a_n = \frac{2(n+1)}{\sqrt{n^2+3n+2}+\sqrt{n^2+n}} - 1.$$

Does the series $\sum_{n=1}^{\infty} a_n$ converge? Justify your answer.

2E Numbers and Sets

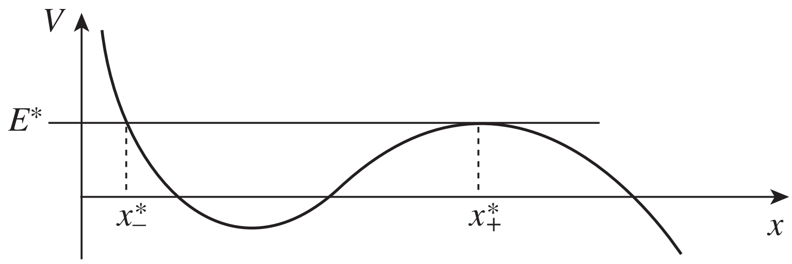
(a) State the fundamental theorem of arithmetic and prove the uniqueness of factorization. [You may use without proof that if p is a prime and a, b are positive integers, then $p|ab$ implies $p|a$ or $p|b$.]

(b) Find a positive integer n such that $n/2$ is a square, $n/3$ is a cube and $n/5$ is a 5th power. [You do not need to compute the decimal representation of your example.]

3B Dynamics and Relativity

A particle of mass m moves in one dimension with position $x(t)$, subject to a potential $V(x)$. Write down the equation of motion and also its first integral for a trajectory of total energy E.

Suppose the potential $V(x)$ takes the form shown in the figure below with a local maximum at x_+^* where $V = E^*$ and a second point $x_-^* < x_+^*$ where also $V = E^*$.



The particle is released from rest at a point x_- in (x_-^*, x_+^*) where $V = E < E^*$ and $V' < 0$. Express the period T of the motion as an integral over the interval $[x_-, x_+]$, where $V(x_+) = V(x_-) = E$.

Consider T as a function of $E < E^*$. Setting $\delta = 1 - E/E^*$, show that

$$T = \sqrt{\frac{m}{-V''(x_+^*)}}\Big(\log(1/\delta) + O(1)\Big) \quad \text{as} \quad \delta \to 0\,.$$

[*Hint: It is helpful to approximate the integrand by Taylor expanding it around* $x = x_+^*$.]

Explain physically why T diverges as E approaches E^*.

4B Dynamics and Relativity

Two ice-hockey pucks A and B consist of rigid disks of radii R_A and R_B and masses M_A and M_B moving freely, without friction, in two dimensions. Initially Puck A is located at rest at the origin and Puck B is incident with speed u from $x = -\infty$ along the line $y = b$, with $0 < b < R_A + R_B$. The two pucks undergo an elastic collision during which the reaction force acts along the line between the two centres. Determine the velocities of the pucks after the collision.

SECTION II

5F Numbers and Sets

Prove that $\sqrt{2} + \sqrt{3}$ is an irrational number.

Prove that $\tan(\pi/8)$ is an irrational number. Is it an algebraic number?

Consider the grid $\mathbb{Z}^2$ as a subset of the plane $\mathbb{R}^2$. Show that the area of any triangle with vertices on the grid $\mathbb{Z}^2$ is a rational number. Deduce that the area of any convex polygon with vertices on the grid $\mathbb{Z}^2$ is a rational number.

Is it possible to draw a regular octagon with vertices on the grid $\mathbb{Z}^2$? If so, give an example; if not, justify your answer.

6E Numbers and Sets

(a) State the Chinese remainder theorem.

(b) Define *Euler's totient function* $\varphi(n)$.

(c) State the Fermat–Euler theorem.

(d) Let n, m be coprime positive integers. Prove $\varphi(nm) = \varphi(n)\varphi(m)$.

(e) Let n be a positive integer that is not divisible by the square of a prime. Let k be a positive integer such that $k \equiv 1 \mod \varphi(n)$. Show that $a^k \equiv a \mod n$ for all integers a. [You should *not* assume $(a, n) = 1$.]

(f) Explain briefly the *RSA public-key cryptography scheme.*

(g) Let n be a positive integer that is divisible by the square of a prime. Prove that there exist integers a, b such that $a \not\equiv b \mod n$, but $a^k \equiv b^k \mod n$ for all integers $k > 1$.

7D Numbers and Sets

(a) What does it mean for a function to be *injective*? Let $f\colon A \to A$ be an injective function on a set A. Is f necessarily surjective? Justify your answer.

(b) Let $f\colon A \to B$ be a function and suppose $S \subset A$ and $T \subset B$. Denote $f(S) = \{f(s)\colon s \in S\} \subset B$ and $f^{-1}(T) = \{a \in A\colon f(a) \in T\}$. Prove that

$$f\big(f^{-1}(T) \cap S\big) = T \cap f(S)\,.$$

If f is surjective, determine whether the following equality necessarily holds:

$$f(S \cap S') = f(S) \cap f(S')\,.$$

(c) Let $f\colon A \to B$ and $g\colon A' \to B$ be functions. Define

$$A \times_B A' = \big\{(a, a') \in A \times A'\colon f(a) = g(a')\big\}$$

and let $p\colon A \times_B A' \to A'$ be the function that maps (a, a') to a'. Determine whether the following statements are true or false. Justify your answers.

(i) If $f\colon A \to B$ is injective then p is injective.

(ii) If $f\colon A \to B$ is surjective then p is surjective.

8D Numbers and Sets

What does it mean for a set to be *countable*?

Let $A \subset \mathbb{R}$ be a countable subset of the real numbers and let $A[x]$ be the set of all polynomials in one variable with coefficients in A. Show that $A[x]$ is countable. Deduce that there exist uncountably many transcendental numbers.

Let N be a countable set and let $\mathcal{Q}$ denote the set of bijections $f\colon N \to N$ with the property that $f(x) \neq x$ for all $x \in N$. Is the set $\mathcal{Q}$ countable? Justify your answer.

Let $\{L_1, L_2, \ldots\}$ be a countable infinite set of straight lines in the plane $\mathbb{R}^2$. Prove that $\mathbb{R}^2$ is not equal to the union $\bigcup_{i=1}^{\infty} L_i$.

9B Dynamics and Relativity

A particle of mass m moves in three dimensions with position $\mathbf{x}(t)$ and velocity $\dot{\mathbf{x}}(t)$ subject to a force

$$\mathbf{F}_1 = \dot{\mathbf{x}} \times \mathbf{B}\,,$$

where $\mathbf{B} = (0, 0, B)$ is a constant vector. Find two constants of motion and explain physically why they are constant.

If the particle starts from $\mathbf{x} = \mathbf{0}$ at $t = 0$ with velocity $\dot{\mathbf{x}}(0) = (u, 0, 0)$, show that the resulting trajectory is a circle whose radius and centre you should determine.

Suppose now that the particle is also subject to a frictional force

$$\mathbf{F}_2 = -\mu\dot{\mathbf{x}}\,,$$

where μ is a positive constant. Find the trajectory of the particle subject to the force $\mathbf{F}_1 + \mathbf{F}_2$ and starting from the same conditions as before. Sketch the resulting path in the (x, y)-plane.

Consider the final position of the particle as a function of the initial speed u. Show that this position lies on a certain straight line, to be specified, and find the final distance of the particle from the origin.

10B Dynamics and Relativity

A particle of mass m moves in the central potential $V(r) = -km/r$, where k is a positive constant. Show that the general form of its trajectory in polar coordinates is

$$r = \frac{r_0}{1 + e\cos\theta}\,.$$

Find the total energy E of the motion, and its angular momentum per unit mass l, in terms of the two integration constants r_0 and e.

(a) An asteroid approaches the solar system from outer space. At early times its trajectory is asymptotically a straight line with perpendicular distance b from the Sun and its speed is v. At late times its asymptotic trajectory is another straight line. Determine the angle between these lines as a function of b and v.

(b) Another asteroid approaches the solar system following a trajectory with total energy $E = 0$ which reaches a minimum distance d from the Sun. Find the resulting motion in polar coordinates, obtaining the polar angle as

$$\theta(t) = 2\tan^{-1}\big(T(t)\big),$$

where $T(t)$ is the root of a certain cubic equation, which you should identify.

11B Dynamics and Relativity

Consider a collection of N particles with positions $\mathbf{x}_i(t)$ and masses m_i, for $i = 1, 2, \ldots, N$, rotating about an axis passing through the origin with a common angular velocity $\boldsymbol{\omega}$. Show that the total kinetic energy of the system takes the form $T = I\omega^2/2$, where $\omega = |\boldsymbol{\omega}|$, and obtain a formula for the moment of inertia I.

State the corresponding formula for the moment of inertia of a solid body of non-uniform density $\rho(\mathbf{x})$ about the same axis.

A hollow bowling ball of mass M has a uniform density and occupies an annular region bounded by two concentric spheres of radii $R_+ > R_-$. Determine the moment of inertia of the bowling ball about its centre.

The bowling ball rolls from rest down an inclined plane of vertical height h without slipping. Find its resulting speed when it reaches the bottom of the incline in terms of the radius ratio $\mu = R_-/R_+$.

Two bowling balls, labelled A and B, roll down the incline starting from rest at different times but following the same path. The balls have the same mass M and outer radius R_+, but different values of μ, denoted μ_A and μ_B respectively. If $\mu_A > \mu_B$ which ball reaches the bottom of the incline in the least time? Justify your answer.

A defective bowling ball C has the same dimensions as A, but its inner cavity is slightly off-centre relative to its outer boundary. The ball rolls down the incline starting from rest. Without attempting detailed calculation, explain briefly whether it reaches the bottom faster or slower than ball A.

12B Dynamics and Relativity

In some inertial frame S with coordinates $x^\mu = (ct, \mathbf{x})$ a relativistic particle follows a world line $x^\mu(\tau)$ parametrized by its proper time τ.

Define the *four-velocity* U^μ of the particle and find the components of U^μ in frame S in terms of the three-velocity $\mathbf{v} = d\mathbf{x}/dt$ with magnitude $v = |\mathbf{v}|$. Show that $U_\mu U^\mu = c^2$.

The *four-acceleration* of the particle is defined as $A^\mu = dU^\mu/d\tau$. Find the components of A^μ in frame S in terms of $\mathbf{v}$ and the three-acceleration $\mathbf{a} = d\mathbf{v}/dt$.

Let S' be the instantaneous rest frame of the particle at time t. Show that in this frame, the four-acceleration takes the form

$$(A^\mu)' = \begin{pmatrix} 0 \\ \mathbf{a}' \end{pmatrix}.$$

Consider a particle moving along the x-axis in frame S with three-velocity $\mathbf{v} = v\hat{\mathbf{x}}$ and three-acceleration $\mathbf{a} = a\hat{\mathbf{x}}$. In frame S' its three-acceleration is $\mathbf{a}' = a'\hat{\mathbf{x}}$. By performing an appropriate Lorentz transformation, show that

$$a' = \frac{a}{(1 - v^2/c^2)^{3/2}}.$$

Suppose further that the particle starts from rest at $x = t = 0$, and that its acceleration a' in frame S' is constant and positive. Find the trajectory $x(t)$ of the particle. Draw a spacetime diagram of the trajectory. Use your diagram to show that a light signal emitted at $t = 0$ from the point $\mathbf{x} = (x_0, 0, 0)$ can never intercept the particle if $x_0 < -c^2/a'$.

END OF PAPER