UNIVERSITY OF CAMBRIDGE

MAT0

MATHEMATICAL TRIPOS Part IA

Monday, 09 June, 2025 9:00am to 12:00pm

PAPER 3

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **all four** questions from Section I and **at most five** questions from Section II. Of the Section II questions, no more than three may be on the same course.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1D Groups

What is the *centre* $Z(G)$ of a group G? Let H be a subgroup of $Z(G)$. Is H necessarily normal in G?

Calculate the centre of the dihedral group D_{2n} for all $n \geqslant 3$.

2D Groups

Let $\mathbb{Z}$ denote the group of integers under addition and let C_n denote the cyclic group of order n. Determine whether each of the following statements is true or false. Justify your answers.

(i) There exists an injective homomorphism from any finite cyclic group to the group $\mathbb{Z}$.

(ii) There exists a surjective homomorphism from C_n to C_m for all $n \geqslant m$.

(iii) Every proper subgroup of the Cartesian product group $\mathbb{Z} \times \mathbb{Z}$ is cyclic.

3A Vector Calculus

Let D be a bounded region in $\mathbb{R}^2$ and $F : D \to \mathbb{R}$ a smooth function. Consider the surface in $\mathbb{R}^3$ defined by

$$S = \big\{ \big(x, y, F(x, y)\big) : (x, y) \in D \big\} .$$

Find an expression for the surface area of S.

Using this formula, compute the surface area in the following cases:

(i) D is the triangle with vertices $(0,0)$, $(2,0)$ and $(0,3)$ and $F(x,y) = 2x + 3y + 23$.

(ii) S is the spherical cap $\{x^2 + y^2 + z^2 = 1 : \; x^2 + y^2 \leqslant a, \; z \geqslant 0\}$, where $a \in (0,1)$.

Suppose that S is the unbounded surface $2z = x^2 - y^2$ in $\mathbb{R}^3$, defined for all x, y. Explain why the following integral exists and compute it:

$$\int_S \frac{dS}{(1 + x^2 + y^2)^2} .$$

4A Vector Calculus

State Green's theorem for integration in the plane.

Consider the region R expressed as $\{(x,y) : x^4 - 2x^3 + 2x^2y^2 - 2xy^2 + y^4 - y^2 \leqslant 0\}$. Express this region more simply in polar coordinates (r,θ).

Let C be the (positively oriented) boundary of R. Using Green's theorem, or otherwise, compute the line integral $\oint_C \mathbf{F} \cdot \mathbf{dr}$ for each of the following vector fields $\mathbf{F} = (F_x, F_y)$:

(i) $\mathbf{F} = \big(-y/\sqrt{x^2+y^2},\, x/\sqrt{x^2+y^2}\big)$ for $r > 0$, $\quad \mathbf{F} = (0,0)$ at $r = 0$,

(ii) $\mathbf{F} = (y^2, xy)$,

(iii) $\mathbf{F} = \big(0,\, \theta(x,y)\big)$ for $r > 0$ with $\theta \in (-\pi, \pi]$, $\quad \mathbf{F} = (0,0)$ at $r = 0$.

[*In parts (i) and (iii) you may assume that Green's theorem applies, despite the irregularity at* $r = 0$.]

SECTION II

5D Groups

(a) Consider a group G acting on a set X.

(i) What is the *orbit* of an element of X? What is the *stabiliser* of an element of X? Suppose every element $x \in X$ has a nontrivial stabiliser. Can the group action be faithful?

(ii) Let x and y be elements of X that lie in the same orbit under the action of G. Let $\mathrm{Stab}_G(x)$ and $\mathrm{Stab}_G(y)$ be the stabilisers of these elements. Prove that $\mathrm{Stab}_G(x)$ is isomorphic to $\mathrm{Stab}_G(y)$.

(b) Let k and n be positive integers with $1 \leqslant k \leqslant n$. Consider the action of the symmetric group S_n on the set of k-element subsets of $\{1, \ldots, n\}$. For which values of k is this action faithful? Justify your answer.

(c) The alternating group A_n does not contain a proper normal subgroup for $n \geqslant 5$. Using this fact, find all the normal subgroups of S_n for $n \geqslant 5$.

6D Groups

(a) Let D_8 be the dihedral group of size 8. Determine all normal subgroups of D_8.

(b) Let G be a group and K a subgroup of G. Prove that K is normal if and only if there exists a surjective homomorphism of groups $\varphi\colon G \to H$ such that $\ker(\varphi)$ is K.

(c) Prove or give a counterexample to the following statement: if G is a group in which every subgroup is normal, then G is abelian.

(d) Let $\varphi\colon G \to H$ be a homomorphism of groups.

(i) Let N be a normal subgroup of H. Prove that

$$\varphi^{-1}(N) = \{g \in G\colon \varphi(g) \in N\}$$

is a normal subgroup of G.

(ii) If φ is surjective and K is a normal subgroup of G, prove that

$$\varphi(K) = \{\varphi(g)\colon g \in K\}$$

is a normal subgroup of H.

7D Groups

Prove directly from the definition that every Möbius transformation has at least one fixed point. Prove that every non-identity Möbius transformation has either one or two fixed points.

Let $m > 1$ be a positive integer. Prove that there exist infinitely many distinct Möbius transformations of order m.

Let $f(z) = \dfrac{az+b}{cz+d}$ and $g(z) = \dfrac{\alpha z+\beta}{\gamma z+\delta}$ be Möbius transformations and let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Prove that if A and B are conjugate in the group $\mathrm{SL}_2(\mathbb{C})$ then $f(z)$ and $g(z)$ are conjugate in the Möbius group.

Prove or give a counterexample to the following statement: if $f(z)$ and $g(z)$ are conjugate in the Möbius group, then either A and B are conjugate in $\mathrm{SL}_2(\mathbb{C})$ or A and $-B$ are conjugate in $\mathrm{SL}_2(\mathbb{C})$.

8D Groups

Let p be a prime number and let $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ denote the set of 2×2 matrices with entries in $\mathbb{Z}/p\mathbb{Z}$ and with non-zero determinant.

Prove that $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ is a group under matrix multiplication. Calculate the order of this group. [You may assume standard facts about matrix multiplication.]

Does $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$ contain an element of order 3? Does it contain an element of order 6? Justify your answers.

Does $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ contain a proper normal subgroup for all primes p? Justify your answer.

Prove that $\mathrm{GL}_2(\mathbb{Z}/11\mathbb{Z})$ contains a non-abelian subgroup of size 55. [It may be useful to consider upper-triangular matrices.]

9A Vector Calculus

Consider the vector field $\mathbf{F} = (3xz, y^2z, x^2 + y^2)$. Let $a, b > 0$ and let V be the region enclosed by the elliptical cylinder $ax^2 + by^2 = 1$ and the planes $z = 0$ and $z = 3$.

(a) State the divergence theorem and use it to compute the flux of $\mathbf{F}$ through the closed surface S, where S is the boundary of V.

(b) Without using the divergence theorem, directly calculate the flux of $\mathbf{F}$ through the top, bottom and side surfaces of S and confirm that the results are consistent with that in part (a).

(c) Is the vector field $\mathbf{F}$ conservative? Let C be the closed curve consisting of the intersection of the plane $z = 1$ and the cylinder $ax^2 + by^2 = 1$. Compute the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{dr}\,.$$

10A Vector Calculus

(a) A polynomial $p(x,y)$ of two variables x and y is called *homogeneous of degree* n if it satisfies $p(\lambda x, \lambda y) = \lambda^n p(x,y)$ for all λ.

By deriving a recurrence relation for the coefficients, prove that there are precisely two linearly independent homogeneous polynomials of each degree $n \geqslant 1$ that satisfy Laplace's equation in two dimensions.

[In parts (b) and (c), you may assume that there is exactly one solution to Poisson's equation with the specified forcing and boundary conditions.]

(b) Consider solving Poisson's equation in spherical polar coordinates,

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = f(r,\theta,\phi)\,,$$

in the region $r > 1$ subject to the boundary conditions $u = g(\theta,\phi)$ on $r = 1$ and $u \to 0$ as $r \to \infty$. In each of the following cases, find u:

(i) $f(r,\theta,\phi) = -2e^{-r}/r + e^{-r} + 12/r^6$ and $g(\theta,\phi) = 1\,.$

(ii) $f(r,\theta,\phi) = \cos(2\theta)/(r^3 \sin\theta)$ and $g(\theta,\phi) = \sin\theta\,.$

(iii) $f(r,\theta,\phi) = -\sin\phi/(r^3 \sin^2\theta)$ and $g(\theta,\phi) = \sin\phi\,.$

[*Hint: Try looking for solutions of the form* $g_1(r)g_2(\theta)g_3(\phi)$ *motivated by the form of the boundary conditions.*]

(c) Consider solving Poisson's equation in Cartesian coordinates,

$$\nabla^2 u = f(x,y,z)\,,$$

in the cube $[0,1]\times[0,1]\times[0,1]$ subject to the boundary condition $u = 0$ on its surface. In each of the following cases, find u:

(i) $f(x,y,z) = x^2y^2 + x^2z^2 + y^2z^2 - x^2y - xy^2 - x^2z - xz^2 - y^2z - yz^2 + xy + xz + yz.$
[*Hint: Try polynomials that satisfy all the boundary conditions.*]

(ii) $f(x,y,z) = \sin(2\pi x)\sin(2\pi y)\sin(4\pi z) - \sin(2\pi x)\sin(\pi y)\sin(5\pi z)$. [*Hint: Apply the Laplacian to* f.]

11A Vector Calculus

Define the *Jacobian matrix* J of a transformation between two sets of curvilinear coordinates in n dimensions.

Let $x_1, \ldots, x_n$ denote the usual Cartesian coordinates in $\mathbb{R}^n$. The *hyperspherical coordinate system* $(r, \theta_1, \theta_2, \ldots, \theta_{n-2}, \phi)$ extends plane-polar and spherical-polar coordinates to higher dimensions and is defined implicitly for $n \geqslant 4$ through

$$\begin{aligned}
x_1 &= r\cos\theta_1\,, \\
x_j &= r\left(\prod_{k=1}^{j-1}\sin\theta_k\right)\cos\theta_j, \quad j = 2, \ldots, n-2, \\
x_{n-1} &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\phi\,, \\
x_n &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\phi\,,
\end{aligned}$$

where $0 \leqslant r < \infty$, $0 \leqslant \theta_i \leqslant \pi$ and $0 \leqslant \phi < 2\pi$.

Find the Jacobian matrix J_n of the transformation from Cartesian coordinates to hyperspherical coordinates in $\mathbb{R}^n$. By expanding about a suitable row or column of J_n, prove that its determinant $|J_n|$ satisfies the recurrence relation

$$|J_n| = r f_n(\theta_1, \theta_2, \ldots, \theta_{n-2})\,|J_{n-1}|\,,$$

for a function f_n that you should specify.

Find a formula for the volume element dV in hyperspherical coordinates.

What is the surface area element dS on the surface $r = R$ in $\mathbb{R}^n$, where R is a positive constant?

(i) By integrating the volume element, find the volume of the region $0 \leqslant r \leqslant R$ and express it in terms of factorials.

(ii) Find the area of the surface $r = R$.

(iii) What is the value of the following surface integral

$$\int_{r=R} x_i x_j \, dS, \quad \text{where} \quad i, j \in \{1, \ldots, n\}\,?$$

12A Vector Calculus

Let $\mathbf{v}$ be a vector field and let T be a second-rank tensor field in $\mathbb{R}^3$, and define

$$[T\mathbf{n}]_i = T_{ij}n_j \quad \text{and} \quad [\mathrm{div}(T)]_i = \frac{\partial T_{ij}}{\partial x_j}\,.$$

Let $V \subseteq \mathbb{R}^3$ be a region with boundary ∂V and outward unit normal $\mathbf{n}$. Prove that

$$\int_V \mathrm{div}(T)\cdot\mathbf{v}\,dV = \int_{\partial V}(T\mathbf{n})\cdot\mathbf{v}\,dS - \int_V T_{ij}\frac{\partial v_i}{\partial x_j}\,dV\,. \qquad (*)$$

Now let V be the unit cube $[0,1]\times[0,1]\times[0,1]$, with

$$T_{ij} = x_i x_j - \delta_{ij} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} x^2 \\ yx \\ zx \end{pmatrix}.$$

By computing the three integrals in $(*)$ separately, verify that the equation holds.

END OF PAPER