UNIVERSITY OF CAMBRIDGE

MAT0

MATHEMATICAL TRIPOS Part IA

Friday, 06 June, 2025 1:30pm to 4:30pm

PAPER 2

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **all four** questions from Section I and **at most five** questions from Section II. Of the Section II questions, no more than three may be on the same course.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1C Differential Equations

Find the stationary points of the function

$$V(x,y) = \tfrac{1}{3}y^3 + x^2y - y$$

and classify them by using the Hessian. Sketch the form of the contours of V.

A particle with position $\big(x(t), y(t)\big)$ moves with velocity $(\dot{x}, \dot{y}) = -\boldsymbol{\nabla} V$. Show that its trajectory $y(x)$ satisfies the equation

$$2xy\frac{dy}{dx} - y^2 = x^2 - 1\,.$$

By use of a suitable integrating factor, or otherwise, find the equation of a general trajectory in the form $y^2 = f(x,c)$, where c is an arbitrary constant.

2C Differential Equations

(a) Find the fixed points of the differential equation

$$\frac{dx}{dt} = f(x), \ \text{where } f(x) = ax + 2x^2 - x^3\,,$$

and determine, graphically or otherwise, the range of a in which each fixed point is stable.

(b) Find the fixed points of the difference equation

$$x_{n+1} = g(x_n), \ \text{where } g(x) = b - \tfrac{1}{4}x^2\,,$$

and determine the finite range of b in which there exists a stable fixed point.

[*In both parts (a) and (b) you do not need to consider stability at the values of a or b where the number of fixed points changes.*]

3F Probability

Let Y be a random variable taking values in $[0,\infty)$ with probability density function f_Y. Show that

$$\mathbb{E}[Y] = \int_0^\infty \mathbb{P}(Y \geqslant y)\,dy\,.$$

Let $X_1,\ldots,X_n$ be independent random variables uniformly distributed on $\{1,2,\ldots,n\}$, and let $M = \min\{X_1,\ldots,X_n\}$. Show that $\mathbb{E}[M] \leqslant e$ and, applying any inequality from the course, deduce that $\mathbb{P}(M \geqslant 6) \leqslant 0.5$. [*Hint:* $(1+x) \leqslant e^x$.]

4F Probability

(a) A train can stop at n stations on a railway line. At each station, the conductor makes a stop with probability $1/n$, independently of previous stations. Let X be the total number of stops made. What is the distribution of X?

For each $k = 0, 1, 2, \ldots$, find $\lim_{n\to\infty} \mathbb{P}(X = k)$.

(b) Consider a modification of this process where, initially, the conductor stops at each station with probability $1/n$, but after the first time that the train does stop, the probability of stopping at subsequent stations increases to $2/n$. Compute $\mathbb{E}[X]$ and find $\lim_{n\to\infty} \mathbb{E}[X]$.

SECTION II

5C Differential Equations

(a) Let $y_1(x)$ and $y_2(x)$ be any two complementary functions for the equation

$$y'' + p(x)y' + q(x)y = f(x)\,. \tag{1}$$

Define the *Wronskian* $W(x)$ of y_1 and y_2. State a differential equation that is satisfied by $W(x)$ for any choice of y_1 and y_2. Assuming that $W(x) \neq 0$ for any x, state briefly why any solution of (1) can be written in the vector form

$$\begin{pmatrix} y \\ y' \end{pmatrix} = u(x)\begin{pmatrix} y_1 \\ y_1' \end{pmatrix} + v(x)\begin{pmatrix} y_2 \\ y_2' \end{pmatrix} \tag{2}$$

for some (as yet unknown) coefficients u and v.

Obtain two coupled first-order differential equations for $u(x)$ and $v(x)$ by requiring consistency between the two components of (2) and between (1) and (2). Deduce that

$$u' = -\frac{y_2 f}{W}\,, \qquad v' = \frac{y_1 f}{W}\,.$$

(b) You are given that $y_1(x) = x^{-1/2}e^{-x}$ is a complementary function for the equation

$$x^2y'' + xy' - (x^2 + \tfrac{1}{4})y = x^{7/2}\,, \qquad x > 0\,. \tag{3}$$

Determine the Wronskian to within a multiplicative factor. Hence find a second complementary function $y_2(x)$, choosing a convenient amplitude for it and thus the Wronskian.

Use the results of part (a) to determine the solution to (3).

6C Differential Equations

(a) For $k, m > 0$, find the solution $x(t)$ of the equation

$$t^2\ddot{x} + t\dot{x} - m^2x = t^k$$

that satisfies the boundary conditions $x(0) = x(1) = 0$.

Show that no such solution exists if $k = m = 0$.

(b) Using matrix methods, or otherwise, find the general solution $x(t)$ and $y(t)$ to the linear system of equations

$$\begin{aligned} t\dot{x} - 4x + \frac{6y}{t} &= t + 2\,, \\ t\dot{y} - 3tx + 4y &= t^2 + t\,. \end{aligned}$$

[*Hint: You may find the substitution $y(t) = t^n z(t)$ helpful for a suitably chosen integer n.*]

7C Differential Equations

(a) The position $x(t)$ of a mass m obeys the equation

$$m\ddot{x} + b\dot{x} + kx = P\sum_{n=1}^{\infty}\delta(t - nt^*)\,,$$

where b, k, P and t^* are positive constants with $b^2 < 4mk$ and δ denotes the Dirac delta function. By comparing the homogenous solutions, or otherwise, identify a frequency ω such that, with $\tau = \omega t$, the scaled position $y(\tau) \equiv m\omega x(t)/P$ satisfies

$$y'' + 2\kappa y' + (\kappa^2 + 1)y = \sum_{n=1}^{\infty}\delta(\tau - nT)\,, \qquad (*)$$

where the dimensionless parameters κ and T should be determined.

(b) Find the solution $\mathbf{x}_n$ to the vector recurrence relation

$$\mathbf{x}_{n+1} = A\mathbf{x}_n + \mathbf{b}\,, \qquad \mathbf{x}_0 = \mathbf{a}\,, \qquad (\dagger)$$

where A is a 2×2 matrix, $\det(A - I) \neq 0$ and $\mathbf{a}$, $\mathbf{b}$ are vectors in $\mathbb{R}^2$.

(c) For each time interval $nT < \tau < (n+1)T$, the solution to $(*)$ is written as a linear combination

$$y(\tau) = C_n f(\tau - nT) + D_n g(\tau - nT)$$

of time-translated homogeneous solutions, where $f(\tau) = e^{-\kappa\tau}\cos\tau$ and $g(\tau) = e^{-\kappa\tau}\sin\tau$.

By considering the variation of y and y' across an infinitesimal interval including $\tau = (n+1)T$, find A and $\mathbf{b}$ such that $\mathbf{x}_n \equiv (C_n, D_n)$ satisfies $(\dagger)$. [*Hint:* $f'(\tau) = -g(\tau) - \kappa f(\tau)$ *and* $g'(\tau) = f(\tau) - \kappa g(\tau)$.]

(d) Explain why y tends to a periodic function for $\kappa\tau \gg 1$, and show that it is given by an amplitude vector $\mathbf{x}_\infty$ with

$$|\mathbf{x}_\infty| = \left\{[1 - f(T)]^2 + g(T)^2\right\}^{-1/2}.$$

Consider the case $0 < \kappa \ll 1$. Find simple leading-order approximations to $|\mathbf{x}_\infty|$ when $T = k\pi$ for an integer k with $\kappa k \ll 1$. Comment physically on this result.

8C Differential Equations

(a) Consider a change of variables from x and t to $\eta = x/t^p$ and $\tau = t$, where p is a constant. Use the chain rule to express $\partial/\partial x$ and $\partial/\partial t$ in terms of η, τ, $\partial/\partial\eta$ and $\partial/\partial\tau$. Show that the substitution $\theta(x,t) = H(\tau)Y(\eta)$ transforms the nonlinear diffusion equation

$$\frac{\partial\theta}{\partial t} = \frac{1}{m+2}\frac{\partial}{\partial x}\left(\theta^m\frac{\partial\theta}{\partial x}\right),$$

where $m > 0$, into the ordinary differential equation

$$-\frac{d(\eta Y)}{d\eta} = \frac{d}{d\eta}\left(Y^m\frac{dY}{d\eta}\right) \qquad (*)$$

provided $H(\tau)$ and p are chosen appropriately. Find the solution to $(*)$ that satisfies $Y(\pm1) = 0$. [You may assume $Y'(0) = 0$ by symmetry.]

(b) The solution in part (a) is perturbed by setting $\theta(x,t) = H(\tau)\big\{Y(\eta)+\epsilon\tau^{-\lambda}y(\eta)\big\}$, where $\epsilon \ll 1$. You are given that the function $y(\eta)$ must then obey the equation

$$-\frac{d(\eta y)}{d\eta} - (m+2)\lambda y = \frac{d^2}{d\eta^2}\left(Y^m y\right). \qquad (\dagger)$$

For the case $m = 1$ show that $(\dagger)$ reduces to

$$(1-\eta^2)y'' - 2\eta y' + 6\lambda y = 0\,.$$

You are given that regularity at $\eta = \pm1$ requires $y(\eta)$ to be a polynomial. By considering series solutions around $\eta = 0$, show that y is a polynomial if $\lambda = \frac{1}{6}k(k+1)$ for some positive integer k.

9F Probability

In a group of $n > 3$ people, each pair is friends with probability $1/2$, independently of every other pair.

(a) A *triad* is a set of three people who are all friends with each other. Show that the probability that there are at least $n^3/24$ triads is at most $1/2$.

(b) The person, or persons, with the most friends has M friends, and the person, or persons, with the least friends has L friends. Show that the mean and median of $M+L$ are both equal to $n-1$. [The median is defined by $\inf\{x : \mathbb{P}(M+L \leqslant x) \geqslant 1/2\}$].

(c) Now suppose that each pair of people in the group are friends with probability $\dfrac{(1+\varepsilon)\log n}{n-1}$ for some $\varepsilon > 0$, independently of every other pair. Show that

$$\mathbb{P}(L = 0) \leqslant n^{-\varepsilon}.$$

10F Probability

Write $\mathbb{N} = \{1, 2, 3, \dots\}$. For any sequence $x = (x_1, \dots, x_n) \in \mathbb{N}^n$, let $N_i(x)$ denote the number of times that $i \in \mathbb{N}$ appears in the sequence x and let $M(x) = \max\{x_1, \dots, x_n\}$. Let (X_i) be a sequence of random variables taking values in $\mathbb{N}$, with $\mathbb{P}(X_1 = 1) = 1$ and

$$\mathbb{P}\big(X_{n+1} = i \mid X_1 = x_1, \dots, X_n = x_n\big) = \begin{cases} \dfrac{N_i(x) - \frac{1}{2}}{n+1} & \text{if } 1 \leqslant i \leqslant M(x)\,, \\ \dfrac{1 + \frac{1}{2}M(x)}{n+1} & \text{if } i = M(x) + 1\,, \\ 0 & \text{if } i > M(x) + 1 \end{cases}$$

for all $n \geqslant 1$.

(a) Let $A_n \subseteq \mathbb{N}^n$ be the set of sequences such that, if $x \in A_n$ then $x_1 = 1$ and $x_i \leqslant \max\{x_1, \dots, x_{i-1}\} + 1$ for all $i \leqslant n$. Prove that

$$\mathbb{P}\big((X_1, \dots, X_n) \in A_n\big) = 1\,.$$

(b) Letting $X = (X_1, \dots, X_n)$ and $M_n = M(X)$, show that

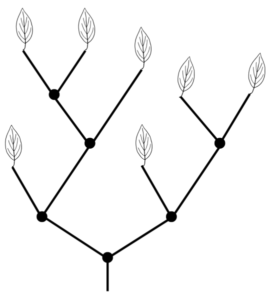
$$\mathbb{P}\big(N_1(X) = 1\big) = \frac{\prod_{i=1}^{n-1}(i + \frac{1}{2})}{n!} \quad \text{and} \quad \mathbb{P}\big(N_{X_n}(X) = 1\big) = \frac{1 + \frac{1}{2}\mathbb{E}[M_{n-1}]}{n}\,.$$

(c) Let x and y be two sequences in A_n, with $M(x) = M(y)$ and such that $\big(N_1(x), \dots, N_{M(x)}(x)\big)$ is a permutation of $\big(N_1(y), \dots, N_{M(y)}(y)\big)$. Show that

$$\mathbb{P}(X_1 = x_1, \dots, X_n = x_n) = \mathbb{P}(X_1 = y_1, \dots, X_n = y_n)\,.$$

(d) Using the result in part (c), prove that $\mathbb{P}\big(N_1(X) = 1\big) = \mathbb{P}\big(N_{X_n}(X) = 1\big)$. Obtain an expression for $\mathbb{E}[M_n]$. [*Hint: Consider a bijection $\pi : A_n \to A_n$ such that the frequency of the first element of x is the same as the frequency of the last element in $\pi(x)$.*]

11F Probability



A *tree*, like the example shown, has a root and splits into two branches at every branching point; branches never rejoin. Consider a random path from the root which, at each branching point, goes left or right with equal probability, and independently, until it reaches a leaf. Let X_1 be the number of branching points traversed by this path, and let (X_i) be a sequence of independent random variables with the same distribution as X_1.

Let ℓ be the number of leaves in the tree, and suppose there are at most b branching points on the path between the root and any leaf. Define a random variable $L = n^{-1} \sum_{i=1}^{n} 2^{X_i}$.

(a) State and prove Markov's inequality.

(b) Show that $\mathbb{E}[L] = \ell$.

(c) Prove that for all $\alpha > 0$,

$$\mathbb{P}\left(\left| \frac{L-\ell}{\ell} \right| \geqslant \alpha \sqrt{\frac{2^b}{n\ell}} \right) \leqslant \frac{1}{\alpha^2} \, .$$

(d) Let Φ be the distribution function of a standard normal random variable. Show that for all $\alpha > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\left| \frac{L-\ell}{\ell} \right| \geqslant \alpha \sqrt{\frac{2^b}{n\ell}} \right) \leqslant 2 - 2\Phi(\alpha).$$

[You may use without proof any theorem from the course in this part.]

12F Probability

Let $E_1, \ldots, E_n$ be independent random variables with an exponential distribution of mean 1, with $n \geqslant 2$. Let S be an independent random variable with probability density function

$$f(s) = \frac{s^{n-1} e^{-s}}{(n-1)!} \quad \text{for } s > 0\,.$$

Define

$$(X_1, \ldots, X_n) = \left(\left(1 + \frac{E_1}{S}\right)^{-n}, \ldots, \left(1 + \frac{E_n}{S}\right)^{-n} \right).$$

(a) Show that the moment generating function of S is $M_S(t) = (1-t)^{-n}$ for $t < 1$.

(b) Find the joint distribution function and joint probability density function of (X_1, X_2). What is the marginal distribution of X_1?

(c) Let K be the least integer $k \in \{1, \ldots, n-1\}$ such that $X_k < X_{k+1}$, with $K = n$ if there is no such integer. Show that

$$\mathbb{E}\left[\sum_{i=1}^{K} S^i \right] = \sum_{i=1}^{n} \binom{n+i-1}{i}.$$

END OF PAPER