UNIVERSITY OF CAMBRIDGE

MAT0

MATHEMATICAL TRIPOS Part IA

Thursday, 05 June, 2025 9:00am to 12:00pm

PAPER 1

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **all four** questions from Section I and **at most five** questions from Section II. Of the Section II questions, no more than three may be on the same course.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1B Vectors and Matrices

(a) Using the properties of complex numbers, show that a triangle inscribed in a circle is a right-angled triangle if and only if one of the sides is a diameter of the circle.

(b) A *chord* of a polygon is defined to be a line joining any pair of distinct vertices.

A regular N-gon is inscribed in the circle $|z+1|=1$, for $z \in \mathbb{C}$, with one vertex chosen to lie at $z=0$. Write down an Nth-order polynomial equation in z whose roots are the N vertices and thus determine the product of the lengths of all the chords emanating from the vertex at the origin. Hence show that the product of the lengths of all the chords of a regular N-gon inscribed in a circle of radius R is equal to $N^{N/2}R^{N(N-1)/2}$.

2C Vectors and Matrices

Consider the equation

$$A\,\mathbf{x} = \mathbf{b}\,, \text{ where } A = \begin{pmatrix} 2 & 0 & a \\ a & 1 & 1 \\ 1 & 1 & -2 \end{pmatrix} \text{ and } \mathbf{b} \in \mathbb{R}^3\,. \qquad (*)$$

(a) For which values of a does $(*)$ not have a unique solution for $\mathbf{x}$?

(b) For each of these values of a, find $\mathbf{n} \in \mathbb{R}^3$ such that $\mathbf{n} \cdot \mathbf{b} = 0$ is necessary for solutions to $(*)$ to exist.

(c) For $\mathbf{b} = (2, b, 0)^T$ and the same values of a, find the general solution of $(*)$ when solutions do exist.

3E Analysis I

(a) Write down without proof the power series that converge to the functions $\exp(z)$ and $\sin(z)$, respectively.

(b) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$. Give without proof a power series converging to $f'(z)$ and state its radius of convergence.

(c) Using the power series of $\exp(z)$, prove that $\exp(a+b) = \exp(a)\exp(b)$ for all $a, b \in \mathbb{C}$. [You may not use any other property of $\exp(z)$ without proving it. You may use without proof that a function $f : \mathbb{C} \to \mathbb{C}$ with $f'(z) = 0$ for all $z \in \mathbb{C}$ is constant.]

(d) Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \sin(z)/z$ for $z \neq 0$ and by $f(0) = 1$. Find $d^k f/dz^k$ at $z = 0$ for $k = 1, 2, 3, \ldots$. Justify your answer.

4E Analysis I

(a) State Riemann's integrability criterion and prove that it implies (Riemann) integrability.

(b) Let $a \leqslant b < c \leqslant d$ be real numbers and let $f : [a, d] \to \mathbb{R}$ be an integrable function. Using Riemann's integrability criterion, prove that f is also integrable on $[b, c]$. [You may not use any other criterion for integrability.]

SECTION II

5B Vectors and Matrices

Consider a non-degenerate tetrahedron T in $\mathbb{R}^3$ with four vertices specified by the position vectors $\mathbf{v}$ and $\mathbf{v}_i$ for $i = 1, 2, 3$. Let $\mathbf{e}_i = \mathbf{v}_i - \mathbf{v}$ denote the vectors along the three edges emanating from the vertex $\mathbf{v}$. Show that the volume of T is given by

$$V = \frac{1}{3!}\,|\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)|\ .$$

[You may assume that the volume of a tetrahedron is one third its base area times its height.]

Given the three vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ defined above, find explicit formulae for three vectors $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ such that $\mathbf{f}_i \cdot \mathbf{e}_j = \delta_{ij}$.

Now suppose that the three faces of T intersecting at the vertex $\mathbf{v}$ lie in planes defined for $\mathbf{x} \in \mathbb{R}^3$ by the vector equations $\mathbf{a}_i \cdot \mathbf{x} + b_i = 0$ and that the final face corresponds to the plane $\mathbf{c} \cdot \mathbf{x} + d = 0$. Here $\mathbf{a}_i$ and $\mathbf{c}$ are non-zero vectors and b_i and d are constants.

Show the following.

(i) $\mathbf{e}_i \cdot \mathbf{c} = -(\mathbf{c} \cdot \mathbf{v} + d)\,.$

(ii) There exists a real number λ_i such that $\mathbf{a}_i = \lambda_i \mathbf{f}_i\,.$

(iii) If γ_1, γ_2 and γ_3 are real numbers such that $\mathbf{c} = \gamma_1\mathbf{a}_1 + \gamma_2\mathbf{a}_2 + \gamma_3\mathbf{a}_3$ then

$$\gamma_i = \frac{\mathbf{e}_i \cdot \mathbf{c}}{\lambda_i}\,.$$

Hence show that the volume of T can be expressed in terms of the planar faces (and the vertex $\mathbf{v}$) of the tetrahedron as

$$V = \frac{1}{3!}\left|\frac{(\mathbf{c} \cdot \mathbf{v} + d)^3}{(\gamma_1\gamma_2\gamma_3)\,\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)}\right|\ .$$

6C Vectors and Matrices

(a) Write the real matrix

$$A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$$

as the sum of symmetric and antisymmetric parts. Hence, or otherwise, show that A can be written as $A_{ij} = \alpha\delta_{ij} + \beta n_i n_j + \gamma\epsilon_{ijk}n_k$, where $\mathbf{n}$ is a unit vector with $n_1 > 0$, and determine the constants α, β and γ in terms of a, b and c.

(b) Describe geometrically the action in $\mathbb{R}^3$ of A on the line through the origin parallel to $\mathbf{n}$ and on the plane through the origin perpendicular to $\mathbf{n}$. By what factor is any area in this plane multiplied?

(c) Deduce necessary and sufficient conditions on a, b and c for A to act on $\mathbb{R}^3$ either (i) as a reflection or (ii) as a rotation.

(d) Assume only $\det A \neq 0$. By considering the geometrical action of A^{-1}, or otherwise, find an expression for $(A^{-1})_{ij}$ in terms of α, β and γ.

(e) Show that $\det A = \frac{1}{2}(a+b+c)\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right]$.

7A Vectors and Matrices

(a) Define what is meant by the *eigenvalues* and *eigenspaces* of an $n \times n$ complex matrix A. Prove that any such matrix has at least one eigenvalue. What are the possible dimensions of the corresponding eigenspace?

(b) By computing a determinant, find the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 0 & -1 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 \\ -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and hence find the eigenvalues of A. What is the dimension of each eigenspace?

[*Hint: The sum of the coefficients of the characteristic polynomial should be zero.*]

(c) The *infinite* matrix B has entries $B_{ij} = 1$ if $j = i + 1$ and $B_{ij} = 0$ otherwise, for $i, j \in \{1, 2, 3, \ldots\}$. We say that a vector $\mathbf{u} = (u_1, u_2, u_3, \ldots)^T$, where $u_j \in \mathbb{C}$, is an *eigenvector* of B with *eigenvalue* λ if

$$\sum_{j=1}^{\infty} B_{ij} u_j = \lambda u_i \text{ for } i = 1, 2, \ldots \text{ and also } 0 < \sum_{i=1}^{\infty} |u_i|^2 < \infty\,.$$

Find all of the eigenvalues and eigenvectors of the matrix B.

(d) Let C be the transpose of the matrix B in part (c), as defined by $C_{ij} = B_{ji}$. Find all of the eigenvalues and eigenvectors of the matrix C.

[*Hint: In parts (c) and (d) do not assume that results for finite matrices necessarily carry over to infinite matrices.*]

8A Vectors and Matrices

(a) What does it mean for an $n \times n$ complex matrix to be *diagonalisable*?

(b) Let A be an $n \times n$ complex matrix (not necessarily diagonalisable) and $p(z) = \sum_{j=0}^{m} a_j z^j$ a polynomial. The $n \times n$ matrix $p(A)$ is defined by $p(A) = \sum_{j=0}^{m} a_j A^j$. Prove that the set of eigenvalues of $p(A)$ is $\{p(\lambda) : \lambda \text{ is an eigenvalue of } A\}$.

(c) The *exponential* of an $n \times n$ complex matrix A is defined by

$$\exp(A) = \sum_{j=0}^{\infty} \frac{A^j}{j!} .$$

[You may assume that the series converges.] Suppose that A is diagonalisable. Without considering a differential equation, prove that $\det\big(\exp(A)\big) = \exp\big(\mathrm{Tr}(A)\big)$.

(d) Let

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} .$$

Using ideas from part (c), or otherwise, find $\exp(B)$.

9E Analysis I

(a) What is a *Cauchy sequence*?

(b) State and prove the general principle of convergence. [You may use without proof the Bolzano–Weierstrass theorem.]

(c) Let (a_n) be a sequence of real numbers and define the sequence (d_n) by $d_n = |a_{n+1} - a_n|$.

Determine whether each of the following two statements is true or false. Give a proof or a counterexample as appropriate.

(i) If the series $\sum_{n=1}^{\infty} d_n$ is convergent, then the sequence (a_n) is convergent.

(ii) If the sequence (a_n) is convergent, then the series $\sum_{n=1}^{\infty} d_n$ is convergent.

(d) Decide whether the sequence (b_n) defined as

$$b_n = \sum_{j=n+1}^{2n} \frac{j}{j^2 + j + 1}$$

is convergent. Justify your answer.

[*In parts (c) and (d) you may use without proof any convergence tests from lectures.*]

10E Analysis I

(a) Let $f : [0,1] \to \mathbb{R}$ be a function. Define what it means that the *limit* of f at a point $x \in [0,1]$ is ℓ for some $\ell \in \mathbb{R}$.

(b) Give a criterion in terms of limits for f to be *continuous* at a point $x \in [0,1]$. [You need not prove the criterion.]

(c) Let $A, B, C \subset \mathbb{R}$ be intervals, and let $f : A \to B$ and $g : B \to C$ be functions. Suppose f is continuous at some point $x \in A$, and g is continuous at $f(x)$. Prove that $g \circ f : A \to C$ is continuous at x. [You may use without proof the sequential characterisation of continuity.]

(d) Consider the functions $f_1, f_2 : [0,1] \to \mathbb{R}$ defined by $f_1(x) = \sin(1/x)$ and $f_2(x) = x \cdot \sin(1/x)$ for $x > 0$ and $f_1(0) = f_2(0) = 0$. Determine the set of points where these functions are continuous. [You may use without proof the continuity of functions constructed from continuous functions using arithmetic operations, provided you state the results you use clearly. You may also assume that $\sin(x)$ is continuous.]

(e) Give examples of functions $f : [0,1] \to \mathbb{R}$ that are continuous at the points given in each of the following sets and nowhere else:

(i) $\{0\} \cup \{1/n : n \in \mathbb{Z}, n > 0\}$,

(ii) $\{1/n : n \in \mathbb{Z}, n > 0\}$.

Justify your answers.

11E Analysis I

Let $a < b$ be real numbers and let $f : [a,b] \to \mathbb{R}$ be a function.

(a) What does it mean that f is *differentiable* at a point $x \in (a,b)$? Define the *derivative*.

(b) State the mean value theorem.

(c) Now suppose that f is continuous on $[a,b]$ and differentiable on (a,b). Prove that f is increasing on (a,b) if and only if $f'(x) \geqslant 0$ for all $x \in (a,b)$. [A function f is increasing on an interval (a,b) if $a < x \leqslant y < b$ implies $f(x) \leqslant f(y)$.]

(d) Suppose further that $a = 0$, $f(0) = 0$ and f' is increasing on (a,b). Prove that $f(x)/x$ is increasing on (a,b). [*Hint: You may find parts (b) and (c) useful.*]

12E Analysis I

(a) State the fundamental theorem of calculus.

(b) Let $f, g : [a, b] \to \mathbb{R}$ be differentiable functions with continuous derivatives. Suppose that $f(a) \leqslant g(a)$ and $f'(x) \leqslant g'(x)$ for all $x \in [a, b]$. Prove that $f(b) \leqslant g(b)$. [You may use without proof any version of the fundamental theorem of calculus provided you state it clearly.]

(c) Let $f : [0, \infty) \to \mathbb{R}$ be a differentiable function with continuous derivative. Suppose that $f(0) = 0$ and $0 \leqslant f'(x) \leqslant 1$ for all $x \in [0, \infty)$. Prove that

$$\int_0^x \big(f(t)\big)^3 dt \leqslant \Big(\int_0^x f(t)\,dt\Big)^2$$

for all $x \in [0, \infty)$.

(d) Does the conclusion in part (c) necessarily hold without the condition $f(0) = 0$? Justify your answer.

END OF PAPER