UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II 2025

List of Courses

Algebraic Geometry
Algebraic Topology
Analysis of Functions
Applications of Quantum Mechanics
Applied Probability
Asymptotic Methods
Automata and Formal Languages
Classical Dynamics
Coding and Cryptography
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Waves

Paper 1, Section II

25J Algebraic Geometry

Define what it means to be a *rational map* between irreducible projective varieties. Define what it means to be a *regular point* of a rational map between irreducible projective varieties.

Consider the rational map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by

$$(x : y : z) \mapsto (xy : xz : z^2).$$

Show that φ is not regular at the points $(0 : 1 : 0), (1 : 0 : 0)$ and is regular at every other point. Show that φ is a birational map which is an isomorphism on $\mathbb{P}^2 \setminus Z(xyz)$, the complement of the union of the coordinate hyperplanes.

Let $V \subset \mathbb{P}^2$ be the subvariety given by the vanishing of $x^2z^4 - x^3y^3 + z^6$. Show that V is irreducible, and that φ determines a birational equivalence between V and a non-singular plane cubic.

Paper 2, Section II

25J Algebraic Geometry

In this question, all varieties are over an algebraically closed field k of characteristic zero.

Let $X \subset \mathbb{A}^n$ be an affine algebraic variety defined over k. Define the *tangent space* $T_{X,P}$ of X at a point $P \in X$, and define the *dimension* of X in terms of the tangent spaces of X.

Suppose that $X = Z(f) \subset \mathbb{A}^n$ where f is a non-constant polynomial. Show from your definition that X has dimension $n - 1$. [Any form of the Nullstellensatz may be used if you state it clearly.]

Now suppose that $n \geqslant 3$ and $X = Z(f) \subset \mathbb{A}^n$ where f is a non-constant irreducible polynomial of degree at least 2. Let $P \in X$ be a smooth point of X, and translate $T_{X,P}$ by P to view it as an embedded hyperplane with $P \in T_{X,P} \subset \mathbb{A}^n$. Show that $X \cap T_{X,P}$ is singular at P.

Now let $Y := \{\varphi : \mathbb{A}^2 \to \mathbb{A}^3 \mid \varphi \text{ is linear but not injective}\}$. Show that Y is the zero locus of an ideal I which is generated by three quadrics. Compute the dimension of Y and identify any singular points of Y. [You may assume without proof that I is a radical ideal.]

Paper 3, Section II

24J Algebraic Geometry

In this question, all algebraic varieties are defined over a field k of characteristic zero. Let $V \subset \mathbb{P}^n$ be a curve.

Define the *degree* $\deg(V)$ of $V \subset \mathbb{P}^n$, and prove that it is well-defined.

Suppose $n < m$ and let $\varphi : \mathbb{P}^n \to \mathbb{P}^m$ be the linear embedding

$$(x_0 : \cdots : x_n) \mapsto (x_0 : \cdots : x_n : 0 : \cdots : 0).$$

For a curve $V \subset \mathbb{P}^n$, show that the degree of V in $\mathbb{P}^n$ agrees with the degree of $\varphi(V)$ in $\mathbb{P}^m$.

Prove that the degree is not an isomorphism invariant by providing an example of isomorphic curves $V_1, V_2 \subset \mathbb{P}^n$ with $\deg(V_1) \neq \deg(V_2)$.

Let $S = (x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_1x_3 - x_2^2) \subset k[x_0, x_1, x_2, x_3]$ and define $V = Z(S)$ to be the zero locus of S in $\mathbb{P}^3$. By considering an affine piece or otherwise, show that V is a curve in $\mathbb{P}^3$, and compute its degree. Prove that there do *not* exist homogeneous polynomials $F_1, \ldots, F_r$ such that $V = Z(F_1, \ldots, F_r)$ and $\deg(V) = \prod_{i=1}^r \deg(Z(F_i))$.

Give an example of two irreducible curves in a projective space $\mathbb{P}^n$ which have the same degree but are not isomorphic. [You may use without proof the fact that a smooth projective curve in $\mathbb{P}^2$ of degree $d \geqslant 2$ has genus $g = (d-1)(d-2)/2$.]

Paper 4, Section II

24J Algebraic Geometry

In this question, all algebraic varieties are defined over an algebraically closed field k of characteristic zero.

State the Riemann–Roch theorem, giving a brief explanation of each term.

A smooth projective curve X is covered by two affine pieces (with respect to different embeddings) which are affine plane curves with equations $y^2 = f(x)$ and $v^2 = g(u)$ respectively, with f a square-free polynomial of even degree $2n > 4$ and $u = 1/x, v = y/x^n$ in the function field of X.

Determine the polynomial $g(u)$.

Using a well-chosen rational differential, compute the canonical divisor K_X of X, and show that it has degree $2n - 4$.

Compute the genus of X.

Write down a basis for the space $L(K_X)$ of rational functions with poles bounded by K_X. Conclude that X cannot be embedded into $\mathbb{P}^2$.

Paper 1, Section II
21F Algebraic Topology
State the Mayer-Vietoris theorem for a simplicial complex K which is the union of subcomplexes M and N.

Let K be a non-empty simplicial complex in $\mathbb{R}^m$, where we consider $\mathbb{R}^m$ as lying in $\mathbb{R}^{m+2}$ via the vectors $(x_1, \ldots, x_m, 0, 0)$. Let $c_1 = (0, \ldots, 0, 1, 0) \in \mathbb{R}^{m+2}$, $c_2 = (0, \ldots, 0, 0, 1) \in \mathbb{R}^{m+2}$ and $c_3 = (0, \ldots, 0, -1, -1) \in \mathbb{R}^{m+2}$. Let L be the collection of simplices in $\mathbb{R}^{m+2}$ given by

$$L := K \cup \{\langle v_0, v_1, \ldots, v_n, c_i\rangle \mid \langle v_0, v_1, \ldots, v_n\rangle \in K, i = 1, 2, 3\}.$$

Show that L is a simplicial complex. Find expressions for the simplicial homology groups of L in terms of the simplicial homology groups of K. [*You may use any results from lectures provided they are clearly stated.*]

Paper 2, Section II
21F Algebraic Topology
Let (X, x_0) be a based topological space. Define the *fundamental group* $\pi_1(X, x_0)$, and show that the composition law is well-defined and satisfies the group axioms.

Let $U(2)$ be the group of unitary 2×2 matrices, with the subspace topology from $\mathbb{C}^{2\times 2}$. Let $I \in U(2)$ denote the identity matrix.

(a) Let $\gamma : [0, 1] \to U(2)$ be given by $\gamma(t) = \begin{pmatrix} e^{2\pi it} & 0 \\ 0 & 1 \end{pmatrix}$. Show that for non-zero $k \in \mathbb{Z}$, $[\gamma]^k$ is never the identity in $\pi_1(U(2), I)$. [You may use without proof a description of $\pi_1(S^1, *)$ provided it is clearly stated.]

(b) Show that $\pi_1(U(2), I)$ is abelian. [You may use without proof the fact that matrix multiplication gives a continuous map $U(2) \times U(2) \to U(2)$.]

Paper 3, Section II

20F Algebraic Topology

(a) State a version of the Seifert–van Kampen theorem. Let (X, x_0) be a based topological space, and suppose that $\alpha : (S^1, *) \to (X, x_0)$ is a based map. Prove that there is an isomorphism

$$\pi_1(X \cup_\alpha D^2, x_0) \cong \pi_1(X, x_0)/\langle\langle[\alpha]\rangle\rangle.$$

Use this to construct a connected cell complex Y such that

$$\pi_1(Y, y_0) \cong \langle a, b \,|\, a^2 b^{-3}\rangle.$$

[You may assume a description of $\pi_1(S^1 \vee S^1, *)$ provided it is clearly stated.]

(b) What does it mean for $p : \widetilde{X} \to X$ to be a *covering space*? For the cell complex Y constructed in part (a), suppose we have a covering space $p : \widetilde{Y} \to Y$ such that $\widetilde{Y}$ is path-connected, and, for $\widetilde{y}_0 \in p^{-1}(y_0)$, we have that $p_*\pi_1(\widetilde{Y}, \widetilde{y}_0)$ is the normal subgroup of $\pi_1(Y, y_0)$ generated by a. Given any $y \in Y$, how many points are in $p^{-1}(y)$? Give an explicit description of $\widetilde{Y}$ as a cell-complex.

Paper 4, Section II

21F Algebraic Topology

Let X be a triangulable space. State a formula for the rational homology groups $H_i(X; \mathbb{Q})$ given the ordinary homology groups $H_i(X)$. Define the *Euler characteristic* $\chi(X)$ of X. For K a simplicial complex triangulating X, state and prove a formula relating $\chi(X)$ to numbers of simplices in K.

Let Δ^3 denote the simplicial complex given by a standard 3-simplex together with all its faces. For each i, which standard abelian group is isomorphic to $H_i(\Delta^3)$?

Let $(\Delta^3)'$ be the barycentric subdivision of Δ^3, and let M be the 2-skeleton of $(\Delta^3)'$.

(i) Calculate the Euler characteristic of $|M|$.

(ii) Use this to compute the simplicial homology groups $H_i(M)$.

(iii) Suppose $f : |M| \to |M|$ is a homeomorphism. Must f have a fixed point? Briefly justify your answer.

Paper 1, Section II

23H Analysis of Functions

[You may use results from Linear Analysis and Probability and Measure without proof provided they are clearly stated.]

(a) Let μ and ν be finite measures on a measurable space $(E, \mathcal{E})$ that are mutually absolutely continuous.

(i) Show carefully that there exists a μ-integrable function $w : E \to [0, \infty]$ such that $\nu(A) = \int_A w \, d\mu$ for every $A \in \mathcal{E}$.

(ii) For which values of $0 < p < \infty$ must $\int_E |w|^p \, d\mu$ be finite? Justify your answer.

(b) Let ν_n be a sequence of probability measures on $(E, \mathcal{E})$ for $n \geqslant 1$. Does there always exist a probability measure μ on $(E, \mathcal{E})$ such that all ν_n are absolutely continuous with respect to μ? Give a proof or counter-example.

Paper 2, Section II

23H Analysis of Functions

(a) State and prove the Rellich-Kondrashov compactness theorem for the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$, where Ω is a bounded open subset of $\mathbb{R}^d$.

[You may use the Banach-Alaoglu and Plancherel theorems without proof.]

(b) Is the embedding of $H^1(\mathbb{R})$ into $L^2(\mathbb{R})$ compact? Justify your answer.

(c) Consider a bounded sequence f_n in $H^1(\mathbb{R})$ such that: (i) there is $C > 0$ so that $|f_n(x)| \leqslant C(1 + x^2)^{-1}$ for all $x \in \mathbb{R}$ and $n \geqslant 1$, and (ii) f_n converges weakly to zero in $L^2(\mathbb{R})$. Prove that f_n converges strongly to zero in $L^2(\mathbb{R})$.

Paper 3, Section II

22H Analysis of Functions

Let dx denote the Lebesgue measure on $\mathbb{R}^n$ and $L^1(\mathbb{R}^n)$ be the space of measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $\int_{\mathbb{R}^n} |f(x)|\, dx < \infty$.

(a) Denote by $[f] = \{g \in L^1(\mathbb{R}^n) : f = g$ almost everywhere$\}$ the equivalence classes for the almost everywhere equality relation, and show that $\|[f]\|_1 = \int_{\mathbb{R}^n} |f(x)|\, dx$ defines a complete norm on $\mathcal{L}^1(\mathbb{R}^n) := \{[f] : f \in L^1(\mathbb{R}^n)\}$.

[You may use the Riesz-Fischer theorem without proof if clearly stated.]

(b) Let $(X, \|\cdot\|_X)$ be a Banach space such that: (i) the inclusion $X \subset \mathcal{L}^1(\mathbb{R}^n)$ holds, and (ii) the convergence in $\|\cdot\|_X$ implies convergence almost everywhere on $\mathbb{R}^n$ along a subsequence.

(i) Show that there exists a constant $C > 0$ such that $\|x\|_1 \leqslant C\|x\|_X$ for all $x \in X$.

(ii) Must X be complete for $\|\cdot\|_1$? Justify your answer.

[*You may use results from Linear Analysis without proof if correctly stated.*]

Paper 4, Section II

23H Analysis of Functions

(a) Let H be a (real) Hilbert space and x_n a sequence in H that converges weakly to x in H as $n \to \infty$.

(i) Prove that $\|x\|_H \leqslant \liminf_n \|x_n\|_H$.

(ii) Prove that $\|x_n\|_H \to \|x\|_H$ if and only if $\|x_n - x\|_H \to 0$.

(iii) Must x_n converge strongly to x along a subsequence? Justify your answer.

(b) Let $\Omega \subset \mathbb{R}^n$ be open and bounded. For $u \in H_0^1(\Omega)$ and $V \in L^\infty(\Omega)$, we define the functional

$$E(u) := \int_\Omega \left(|Du|^2 + Vu^2 \right) dx.$$

(i) Show that for any sequence $u_n \in H_0^1(\Omega)$ that converges weakly to u in $H_0^1(\Omega)$, we must have $E(u) \leqslant \liminf_n E(u_n)$.

[You may use the Rellich-Kondrashov theorem without proof.]

(ii) Let $\lambda = \inf \mathcal{E}$ where

$$\mathcal{E} = \left\{ E(u) \ : \ u \in H_0^1(\Omega), \, \|u\|_{L^2(\Omega)} = 1 \right\}.$$

Show that there exists $w \in H_0^1(\Omega)$ such that $\|w\|_{L^2(\Omega)} = 1$ and $E(w) = \lambda$.

(iii) Prove that

$$\inf \left\{ \int_{\mathbb{R}} (Du)^2 \, dx \ : \ u \in H^1(\mathbb{R}), \, \|u\|_{L^2(\mathbb{R})} = 1 \right\} = 0.$$

Is this infimum attained? Justify your answer.

Paper 1, Section II

35B Applications of Quantum Mechanics

In this question you will study a one-dimensional particle of mass m governed by the Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\,\psi(x)\,,$$

where $V(x)$ is a one-dimensional potential with $V(x) \to 0$ as $x \to \pm\infty$, and $E > 0$ is the energy of the particle.

(a) For a particle incident from the negative x-direction, show that

$$\psi(x) = e^{ikx} - \frac{im}{\hbar^2 k}\int_{-\infty}^{\infty} dx' e^{ik|x-x'|} V(x')\psi(x') \qquad (*)$$

solves the Schrödinger equation, where $k = \sqrt{2mE}/\hbar$.

(b) Consider the following potential

$$V(x) = -\lambda\left[\delta(x+a) + \delta(x) + \delta(x-a)\right]\,,$$

where $a > 0$, $\lambda > 0$ are constants and $\delta(x)$ is the Dirac delta function. For this potential, write down a solution to the Schrödinger equation using equation $(*)$. Write explicitly the set of three algebraic equations determining $\psi(a)$, $\psi(0)$, and $\psi(-a)$, and express these equations in matrix form.

By inspecting the asymptotic solution at $x \to +\infty$ and writing it as $S_{++}e^{ikx}$, find the scattering amplitude $S_{++}(k)$ in terms of $\psi(a)$, $\psi(0)$, and $\psi(-a)$.

Show that solutions to the algebraic equation

$$1 - \gamma - 2e^{-2ika}(1+\gamma) + e^{-4ika}(1+\gamma)^3 = 0$$

correspond to singularities of S_{++}, where $\gamma = ik\hbar^2/(\lambda m)$. By looking at limiting values of a, argue that there are solutions to this algebraic equation on the imaginary k axis. What is the interpretation of these singularities?

Paper 2, Section II

36B Applications of Quantum Mechanics

(a) A particle moving in an attractive potential $V_1(\mathbf{x})$ has a ground-state energy E_1, while a particle moving in an attractive potential $V_2(\mathbf{x})$ has a ground-state energy E_2. Using the variational method, show that $E_1 \geqslant E_2$ if $V_1(\mathbf{x}) \geqslant V_2(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^3$.

[*Hint: use the wavefunction of the particle in* $V_1(\mathbf{x})$ *as a trial function for* $V_2(\mathbf{x})$.]

(b) Consider a one-dimensional Hamiltonian $H = T + V$, with kinetic energy T and the attractive potential

$$V(x) = -\frac{\alpha}{|x|^n} ,$$

where α and n are positive constants. The exact ground state of the Hamiltonian H is $\psi_0(x)$. By considering the trial function $\psi(x) = \psi_0(\lambda x)$, use the variational method to show that there are no bound states for $n > 2$.

(c) Consider a two-level quantum system, where the Hamiltonian H_0 admits two eigenstates: $|\psi_1\rangle$ with energy E_1, and $|\psi_2\rangle$ with energy E_2. You may assume that the states are orthogonal, normalised, and non-degenerate, and that $E_1 < E_2$.

Consider the perturbation H_{p}, with matrix elements

$$\langle\psi_1|H_{\text{p}}|\psi_1\rangle = \langle\psi_2|H_{\text{p}}|\psi_2\rangle = 0 ,$$

and

$$\langle\psi_1|H_{\text{p}}|\psi_2\rangle = \langle\psi_2|H_{\text{p}}|\psi_1\rangle = h ,$$

and h constant. Find the exact eigenvalues of the Hamiltonian $H = H_0 + H_{\text{p}}$.

Estimate the ground-state energy of H using the variational method, where the trial function is

$$|\psi_\beta\rangle = \sin\beta|\psi_1\rangle + \cos\beta|\psi_2\rangle ,$$

and β is an adjustable parameter. How does your answer compare to the exact ground state?

Paper 3, Section II

34B Applications of Quantum Mechanics

Consider a system of three atoms arranged on a circle, which can also be viewed as a one-dimensional crystal with atoms equally spaced by a distance a under periodic boundary conditions. The atoms are labelled 0, 1 and 2, and the wavefunction corresponding to an electron bound to the n-th atom is denoted by $\psi_n(x)$. The periodic boundary conditions imply that ψ_n is identified with ψ_{n+3}.

The atomic Hamiltonian is H_0 with $H_0|\psi_n\rangle = E_0|\psi_n\rangle$. The tunnelling between atomic sites is represented by a potential V, which has matrix elements

$$\begin{aligned} \langle\psi_n|V|\psi_n\rangle &= \alpha \qquad \forall\ n\ , \\ \langle\psi_n|V|\psi_{n'}\rangle &= -A \qquad \text{for } n \neq n'\ , \end{aligned}$$

where α and A are real constants and $n, n' = 0, 1, 2$. The stationary state of the total Hamiltonian, $H = H_0 + V$, is denoted by Ψ and can be written as

$$|\Psi\rangle = \sum_{n=0}^{2} c_n |\psi_n\rangle\ ,$$

with $H|\Psi\rangle = E|\Psi\rangle$ and $c_n \in \mathbb{C}$ for $n = 0, 1, 2$.

(a) Assuming $\langle\psi_n|\psi_m\rangle = \delta_{nm}$, show that the coefficients (c_0, c_1, c_2) satisfy the linear equations

$$\begin{pmatrix} E_0+\alpha & -A & -A \\ -A & E_0+\alpha & -A \\ -A & -A & E_0+\alpha \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = E \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} .$$

Write out explicitly the energy eigenvalues of the system. Show that the possible solutions for (c_0, c_1, c_2) are

$$(1,1,1)\ , \qquad (1,\omega,\omega^2)\ , \qquad (1,\omega^2,\omega)\ ,$$

where ω is a cube root of unity. [*Hint:* $x^3 - 3A^2x - 2A^3 = 0$ *has a double root at* $x = -A$.]

(b) Interpret these solutions in terms of a wavenumber k and determine the possible values of k. Write the corresponding eigenvectors $|\Psi_k\rangle$ in terms of $|\psi_n\rangle$. What is the Brillouin zone for this system?

(c) Let $\psi_n(x) = \psi(x - na)$. Show that, for each value of k, $\Psi_k(x)$ can be written as

$$\Psi_k(x) = u_k(x)\, e^{ikx}\ ,$$

$$u_k(x) = \sum_{n=0}^{2} \psi(x - na)\, e^{-ik(x-na)}\ .$$

Check that $u_k(x + a) = u_k(x)$. State Bloch's theorem in one dimension and justify why it applies to this system.

Paper 4, Section II

34B Applications of Quantum Mechanics

Consider a particle of mass m and electric charge e under the influence of a magnetic field $\mathbf{B}$ and a periodic potential. In these circumstances, the Hamiltonian is

$$H = \frac{1}{2m}\left[\mathbf{p} - e\mathbf{A}(\mathbf{x})\right]^2 + V(\mathbf{x}) \ .$$

Here $\mathbf{p} = -i\hbar\boldsymbol{\nabla}$ is the canonical momentum and $V(\mathbf{x})$ is a periodic potential dictated by the lattice Λ, that is, $V(\mathbf{x}) = V(\mathbf{x}+\mathbf{r})$ for all $\mathbf{r} \in \Lambda$. The magnetic field $\mathbf{B}$ is constant and we adopt the gauge

$$\mathbf{A}(\mathbf{x}) = \frac{1}{2}\mathbf{B} \times \mathbf{x} \ .$$

(a) Evaluate the commutators

$$[p_i + eA_i, p_j - eA_j] \qquad \text{and} \qquad [p_i + eA_i, p_j + eA_j] \ .$$

(b) The translation operator is defined as $T_{\mathbf{r}} = e^{i\mathbf{r}\cdot\mathbf{p}/\hbar}$. Show that $T_{\mathbf{r}}$ does not commute with H in the presence of a magnetic field.

In the presence of a magnetic field, it is useful to introduce the magnetic translation operator, defined as

$$\mathcal{T}_{\mathbf{r}} = \exp\left\{\frac{i}{\hbar}\mathbf{r}\cdot\left[\mathbf{p} + e\mathbf{A}(\mathbf{x})\right]\right\} \ .$$

Show how $\mathcal{T}_{\mathbf{r}}$ acts on functions, and show that $\mathcal{T}_{\mathbf{r}}$ commutes with the Hamiltonian.

(c) Show that

$$\mathcal{T}_{\mathbf{r}}\mathcal{T}_{\mathbf{r}'} = \exp\left[\frac{ie}{\hbar}(\mathbf{r} \times \mathbf{r}')\cdot\mathbf{B}\right]\mathcal{T}_{\mathbf{r}'}\mathcal{T}_{\mathbf{r}} \ .$$

The magnetic flux through a cell of the lattice is defined as $\Phi = \mathbf{B}\cdot(\mathbf{r} \times \mathbf{r}')$. Under what condition on Φ do the magnetic translations form an abelian group?

[*Hint: you may use, without proof, that* $e^{\mathbf{M}}e^{\mathbf{N}} = e^{\mathbf{M}+\mathbf{N}+\frac{1}{2}[\mathbf{M},\mathbf{N}]}$ *if* $\mathbf{M}$ *and* $\mathbf{N}$ *commute with* $[\mathbf{M},\mathbf{N}]$.]

Paper 1, Section II

28L Applied Probability

Let $(X_t, t \geqslant 0)$ be a Poisson process on $\mathbb{R}^+$ with rate $\lambda > 0$.

(a) Assuming the infinitesimal definition of a Poisson process, find the distribution of X_t.

(b) Now condition on the event $X_t = n$ for some $t > 0$ and $n \in \mathbb{N}$. What is the probability that the last jump before t occurs before $3t/4$? What is the distribution of the number of jumps between $t/4$ and $3t/4$?

(c) Suppose $(X_t, t \geqslant 0)$ describes the arrival of particles into a system. Each particle then lives for a length of time that is independent of the arrival process and independent of the lives of other particles. The particle lifespans are exponentially distributed with mean $1/\mu$. Find the distribution of the number of particles alive at time t.

[*Clearly state all results you use. You may use that if N is a Poisson*(λ) *random variable, then* $\mathbb{E}(e^{\theta N}) = \exp(\lambda(e^{\theta} - 1))$.]

Paper 2, Section II

28L Applied Probability

(a) Consider a right-continuous continuous-time Markov chain X on $\mathbb{Z}$ starting from 0 such that $q_{0,1} = q_{0,-1} = 1/2$ and

$$q_{i,i+1} = \frac{2q_i}{3}, \quad q_{i,i-1} = \frac{q_i}{3}, \quad q_{-i,-i-1} = \frac{2q_{-i}}{3}, \quad q_{-i,-i+1} = \frac{q_{-i}}{3} \quad \forall i \geqslant 1;$$

with $q_i = 3^{|i|}$ for $i \in \mathbb{Z}$.

Is X recurrent? Is X explosive? Does X have an invariant distribution? Justify your answers.

(b) Let $X \sim \text{Markov}(Q)$ be an irreducible right-continuous continuous-time Markov chain on a countable state space with generator Q. Are the following statements true? Prove or give a counterexample.

(i) If the jump chain Y is positive-recurrent, then X is positive-recurrent.

(ii) If X is positive-recurrent, then the jump chain Y is positive-recurrent.

(c) Consider an $M/M/1$ queue with arrival and service rates $\lambda > 0$ and $\mu > 0$ respectively. After service, each customer returns to the beginning of the queue with probability $p \in (0,1)$. Let $(L_t)_{t \geqslant 0}$ denote the queue length.

(i) For which parameters is L transient, and for which is it recurrent?

(ii) When is it positive recurrent?

(iii) Find the invariant distribution when it exists and the expected queue length at equilibrium.

(iv) What is the distribution of the departure process at equilibrium?

[Clearly state all results you use. You may assume the recurrence and transience properties of simple random walks on $\mathbb{Z}$.]

Paper 3, Section II

27L Applied Probability

Let (ξ_i) be a sequence of i.i.d. non-negative random variables with ξ_1 having a probability density function and $\mathbb{E}\xi_1 = 1/\lambda < \infty$.

(a) Define the *renewal process* N_t formed by the sequence (ξ_i). Assuming the law of large numbers, show that $N_t/t \to \lambda$ almost surely as $t \to \infty$.

Let $L(t)$ denote the length of the renewal interval containing t.

(b) Define what it means for a random variable $\hat{\xi}_1$ to have the *size-biased distribution* corresponding to ξ_1. If ξ_1 has an exponential distribution, show that $L(t)$ converges in distribution to $\hat{\xi}_1$ as $t \to \infty$. [Your proof should not use the equilibrium theorem of general renewal processes.]

(c) For all $x, t > 0$, prove that $\mathbb{P}(L(t) \geqslant x) \geqslant \mathbb{P}(\xi_1 \geqslant x)$.

Paper 4, Section II

27L Applied Probability

(a) State the mapping theorem for a non-homogeneous spatial Poisson process on $\mathbb{R}^d$ with intensity function λ and a map $f : \mathbb{R}^d \to \mathbb{R}^s$. You should clearly state all the necessary conditions.

(b) Assume that the positions $(x, y, z) \in \mathbb{R}^3$ of stars in space are distributed according to a homogeneous spatial Poisson process Π with a constant intensity $\lambda > 0$.

(i) Let $f : \mathbb{R}^3 \to [0, \infty)$ be given by $f(x, y, z) = (x^2 + y^2 + z^2)^{3/2}$. Show that $f(\Pi)$ is again a homogeneous Poisson process on $[0, \infty)$. What is its intensity?

(ii) Let $R_1, R_2, \ldots$ be an increasing sequence of positive random variables such that R_k denotes the distance of the k-th closest star from the origin. Find the density function for the distribution of R_k. [*Hint: The sum of n independent Exp(1) random variables has a Gamma(n) distribution with density function $x^{n-1}e^{-x}/(n-1)!$ for $x > 0$.*]

Paper 2, Section II

32D Asymptotic Methods

The incomplete gamma function $\gamma(x, y)$ is defined by

$$\gamma(x, y) = \int_y^\infty t^{x-1} \mathrm{e}^{-t} \, dt,$$

for real positive x and y.

(a) Using integration by parts, show that for fixed finite x,

$$\gamma(x, y) \sim y^{x-1} \mathrm{e}^{-y} \sum_{n=0}^{\infty} a_n(x) \, y^{-n}, \quad \text{as } y \to \infty,$$

where you should determine the coefficients $a_n(x)$.

(b) Give the leading-order term in the asymptotic approximation of $\gamma(x, y)$ for fixed finite y and as $x \to \infty$.

(c) Suppose that $x \to \infty$ and $y \to \infty$ with $y/x = \lambda$, where $\lambda > 1$ is a constant. Calculate the first two terms of the asymptotic expansion of $\gamma(x, y)$, in the form

$$\gamma(x, y) \sim y^{x-1} \mathrm{e}^{-y} \left[f(\lambda) + x^{-1} g(\lambda) \right],$$

where $f(\lambda)$ and $g(\lambda)$ are functions that you should determine.

[*Recall that* $\int_0^\infty t^n e^{-\alpha t} dt = \alpha^{-n-1} n!$, *for* n *a positive integer and* $\alpha > 0$.]

Paper 3, Section II

30D Asymptotic Methods

(a) Consider the function

$$I(x) = \int_C f(z) e^{x\phi(z)} dz, \qquad (\dagger)$$

where C is a complex contour, x is real and positive, f and ϕ are complex-valued functions, ϕ possesses a simple saddle point z_0, and $f(z_0) \neq 0$. Suppose C can be deformed so that it passes through z_0 without changing the value of $I(x)$.

Show that the saddle point's leading order asymptotic contribution to I is

$$f(z_0)\sqrt{\frac{2\pi}{x|\phi''(z_0)|}} e^{x\phi(z_0)+i\alpha}, \qquad \text{as} \quad x \to \infty,$$

where α is the angle of the tangent to the steepest descent curve at $z = z_0$.

[You may quote, without proof, results from Laplace's method for real integrals.]

(b) The Legendre polynomials can be expressed by the Schläfli integral:

$$P_n(t) = \frac{1}{2^{n+1}\pi i} \oint_C \frac{(z^2-1)^n}{(z-t)^{n+1}} dz,$$

where n is a positive integer, $t = \cos\theta$, with $0 < \theta < \pi$, and C is any closed anti-clockwise contour encircling t.

(i) Express $P_n(t)$ in the form

$$P_n(t) = \frac{1}{2^{n+1}\pi i} \oint_C f(z,t) e^{n\phi(z,t)} dz,$$

for some functions f and ϕ, and show that the saddle points of ϕ are located at $z = z_\pm = e^{\pm i\theta}$.

(ii) Show that $\text{Arg}[\phi''(z_\pm)] = \mp(\theta + \pi/2)$. Hence sketch an appropriate contour C that passes through z_+ and z_-, calculating its angle α at each saddle point.

(iii) Find the leading order asymptotic approximation of $P_n(t)$ as $n \to \infty$, in the form

$$P_n(t) \sim A \cos\left(n\theta + \tfrac{1}{2}\theta - \tfrac{1}{4}\pi\right),$$

where you should determine $A(\theta)$.

Paper 4, Section II

31D Asymptotic Methods

(a) Consider the function

$$U(x) = \int_0^\infty \frac{e^{-xt}}{1+t} dt,$$

where x is real and positive. Use Watson's lemma to show $U(x) \sim \sum_{n=0}^{\infty} a_n x^{-n-1}$ as $x \to \infty$, where you should determine the coefficients a_n.

(b) Show that U is a solution to the differential equation

$$xy''(x) + (1-x)y'(x) - y(x) = 0, \qquad (\dagger)$$

for real positive x.

(c) By a suitable transformation, rewrite equation $(\dagger)$ as

$$v''(x) - \tfrac{1}{4}\left(1 + Ax^{-1} + Bx^{-2}\right) v(x) = 0, \qquad (\star)$$

where A and B are constants you need to find. Determine that positive infinity is an irregular singular point of this equation.

(d) Consider Liouville-Green solutions to equation $(\star)$ of the form $v(x) = e^{S(x)}$ with $S'(x) \sim \sum_{n=0}^{\infty} b_n x^{-n}$ as $x \to \infty$. Calculate terms up to, and including, order x^{-1} in the expansion of S. Find the associated asymptotic expansion of y, and compare with your solution from part (a). What is the leading order asymptotic approximation of the other solution to equation $(\dagger)$ as $x \to \infty$?

Paper 1, Section I

4F Automata & Formal Languages

Consider the following table with classes of formal languages in the rows and closure properties in the columns (where "union", "intersection", and "complement" stand for closed under union, intersection, and complement, respectively). Fill the twelve entries of the table with "Yes" and "No", depending on whether the class of formal languages in the row has the closure property given in the column or not. You do not need to give arguments for "Yes" answers. For each "No" answer, either provide a counterexample or an argument why the class is not closed under the operation.

	union	intersection	complement
regular	$\circ$	$\circ$	$\circ$
context-free	$\circ$	$\circ$	$\circ$
computable	$\circ$	$\circ$	$\circ$
computably enumerable	$\circ$	$\circ$	$\circ$

[*If you give a counterexample from the lectures, you do not have to prove that it is a counterexample, provided that you state it correctly.*]
[*Comment.* This is a very easy question for a student who has grasped the course as a whole. Closure properties were a recurring theme. It draws from all parts of the course (some results from all four chapters) and requires a certain amount of bird's eye view. A student who has this type of overview of the entire course deserves a β, even if it is an easy β.]

Paper 2, Section I

4F Automata & Formal Languages

(a) Say what it means for a language L to satisfy the *context-free pumping lemma.*

(b) For each of the following languages over the alphabet $\{\mathbf{0}, \mathbf{1}\}$, either give a context-free grammar that produces the language or prove that the language is not context-free. We write w^{R} for the reverse word of w, i.e., if $w = a_0...a_{n-1}$ then $w^{\mathrm{R}} = a_{n-1}...a_0$.

(i) The language $L := \{\mathbf{0}^n\mathbf{1}^n\mathbf{0}^n\,;\, n > 0\}$.

(ii) The language $L := \{ww^{\mathrm{R}}\,;\, w \in \mathbb{B}^+\}$ of even length palindromes.

(iii) The language $L := \{ww^{\mathrm{R}}\,;\, w \in \mathbb{B}^+$ such that the number of $\mathbf{0}$s in w is equal to the number of $\mathbf{1}$s in $w\}$.

(iv) The language $L := \{\mathbf{0}^n\mathbf{1}^m\mathbf{0}^{n+m}\,;\, n, m > 0\}$.

Paper 3, Section I

4F Automata & Formal Languages

(a) The following register machines M and N given explicitly by their programs compute characteristic functions, i.e., $f_{M,1} = \chi_A$ and $f_{N,1} = \chi_B$. Determine A and B. Justify your answer.

$$\begin{array}{lllclll} & & M & \quad & & & N \\ q_S & \mapsto & ?_\varepsilon(0, q_1, q_0) & & q_S & \mapsto & ?_{\mathbf{0}}(0, q_3, q_0) \\ q_0 & \mapsto & -(0, q_2, q_0) & & q_0 & \mapsto & -(0, q_2, q_0) \\ q_1 & \mapsto & +_{\mathbf{1}}(0, q_H) & & q_1 & \mapsto & +_{\mathbf{1}}(0, q_H) \\ q_2 & \mapsto & +_{\mathbf{0}}(0, q_H) & & q_2 & \mapsto & +_{\mathbf{0}}(0, q_H) \\ & & & & q_3 & \mapsto & -(0, q_1, q_3) \\ q_H & \mapsto & ?_\varepsilon(0, q_H, q_H) & & q_H & \mapsto & ?_\varepsilon(0, q_H, q_H) \end{array}$$

(b) By modifying N or otherwise, give the explicit program of a register machine that computes the characteristic function of the set $\{w\mathbf{1}\,;\, w \in \mathbb{B}\}$. Justify your answer.

(c) Suppose you are given the program of a register machine that computes the characteristic function of a set X. Describe how to explicitly modify the given program in order to obtain the program of a register machine that computes the characteristic function of the complement $\mathbb{B}\backslash X$. Justify your answer.

Paper 4, Section I

4F Automata & Formal Languages

(a) Say what it means for a grammar to be *variable based*.

(b) Given two grammars $G = (\Sigma, V, P, S)$ and $G' = (\Sigma, V', P', S')$, give the definitions of the *concatenation grammar* H and the *regular concatenation grammar* H^{reg}, i.e., grammars H and H^{reg} such that

(i) if G and G' are variable based with disjoint sets of variables, then

$$\mathcal{L}(H) = \mathcal{L}(G)\mathcal{L}(G') \text{ and}$$

(ii) if G and G' are regular with disjoint sets of variables, then H^{reg} is regular and

$$\mathcal{L}(H^{reg}) = \mathcal{L}(G)\mathcal{L}(G').$$

[*You do not need to prove these statements, only to provide the definitions of the grammars.*]

(c) Find examples of regular grammars G and G' with disjoint sets of variables such that H is not a regular grammar. Justify your claim.

(d) Find examples of variable based grammars G and G' with disjoint sets of variables such that $\mathcal{L}(H^{reg}) \neq \mathcal{L}(G)\mathcal{L}(G')$. Justify your claim.

Paper 1, Section II

12F Automata & Formal Languages

(a) Let $C, D \subseteq \mathbb{B}$. Give definitions of the following concepts:

(i) $C \leqslant_{\mathrm{m}} D$;
(ii) $C \equiv_{\mathrm{m}} D$; and
(iii) C is a *nontrivial index set.*

(b) The proof of Rice's theorem shows that nontrivial index sets I are not computable by either proving $\mathbf{K} \leqslant_{\mathrm{m}} I$ or $\mathbb{B}\backslash\mathbf{K} \leqslant_{\mathrm{m}} I$. State when the first or the second option holds according to the proof of Rice's theorem.

[*Define your notation; you do not need to prove your claim.*]

(c) For the nontrivial index sets

$$\mathbf{Emp} := \{w \in \mathbb{B}\,;\, \mathrm{W}_w = \varnothing\} \text{ and } \mathbf{Inf} := \{w \in \mathbb{B}\,;\, \mathrm{W}_w \text{ is infinite}\},$$

state in each case whether the first or the second option of (b) holds.

(d) Is the set $\{w \in \mathbb{B}\,;\, |w| \text{ is even}\}$ an index set? Justify your answer.

(e) Consider the nontrivial index set $\mathbf{Two} := \{w \in \mathbb{B}\,;\, |\mathrm{W}_w| \geqslant 2\}$ and show that $\mathbf{K} \equiv_{\mathrm{m}} \mathbf{Two}$.

(f) Consider the nontrivial index set

$$\mathbf{Cof} := \{w \in \mathbb{B}\,;\, \text{the complement of } \mathrm{W}_w \text{ is finite}\}$$

and show that both $\mathbf{K} \leqslant_{\mathrm{m}} \mathbf{Cof}$ and $\mathbb{B}\backslash\mathbf{K} \leqslant_{\mathrm{m}} \mathbf{Cof}$.

[*In the entire question, you may use any results proved in the lectures, provided that you state them precisely and correctly.*]

Paper 3, Section II

12F Automata & Formal Languages

(a) Let $D = (\Sigma, Q, \delta, q_0, F)$ be a deterministic automaton.

(i) Define what it means that a state $q \in Q$ is *inaccessible.*

(ii) Define what it means that two states $q, q' \in Q$ are *indistinguishable.*

(iii) Define what it means that the automaton D is *irreducible.*

(iv) State the relationship between irreducibility and the size of the smallest automaton for a regular language.

(b) Let $N = (\Sigma, Q, \Delta, q_0, F)$ be a non-deterministic automaton and $w = a_0...a_{n-1} \in \mathbb{W}$. We say that a sequence $(p_0, ..., p_n) \in Q^{n+1}$ is a witnessing sequence for w if for all $i < n$, we have $p_{i+1} \in \Delta(p_i, a_i)$. We say that it starts with q if $p_0 = q$ and that it ends with q' if $p_n = q'$.

(i) Let $q \in Q$ and $w \in \mathbb{W}$. Define $\widehat{\Delta}(q, w)$ and what it means that $w \in \mathcal{L}(N)$.

(ii) Describe the *subset construction* that takes a non-deterministic automaton N and constructs a deterministic automaton D such that $\mathcal{L}(D) = \mathcal{L}(N)$. [You should provide the construction of D but you do not need to prove that $\mathcal{L}(D) = \mathcal{L}(N)$.]

(iii) Prove that for $q, q' \in Q$ and $w \in \mathbb{W}$, we have that $q' \in \widehat{\Delta}(q, w)$ if and only if there is a witnessing sequence for w that starts with q and ends with q'.

(c) We say that a non-deterministic automaton $N = (\Sigma, Q, \Delta, q_0, F)$ is a Brzozowski automaton if

(Br_1) $F = \{q_*\}$ is a singleton;

(Br_2) for every $q \in Q$ there is a $w \in \mathbb{W}$ and a witnessing sequence for w starting from q and ending in q_*;

(Br_3) for every $w \in \mathbb{W}$ there is a unique $q \in Q$ such that there is a witnessing sequence for w starting from q and ending in q_*.

Let N be a Brzozowski automaton, D be the result of the subset construction applied to N, and D' be the automaton D with all inaccessible states removed. Show that D' is an irreducible automaton such that $\mathcal{L}(D') = \mathcal{L}(N)$.

[*In the entire question, you may use any results proved in the lectures, provided that you state them clearly.*]

Comment 1. This question looks long, but the large number of subitems in (a) and (b) will actually help the students since (a) (i) to (iii) and (b) (i) and (ii) are asking for definitions that the student will have to write down for the proof of (c) anyway. (b) (iii) is crucial for (c). One could cut (a) (iv) if the question seems too long.

Comment 2 (irrelevant for the exam; but might be of interest to future supervisors of revision supervisions). If D is any deterministic automaton, reversing all arrows of the transition function gives a non-deterministic automaton N such that $\mathcal{L}(N)$ is the set of reversed words in $\mathcal{L}(D)$. This automaton is a Brzozowski automaton and thus leads to a nice construction of the minimal automaton for the reverse language: reverse the arrows, do the subset construction, remove inaccessible states. This observation is the key ingredient to Brzozowski's algorithm for minimising automata: his algorithm is to do the above construction twice.

Paper 1, Section I

8B Classical Dynamics

This question concerns a linear, triatomic molecule, consisting of two outer atoms of mass m on either side of an inner atom of mass M. All three atoms lie on a vertical line, taken as the y-axis (directed upwards), at heights $y_1 > y_2 > y_3$. The atoms move under the influence of a uniform, downward gravitational acceleration of magnitude g, as well as forces arising from the potential energy

$$\tfrac{1}{2}\left[k(y_1-y_2)^2 + k(y_2-y_3)^2\right] .$$

The constants m, M, k and g are positive.

(a) Write down the Lagrangian $L(y_1, \dot{y}_1, y_2, \dot{y}_2, y_3, \dot{y}_3)$ for the system. Give an expression for the centre of mass Y of the molecule, and determine its time evolution $Y(t)$ assuming $Y(0) = A$, where A is a constant.

(b) Introduce generalised coordinates

$$Q_s = y_1 + y_3 \qquad \text{and} \qquad Q_a = y_1 - y_3$$

and use your answer to part (a) to eliminate y_2 and obtain a Lagrangian $\hat{L}$ in terms of Q_s and Q_a. Hence obtain a differential equation for Q_a.

Paper 2, Section I

8B Classical Dynamics

(a) Explain very briefly how to introduce action-angle variables ϕ, I for a Hamiltonian system determined in the standard way by a Hamiltonian $H(q,p)$ defined for $(q,p) \in \mathbb{R}^2$. [You may assume that all orbits are bounded for your discussion.] Furthermore, briefly explain what is meant by the *principle of adiabatic invariance of the action.*

(b) Consider the case

$$H(q,p) = \frac{p^2}{2m} - \frac{1}{|q|} ,$$

where m is a positive constant. Explain why, for solutions with $H = E = -|E| < 0$, the magnitude of q must remain bounded. Find the smallest possible $q_{\max} = q_{\max}(|E|)$ such that the interval $[-q_{\max}, q_{\max}]$ contains all possible values of $q(t)$ for such a solution.

Calculate the action I in terms of $|E|$ and m. Assuming further that the adiabatic invariance principle holds for this system, if m varies slowly over a long time interval, doubling in magnitude, how does the energy change?

[*Hint: you may make use of the integral* $\int_0^1 \sqrt{1-x^2}dx = \frac{1}{4}\pi$.]

Paper 3, Section I

8B Classical Dynamics

Let $\mathcal{C}$ be a solid cone of height l with circular cross-section of radius R at the base. (The height is the distance between the base and the vertex along the axis of symmetry.) Denote the axis of symmetry by $\mathbf{e}_3$, which is directed from the vertex of $\mathcal{C}$ to its base. The cone has uniform density ρ, so that the total mass is $M = \frac{1}{3}\pi R^2 l\rho$.

The principal moments of inertia of $\mathcal{C}$ with respect to its centre of mass are

$$I_1^{\rm CM} = I_2^{\rm CM} = \frac{1}{80}\pi R^2 l(4R^2 + l^2)\rho \quad \text{and} \quad I_3^{\rm CM} = \frac{1}{10}R^4 l\rho\,.$$

Using standard Euler angles (ψ, θ, ϕ) to describe the orientation of $\mathcal{C}$, the angular velocity has components

$$\boldsymbol{\omega} = (\dot\psi + \cos\theta\,\dot\phi)\mathbf{e}_3 + (\cos\psi\sin\theta\,\dot\phi - \sin\psi\,\dot\theta)\mathbf{e}_2 + (\sin\psi\sin\theta\,\dot\phi + \cos\psi\,\dot\theta)\mathbf{e}_1$$

with respect to the principal axes $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

(a) Compute the centre of mass of $\mathcal{C}$. State the parallel axis theorem. Find the principal moments of inertia of $\mathcal{C}$ about its vertex.

(b) Let the vertex of $\mathcal{C}$ be fixed. Using the formulae for the angular velocity with respect to the principal axes given above, write down the Lagrangian for the dynamics of $\mathcal{C}$, taking the magnitude of downward gravitational acceleration g to be constant. Identify any ignorable (that is to say, cyclic) coordinates, and find the corresponding conserved quantities. Obtain the Hamiltonian for the system.

(c) Find a Hamiltonian system on a two-dimensional phase space by reducing the number of degrees of freedom by means of the ignorable coordinates you found in part (b). Show that the configuration in which $\mathbf{e}_3$ is oriented vertically upward is an equilibrium which is stable if the angular momentum about this axis is sufficiently large.

Paper 4, Section I

8B Classical Dynamics

(a) State the *Jacobi identity* for the Poisson bracket $\{F, G\}$ on a phase space. Prove that if F and G are both conserved quantities for the flow generated by a Hamiltonian H, then $\{F, G\}$ is also conserved.

(b) Consider the following mappings $(q, p) \mapsto (Q(q, p), P(q, p))$, which depend on the parameter $\lambda \in \mathbb{R}$:

(i) $(Q, P) = (\lambda q, \lambda p)\,;$

(ii) $(Q, P) = (q, \lambda p)\,;$

(iii) $(Q, P) = (p, \lambda q)\,.$

For which values of λ are these mappings canonical? For each value of λ *either* show the mapping is *not* canonical *or*, if it is canonical, find a generating function for the mapping.

[*Hint: the generating function will either be of type $S = S(q, P; \lambda)$, such that the mapping is equivalent to*

$$p = \frac{\partial S}{\partial q}\,, \qquad Q = \frac{\partial S}{\partial P}\,,$$

or of type $\Phi = \Phi(q, Q; \lambda)$ with the mapping equivalent to

$$p = \frac{\partial \Phi}{\partial q}\,, \qquad P = -\frac{\partial \Phi}{\partial Q}\,.$$]

Paper 2, Section II

14B Classical Dynamics

This question concerns a double pendulum, consisting of a simple pendulum of mass M and length l pivoted at the origin with angle θ_1 with respect to the vertical, together with another simple pendulum of mass m, also of length l and angle θ_2 with respect to the vertical, pivoted at the first mass M. The whole system moves freely in a vertical plane under the influence of a downward uniform gravitational acceleration of magnitude g. All the constants m, M, l and g are positive and, in addition, define $\omega_0 = \sqrt{g/l}$.

(a) Show that the Lagrangian for the system is

$$L = \frac{1}{2}Ml^2\dot{\theta}_1^2 + \frac{1}{2}m\left[l^2\dot{\theta}_1^2 + l^2\dot{\theta}_2^2 + 2l^2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2\right] + Mgl\cos\theta_1 + mg(l\cos\theta_2 + l\cos\theta_1)\ .$$

(b) Write down the equations of motion and expressions for any conserved quantities. Furthermore, show that $\theta_1 = 0 = \theta_2$ is an equilibrium point, and derive the linearized equations of motion for small oscillations

$$(\theta_1, \theta_2) = (0, 0) + (z_1, z_2)\,, \qquad |z_1| + |z_2| = o(1)\,,$$

around it.

(c) Find the four normal modes and show that the equilibrium point is stable. Consider the case when $\mu = m/M \ll 1$. Show that the characteristic frequencies are $\pm\omega$ and $\pm\omega'$, for some positive ω and ω' with $\omega - \omega' = \alpha\sqrt{\mu} + O(\mu)$, where α is a constant you should find.

Paper 4, Section II

15B Classical Dynamics

Let $I_1 < I_2 < I_3$ be the three principal moments of inertia of a rigid body that rotates freely with angular velocity $\boldsymbol{\omega}$ according to the Euler equations

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3\,,$$
$$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1\,,$$
$$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2\,,$$

where the components ω_1, ω_2 and ω_3 of the angular velocity are taken with respect to the principal axes of inertia.

(a) Write down expressions for the energy E and the total angular momentum squared L^2, and prove that these are conserved using the Euler equations.

(b) Show that if $L^2 = 2EI_2$ there exist solutions in which the angular velocity is directed along the second principal axis, i.e., ω_1 and ω_3 are zero. What are the possible values for ω_2? Use linearisation to analyse the stability of these solutions.

(c) Still working under the condition $L^2 = 2EI_2$, use your expressions from part (a) to express ω_1 and ω_3 in terms of E and L^2, and hence obtain a first-order differential equation for ω_2. Integrate this equation and show that $\omega_2(t) = \mu \tanh(\lambda t)$ for some constants μ, λ which you should find. Briefly comment on the relation of this solution to your answer to part (b).

Paper 1, Section I

3K Coding & Cryptography

State and prove Kraft's inequality.

Describe Shannon-Fano Coding. Explain why it works and give an upper bound on its expected word length.

Paper 2, Section I

3K Coding & Cryptography

Suppose codewords 000 and 111 are sent with probabilities 1/5 and 4/5 respectively through a Binary Symmetric Channel with error probability $p = 1/4$. If we receive 001 how should we decode if we use (i) ideal observer, (ii) maximum likelihood, (iii) minimum distance decoding? Justify your answers.

In light of this give some positives and negatives of the three decoding methods.

Paper 3, Section I

3K Coding & Cryptography

Explain what is meant by a *Bose-Ray Chaudhuri-Hocquenghem (BCH) code with design distance* δ. Prove that, for such a code, the minimum distance between codewords is at least δ. [Results about the Vandermonde determinant may be quoted without proof provided they are clearly stated.]

How many errors will the code detect? How many errors will it correct? Justify your answers.

Paper 4, Section I

3K Coding & Cryptography

Define a linear feedback shift register (LFSR) and its associated feedback polynomial.

Suppose an LFSR has a feedback polynomial of degree d. Explain why the period produced by this LFSR cannot be longer than $2^d - 1$.

Explain why an LFSR that generates a maximal period must have an odd number of coefficients equal to 1 in its feedback polynomial. (That is, if the feedback polynomial is given by $x^d + a_{d-1}x^{d-1} + \cdots + a_0$, then an even number of the a_i should be equal to 1.)

The output sequence of an LFSR starts with 100000001. Give a minimal LFSR that generates this output, i.e. one whose feedback polynomial has least degree. Justify your answer.

Paper 1, Section II
11K Coding & Cryptography

Describe the Huffman coding scheme and prove that Huffman codes are optimal.

A Huffman code is used to encode letters $a_1, \ldots, a_m$ with respective probabilities $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_m$. Prove that, if $p_1 < 1/3$, all codewords have length at least 2. Prove that, if $p_1 > 2/5$, then there is a codeword of length 1.

Find a probability distribution for which both of the following codes are optimal.

(a) 0, 10, 110, 111

(b) 00, 01, 10, 11

Paper 2, Section II
12K Coding & Cryptography

Consider a cryptosystem $\langle M, K, C \rangle$. Let e, d be the respective encryption and decryption functions. Model the key and messages as random variables k, m taking values in K, M, respectively and such that $m = d(c, k) \in M$ and $c = e(m, k) \in C$.

Define, both in words and formally, the *unicity distance*, U, of a cryptosystem.

Prove that

$$U = \frac{\log |K|}{\log |\Sigma| - H}$$

where Σ is the alphabet of the ciphertext and $H = H(m)$. Make clear any assumptions you make.

Suppose $M = \{0, 1, 2\}$ is emitted from a memoryless source with probabilities

$$P(m = 0) = 1/2, \; P(m = 1) = p \; \text{ and } \; P(m = 2) = 1/2 - p$$

where $0 \leqslant p \leqslant 1/4$. Let the key $k = (k_0, k_1, k_2)$ be chosen uniformly from the set of binary 3-tuples i.e. $K = \{(k_0, k_1, k_2) : k_i \in \{0, 1\}\}$. A sequence of messages $m_1, m_2, \ldots, m_n$ is encrypted to a sequence of ciphertexts $c_1, c_2, \ldots, c_n$ by

$$c_i = m_i + k_{i \bmod 3} \pmod 3$$

for $1 \leqslant i \leqslant n$.

Show that, if the unicity distance of the cryptosystem is at least 20, then we must have $H(2p, 1 - 2p) \geqslant 0.87$ (you may take $\log_2(3) = 1.585$).

Given that $H(2p, 1 - 2p) = 0.87$ is satisfied when $p = 0.15$, find all values of $p \in [0, 1/2]$ that give a unicity distance of at least 20.

Now suppose $p = 0$. Propose a new cipher for this source which has infinite unicity distance.

Paper 1, Section I

9E Cosmology

Consider the motion of light rays in a homogeneous and isotropic expanding universe with scale factor $a(t)$. Light emitted by a distant galaxy at wavelength λ_e is observed on Earth to have wavelength λ_0. The galaxy redshift z is defined by

$$1 + z = \frac{\lambda_0}{\lambda_e}.$$

(a) Assuming that the galaxy remains at a fixed comoving distance, show that the redshift is related to the scale factor by

$$1 + z = \frac{a(t_0)}{a(t_e)},$$

where the light is emitted at time t_e and observed today at time t_0.

(b) Suppose the galaxy is located at comoving position x and let L be the amount of energy emitted by the galaxy in photons per unit time. Show that the total energy per unit time crossing a sphere centred on the galaxy and intercepting the Earth is

$$\frac{L}{(1+z)^n},$$

where n is an integer you should determine. Hence, show that the energy per unit time per unit area reaching the Earth is

$$\frac{L}{4\pi a^2(t_0)\, x^2\, (1+z)^n}.$$

Paper 2, Section I

9E Cosmology

Prior to the synthesis of light elements ($k_{\mathrm{B}}T \gtrsim 1$ MeV), neutrons and protons are kept in equilibrium by the weak interactions

$$n + \nu_e \leftrightarrow p + e^- , \qquad p + \bar{\nu}_e \leftrightarrow n + e^+ .$$

The ratio of the weak interaction rate $\Gamma_W \propto T^5$, which maintains equilibrium, relative to the Hubble expansion rate $H \propto T^2$, is

$$\frac{\Gamma_W}{H} \approx \left(\frac{k_{\mathrm{B}}T}{\kappa}\right)^3 \qquad \text{where} \;\; \kappa = 0.7 \text{ MeV}. \qquad (\dagger)$$

(a) Assuming that the chemical potentials for all leptons are small, $\mu_{e^-} \ll k_{\mathrm{B}}T$ etc., show that, in equilibrium, the neutron-to-proton ratio can be expressed as

$$\frac{n_n}{n_p} \approx e^{-Q/(k_{\mathrm{B}}T)} ,$$

where $Q = (m_n - m_p)c^2 = 1.29$ MeV is the mass difference between a neutron and a proton.

(b) Using equation (†), briefly explain why the neutron-to-proton ratio effectively 'freezes out' once $k_{\mathrm{B}}T < 0.7$ MeV. At this time, the ratio is $n_n/n_p \approx 1/6$, but it decreases to a final value $n_n/n_p \approx 1/7$ when deuterium forms at $k_{\mathrm{B}}T \approx 0.07$ MeV. Briefly specify why.

(c) Briefly explain why eventually almost all neutrons are captured in helium-4, and estimate the resulting helium mass parameter $Y_p = \rho_{\mathrm{He}}/\rho_{\mathrm{B}}$, where ρ_{He} is the helium-4 density, $\rho_{\mathrm{B}} = m_p\, n_{\mathrm{B}}$, and n_{B} is the baryon number density.

(d) Consider an otherwise identical universe where the constant κ in equation (†) is much larger than 0.7 MeV. Describe how this would affect the 'freeze-out' described by equation (†) and the helium mass parameter Y_p. Briefly discuss potential implications for stellar lifetimes and the origin of life in this alternative universe.

Paper 3, Section I

9E Cosmology

The equation governing the evolution of density-perturbation modes $\delta(\mathbf{k},\tau)$ in conformal time τ is

$$\delta''(\mathbf{k},\tau) + \mathcal{H}(\tau)\,\delta'(\mathbf{k},\tau) - \frac{3}{2}\Omega_M(\tau)\,\mathcal{H}(\tau)^2\delta(\mathbf{k},\tau) = 0\,, \qquad (\dagger)$$

where a prime denotes $\frac{\partial}{\partial\tau}$, $\mathcal{H}(\tau) = a'/a$, a is the scale factor, Ω_M is the density of non-relativistic matter relative to the total density, and $\mathbf{k}$ is the comoving wavevector.

In the following, we consider a flat, matter-dominated universe after equal matter-radiation ($\tau \geqslant \tau_{\rm eq}$), for which you may assume that $\Omega_M \approx 1$ and $a(\tau) = (\tau/\tau_0)^2$, where τ_0 is the conformal time today.

(a) By seeking a power-law solution of the form $\delta = \tau^\beta$, show that the general solution of equation ($\dagger$) for the matter-dominated era ($\tau_{\rm eq} \leqslant \tau \leqslant \tau_0$) takes the form

$$\delta(\mathbf{k},\tau) = A(\mathbf{k})\tau^2 + B(\mathbf{k})\tau^{-3}\,, \qquad (*)$$

where $A(\mathbf{k})$, $B(\mathbf{k})$ are arbitrary functions.

(b) Show that a mode with physical wavelength $\lambda(\tau) = 2\pi a(\tau)/k$, corresponding to the comoving wavenumber $k = |\mathbf{k}|$, crosses inside the cosmological horizon at time $\tau_H = 2\pi/(k\,c)$. Now consider a perturbation mode $\delta(\mathbf{k}_{\rm eq},\tau)$ with wavevector $\mathbf{k}_{\rm eq}$ that crosses inside the cosmological horizon at $\tau_H = \tau_{\rm eq}$, that is, at time of equal matter-radiation. Using equation ($*$), show that the linear growth of this perturbation mode is given today by

$$D_{\rm eq} := \frac{\delta(\mathbf{k}_{\rm eq},\tau_0)}{\delta(\mathbf{k}_{\rm eq},\tau_{\rm eq})} = \frac{a(\tau_0)}{a(\tau_{\rm eq})}\,.$$

(c) Assume that for each wavevector $\mathbf{k}$, the amplitude of the corresponding mode at its horizon crossing time τ_H is given by $|\delta(\mathbf{k},\tau_H)| = \tau_H^2\,\hat{A}\,k^{1/2}$ with constant $\hat{A}$. Show that the power spectrum today takes the form

$$|\delta(\mathbf{k},\tau_0)|^2 = \frac{C}{k_{\rm eq}^4}k\,, \qquad k_{\rm eq} \leqslant k \leqslant k_0\,,$$

where the amplitude C should be specified in terms of $\hat{A}$ and $D_{\rm eq}$. Here, k_0 is the wavenumber of a mode crossing inside the cosmological horizon at τ_0.

Paper 4, Section I

9E Cosmology

An inflationary Friedmann-Lemaître-Robertson-Walker universe is governed by the following slow-roll equations for the scale factor $a(t)$ and the scalar field $\phi(t)$,

$$H^2 = \frac{1}{3M_{\rm Pl}^2} V(\phi)\,, \qquad 3H\dot{\phi} = -V'(\phi)\,,$$

where a dot denotes d/dt, $H = \dot{a}/a$, $V'(\phi) = dV/d\phi$ and $M_{\rm Pl}$ is the Planck mass.

(a) Defining the slow-roll parameter

$$\epsilon(\phi) \; := \; \frac{M_{\rm Pl}^2}{2}\left[\frac{V'(\phi)}{V(\phi)}\right]^2 ,$$

verify that the condition $\epsilon(\phi) \ll 1$ is consistent with the slow-roll condition $\dot{\phi}^2 \ll V$. Show that

$$\frac{H}{\dot{\phi}} = -\frac{1}{M_{\rm Pl}^2}\frac{V}{V'} = -\frac{1}{\sqrt{2}M_{\rm Pl}}\frac{1}{\sqrt{\epsilon}}\,.$$

(b) The amount of inflation is given by the number of e-folds by which the scale factor grows, $N = \log[a(t_f)/a(t_i)]$, where t_i and t_f are the start and end times of inflation. Denoting $\phi_i = \phi(t_i)$ and $\phi_f = \phi(t_f)$, show that in the slow-roll regime,

$$N = \int_{t_i}^{t_f} H\,dt \;\approx\; \frac{1}{\sqrt{2}M_{\rm Pl}}\int_{\phi_f}^{\phi_i}\frac{d\phi}{\sqrt{\epsilon(\phi)}}\,.$$

(c) Consider the potential $V(\phi) = V_0\,[1+\cos(\phi/f)]$, where V_0 and f are positive constants. Show that

$$\sqrt{\epsilon(\phi)} = \frac{M_{\rm Pl}}{\sqrt{2}f}\,\tan\frac{\phi}{2f}\,.$$

Hence, find that the number of e-foldings for this model is given by

$$N = \frac{2}{M_{\rm Pl}^2}\,f^2\left[\log\left(\sin\frac{\phi_i}{2f}\right) - \log\left(\sin\frac{\phi_f}{2f}\right)\right]\,.$$

Paper 1, Section II

15E Cosmology

(a) Consider the Friedmann-Lemaître-Robertson-Walker (FLRW) metric with co-moving curvature constant k (not normalised to unity),

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[\frac{dr^2}{1 - k\, r^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right] .$$

(i) Briefly comment on the three geometries described by this metric and, in each case, calculate the proper distance between the points $r = 0$ and $r = \Delta r$ along curves with $dt = d\theta = d\phi = 0$.

(ii) For each geometry give new time and radial coordinates τ and χ that transform the metric to

$$ds^2 = a^2(\tau) \left[-c^2 d\tau^2 + d\chi^2 + f^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2) \right] ,$$

where the function $f(\chi)$ should be specified. Along which trajectories do radial light rays ($d\theta = d\phi = 0$) propagate in these coordinates?

(b) For an FLRW universe with vanishing cosmological constant the Friedmann and continuity equations are

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3c^2}\, \rho , \qquad \dot{\rho} = -3\frac{\dot{a}}{a} (\rho + P) ,$$

where ρ is the energy density, P is the pressure and $\dot{a} = \frac{da}{dt}$. Consider an open universe ($k < 0$) filled with dark energy 'quintessence', which has an equation of state $P = -\frac{2}{3}\rho$. At $t = t_0$ we take $\rho(t_0) = \rho_0$ and $a(t_0) = 1$.

(i) Use the continuity equation to determine the rate at which the energy density falls as the universe expands and show that

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\gamma}{a} + \frac{\beta}{a^2} ,$$

where γ and β are positive parameters you should determine.

(ii) Solve the Friedmann equation for initial conditions $a(0) = 0$ to find the scale factor $a(t) = t(\sqrt{\beta} + \gamma t/4)$.

(iii) Calculate the age of the universe t_0 when $a(t_0) = 1$. Compare t_0 with the inverse Hubble parameter H_0^{-1} at t_0 in the limiting cases $\beta \gg \gamma$ and $\gamma \gg \beta$.

(iv) Our present universe is observed to be accelerating, through measurements of the deceleration parameter $q_0 := -\ddot{a}(t_0)\, a(t_0)/\dot{a}(t_0)^2 \approx -0.55$. Can the quintessence model outlined in this part of the question have $q_0 \leqslant -0.5$ for any parameter values?

Paper 3, Section II
14E Cosmology

(a) Consider non-relativistic particles of mass m in equilibrium at temperature T with chemical potential μ. Assuming $k_{\rm B}T \ll mc^2$ and $\mu \ll mc^2$, show that both Bose-Einstein and Fermi-Dirac distributions reduce to the Maxwell-Boltzmann distribution

$$n = \left(\frac{4\pi g_s}{h^3}\right) \int_0^\infty dp\; p^2\; e^{-[E(p)-\mu]/(k_{\rm B}T)}\,.$$

Using $\int_0^\infty dp\; e^{-p^2/\sigma^2} = \frac{1}{2}\sigma\sqrt{\pi}$, show that for $E(p) = mc^2 + p^2/(2m)$,

$$n = g_s \left(\frac{2\pi m k_{\rm B} T}{h^2}\right)^{\frac{3}{2}} e^{(\mu - mc^2)/(k_{\rm B}T)}\,. \qquad (\dagger)$$

(b) The recombination of free electrons in the early universe is significantly affected by the abundance of helium-4 in the universe, which, in terms of the baryon density $n_{\rm B}$, is given by the parameter

$$Y_p = \frac{m_{\rm He}}{m_{\rm H}}\frac{n_{\rm He}}{n_{\rm B}} = 4\,\frac{n_{\rm He}}{n_{\rm B}} \;\approx\; \frac{1}{4}\,.$$

In the following we neglect doubly-ionized helium ($n_{\rm He^{++}} \approx 0$). Then the recombination of hydrogen and helium proceeds with ionization energies $I_{\rm H}$ and $I_{\rm He}$ according to

$$\begin{aligned} {\rm H}^+ + {\rm e}^- &\leftrightarrow {\rm H}^0 + \gamma\,, \qquad & I_{\rm H} &= (m_{\rm H^+} + m_{\rm e} - m_{\rm H^0})\,c^2 \;\approx\; 13.6\,{\rm eV}\,, \\ {\rm He}^+ + {\rm e}^- &\leftrightarrow {\rm He}^0 + \gamma\,, \qquad & I_{\rm He} &= (m_{\rm He^+} + m_{\rm e} - m_{\rm He^0})\,c^2 \;\approx\; 25.6\,{\rm eV}\,. \end{aligned}$$

(i) Using these equilibrium processes and equation ($\dagger$) together with $g_e = 2$, $g_{\rm H^+}/g_{\rm H^0} = \frac{1}{2}$ and $g_{\rm He^+}/g_{\rm He^0} = 1$, show that

$$\begin{aligned} \frac{n_{\rm e}\, n_{\rm H^+}}{n_{\rm H^0}} &= \left(\frac{2\pi\, m_{\rm e}\, k_{\rm B}T}{h^2}\right)^{3/2} e^{-I_{\rm H}/(k_{\rm B}T)}\,, \\ \frac{n_{\rm e}\, n_{\rm He^+}}{n_{\rm He^0}} &= 2\left(\frac{2\pi\, m_{\rm e}\, k_{\rm B}T}{h^2}\right)^{3/2} e^{-I_{\rm He}/(k_{\rm B}T)}\,. \end{aligned}$$

(ii) The hydrogen and helium ionization fractions are $X_{\rm H^+} := n_{\rm H^+}/n_{\rm H}$ and $X_{\rm He^+} := n_{\rm He^+}/n_{\rm He}$ (with $n_{\rm H} = n_{\rm H^0} + n_{\rm H^+}$, $n_{\rm He} = n_{\rm He^0} + n_{\rm He^+}$). Show that the free electron density is

$$\mathcal{F} \;\equiv\; \frac{n_{\rm e}}{n_{\rm B}} = \alpha\, X_{\rm H^+} + \beta\, X_{\rm He^+}\,,$$

where α and β should be specified in terms of Y_p.

(iii) Given the relation $n_B = \eta\, n_\gamma = \eta\left[16\pi\zeta(3)/(hc)^3\right](k_{\rm B}T)^3$ (with baryon-to-photon ratio η), use the fractional densities $X_{\rm H^+}$, $X_{\rm He^+}$ and $\mathcal{F}$ to obtain a closed set of two equations that describes recombination for both hydrogen and helium. Verify that taking the $Y_p \to 0$ limit yields the usual expression for Saha's equation with only hydrogen.

Paper 1, Section II

26I Differential Geometry

(a) Define the terms *critical point, critical value* and *regular value.* Let $A \in S_n(\mathbb{R})$ be a symmetric $n \times n$ matrix with real entries and $f : \mathbb{R}^n \to \mathbb{R}$ the map $f(X) = X^T A X$ for a column vector $X \in \mathbb{R}^n$. Show that f has only one critical value. Can it have more than one critical point? Justify your answer.

(b) Let $M_{2n}(\mathbb{R})$ be the set of $2n \times 2n$ matrices with real entries, and $\mathrm{Sp}_n(\mathbb{R}) \subset M_{2n}(\mathbb{R})$ the set of matrices A such that $A^T J A = J$ with $J := \left(\begin{smallmatrix} 0_n & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0_n \end{smallmatrix}\right)$. Prove that $\mathrm{Sp}_n(\mathbb{R})$ is a submanifold of $M_{2n}(\mathbb{R})$ with dimension $2n^2 + n$.

[*You can use the pre-image theorem if properly stated.*]

(c) Let $\mathrm{Gr}_{1,3}(\mathbb{R})$ be the set of 3×3 symmetric matrices with real entries P so that $P^2 = P$ and $\mathrm{Trace}(P) = 1$.

(i) Prove that $\mathrm{Gr}_{1,3}(\mathbb{R})$ is a submanifold of $S_3(\mathbb{R})$ with dimension 2.

[*Hint: You might want to first prove that given $P \in \mathrm{Gr}_{1,3}(\mathbb{R})$, there exists $X \in \mathbb{R}^3$ such that the solutions to $P = YY^T$ with $|Y|^2 = 1$ for the Euclidean norm are exactly $Y \in \{X, -X\}$. Then use this fact to construct local parametrisations.*]

(ii) Does $\mathrm{Gr}_{1,3}(\mathbb{R})$ admit a global parametrisation by an open subset of $\mathbb{R}^2$?

Paper 2, Section II

26I Differential Geometry

(a) Define what is a *regular curve* and its *arc-length*, and prove that a regular curve can always be parametrised by arc-length. When so, define its *torsion*, and prove that the torsion is zero for planar curves.

(b) Consider a regular simple planar closed curve $\alpha : I = [a,b] \to \mathbb{R}^2$ enclosing an open bounded convex set Ω. We consider a line $L \subset \mathbb{R}^2$ outside Ω. Without loss of generality we assume $L = \{y = 0\}$ and $\Omega \subset \{y > 0\}$ and denote by x_0 and x_1 the minimum and maximum x-coordinate of $\alpha(I)$. You may assume that there are two smooth functions $u_\pm : [x_0, x_1] \to \mathbb{R}$ so that

$$\Omega = \{(x,y) \,:\, x \in (x_0,x_1) \text{ and } y \in (u_-(x), u_+(x))\}$$

with $u_- \leqslant u_+$ and u_- convex and u_+ concave. Then we define the following symmetrised set

$$S_L(\Omega) := \left\{(x,y) \,:\, x \in (x_0,x_1), \text{ and } y \in \left(-\frac{u_+(x)-u_-(x)}{2}, \frac{u_+(x)-u_-(x)}{2}\right)\right\}.$$

(i) Prove that the areas enclosed satisfy $\mathcal{A}(S_L(\Omega)) = \mathcal{A}(\Omega)$.

[*Hint: Decompose the area into a trapezoid and two caps and figure out how they are transformed by the symmetrisation.*]

(ii) Prove that the perimeter of $S_L(\Omega)$ is at most the perimeter of Ω with equality if and only if Ω has an axis of symmetry parallel to L.

[*Hint: Calculate the perimeters following the decomposition of the previous hint, and reduce the inequality to be proven to a Minkowski inequality.*]

(iii) Deduce that any convex perimeter-minimizing domain Ω with fixed area and whose boundary is a regular curve must admit axes of symmetry in all directions.

Paper 3, Section II

25I Differential Geometry

(a) Given a surface S (2-manifold) in $\mathbb{R}^3$, define the *first fundamental form* and express it in a local parametrisation, then define the *Gauss map* and the *second fundamental form*, and express them in a local parametrisation. Define the *Gauss curvature* and the *mean curvature*.

(b) Let $s \mapsto (Y(s), Z(s))$ be a planar curve parametrised by arc-length in the yz-plane with $Y(s) > 0$ for all s. The surface S of revolution attained by rotating this curve about the z-axis is parametrised by $\phi(u, v) = (Y(u)\cos v, Y(u)\sin v, Z(u))$.

(i) Calculate the first fundamental form, the Gauss map and the second fundamental form in the parametrisation ϕ. Deduce that the Gauss curvature K is equal to $-Y''/Y$ and give an expression for the mean curvature H in terms of Y and Z.

(ii) Given a curve $\alpha : I \to S$ on a surface S parametrised by arc-length, define what it means for the curve to be a *geodesic*, the *Christoffel symbols*, and the *geodesic equations* in terms of the Christoffel symbols.

(iii) Given a surface of revolution S with the above parametrisation, $\alpha(t) = \phi(u(t), v(t))$ a curve on S, prove that if α is a geodesic then $[Y(u)]^2\,\dot{v}$ is constant.

Paper 4, Section II

25I Differential Geometry

(a) Given a surface $S \subset \mathbb{R}^3$, define the *exponential map* around a point, and state and prove the Gauss lemma (expressing the first fundamental form in the local parametrisation ϕ that maps the polar coordinates on the tangent space to the surface by the exponential map).

(b) Given a surface S and a smooth curve α on S parametrised by arc-length, define the *Gauss map* and the *geodesic curvature* of α in terms of a covariant derivative. What does it mean for α to have zero *geodesic curvature*?

(c) Give a statement of the global Gauss-Bonnet theorem with boundary terms.

(d) Let $S \subset \mathbb{R}^3$ be a compact connected oriented surface diffeomorphic to a sphere and with positive Gauss curvature everywhere. Prove that any two closed geodesic curves on S must intersect.

Paper 1, Section II

32A Dynamical Systems

(a) Define a *Lyapunov function* for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ around a fixed point at the origin. State both the first and second Lyapunov Theorems and prove the first.

(b) Consider the system

$$\dot{x} = -x + 2x^2 + y^2 + xy^2,$$
$$\dot{y} = -y + 4x^2 + 2y^2 - 2x^2 y.$$

(i) Show that the fixed point at the origin is asymptotically stable.

(ii) Show that the basin of attraction of the origin includes the region

$$12x^2 + 6y^2 < 1.$$

(iii) Can the strict inequality in part (ii) be extended to include equality? Justify your answer, stating carefully any results you need.

Paper 2, Section II

33A Dynamical Systems

State the Centre Manifold Theorem for the dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu)$ in $\mathbb{R}^n$ where μ is a real parameter. What is the key step in generating an extended centre manifold?

Consider the system

$$\dot{x} = x(\mu - y^2 - 2x^2),$$
$$\dot{y} = y(1 - x^2 - y^2),$$

where $x, y \geqslant 0$ and $\mu > 0$.

(a) Show that the fixed point $(0, 1)$ has a bifurcation at $\mu = 1$. Find a fixed point on the x-axis and determine the value of $\mu = \mu_c > 1$ at which it has a bifurcation.

(b) By finding the first approximation to the extended centre manifold, construct the normal form near the bifurcation point $(0, 1)$ when $\mu \approx 1$. Hence identify the type of bifurcation. By appealing to a symmetry of the system, explain why this bifurcation is expected.

(c) Show that there is another fixed point with $x > 0$, $y > 0$ and show how its structure is consistent with your normal form in part (b).

(d) Draw a sketch of the values of x at the fixed points as functions of μ indicating the bifurcation points and the regions where each branch is stable. [Detailed calculations are not required.]

Paper 3, Section II

31A Dynamical Systems

(a) State and prove Dulac's Theorem. What is the divergence test?

(b) Consider the system

$$\dot{x} = \mu x - y - (x^3 + xy^2 - \lambda x)(x^2 + y^2),$$
$$\dot{y} = x + \mu y - (y^3 + x^2 y - \lambda y)(x^2 + y^2),$$

for a real parameter μ and constant $\lambda > 0$.

(i) Show that there is a fixed point at the origin and classify its stability.

(ii) Find how the number of periodic orbits varies with the value of μ and hence identify two bifurcation points.

[*Hint: use polar coordinates* (r, θ).]

(iii) Identify the type of bifurcation occurring at the larger value of μ and, without detailed computation, write down its normal form. Draw the steady and periodic solutions in a (μ, r) diagram.

(iv) Show that, as μ varies, the locations of the periodic orbits are consistent with the divergence test.

Paper 4, Section II

32A Dynamical Systems

Define what it means for a map $F : I \to I \subset \mathbb{R}$ to be chaotic according to Devaney.

Show that the sawtooth map,

$$F(x) = 2x \text{ [mod 1]},$$

satisfies this definition for $x \in [0, 1)$.

(a) In the following use binary representation to describe the action of F.

(i) Give a value for x that produces a chaotic sequence.

(ii) Show that there is only one fixed point of F.

(iii) Find all 2-cycles and 4-cycles of F and express them as fractions.

(b) Now consider $F^n(x)$, the map F applied n times.

(i) Show that $F^n(x) = 2^n x$ [mod 1].

(ii) Use part (b)(i) to determine the number of fixed points of F^n. Explain how this is consistent with your answers in part (a)(iii).

(iii) Hence show that the number of 2^k cycles of F is $2^{2^k} - 2^{2^{k-1}}$ when $k \geqslant 1$. [Cycles starting a different points count as different cycles.]

Paper 1, Section II

37B Electrodynamics

A relativistic particle of mass m and charge q moves with four-velocity u^μ in the presence of a background electromagnetic field with field-strength tensor $F_{\mu\nu}$ according to the Lorentz force law,

$$\frac{du^\mu}{d\tau} \;=\; \frac{q}{m}\,F^\mu{}_\nu u^\nu \; .$$

Here τ is the proper time.

Assume that the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ are constant and homogenous and that the particle starts from rest at the origin $\mathbf{x} = \mathbf{0}$ at time $t = 0$ in some inertial frame. Find the subsequent trajectory of the particle, giving its spacetime position $(ct, \mathbf{x})$ explicitly as a function of τ, in the following special cases:

(i) $\mathbf{E} = (E, 0, 0)$ and $\mathbf{B} = \mathbf{0}$.

In this case consider a light signal directed along the positive x-axis emitted from the point $\mathbf{x} = (-h, 0, 0)$ at time $t = 0$, where $h > 0$. Find the time taken for the light signal to catch up with the particle and show that it never catches the particle if h exceeds a critical value that you should determine.

(ii) $\mathbf{E} = (E, 0, 0)$ and $\mathbf{B} = (0, 0, E/c)$.

In this case you should show that the particle trajectory lies on a cubic curve in the (x, y) plane that you should determine explicitly.

Paper 3, Section II
36B Electrodynamics

Starting from a suitable general solution of Maxwell's equations, which you may state without derivation, find the total power $\mathcal{P}$ emitted through a large spherical surface of radius R by a localised source with time-dependent electric dipole moment $\mathbf{p}(t)$ in the dipole approximation,

$$\mathcal{P} \simeq \frac{\mu_0}{6\pi c}|\ddot{\mathbf{p}}(t-R/c)|^2 \ .$$

You should state clearly the conditions under which the approximation is valid.

A simple model of a pulsar consists of a solid uniform sphere of mass M and radius $\mathcal{R}$ spinning with angular frequency Ω around an axis $\hat{\mathbf{z}}$. The body has a time-dependent magnetic dipole moment $\mathbf{p}$ that is inclined at a constant angle α to the z-axis and rotates according to

$$\mathbf{p}(t) = p_0 \left[\sin\alpha\ \cos(\Omega t)\,\hat{\mathbf{x}} + \sin\alpha\ \sin(\Omega t)\,\hat{\mathbf{y}} + \cos\alpha\ \hat{\mathbf{z}}\right] \ ,$$

where p_0 is a real constant. Calculate the total power $\mathcal{P}$ emitted by the pulsar in the dipole approximation. Assuming that the angular frequency of rotation $\Omega(t)$ slowly varies with time so that energy is conserved, calculate the time taken for this system to lose half its initial rotational energy E_0 due to emission of radiation. [You may assume that the rotational energy of a solid uniform sphere of radius $\mathcal{R}$ and mass M is $E = I\Omega^2/2$ with moment of inertia $I = 2M\mathcal{R}^2/5$.]

Paper 4, Section II

36B Electrodynamics

(a) State Maxwell's equations for the fields $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D} = \epsilon\,\mathbf{E}$ and $\mathbf{B} = \mu\,\mathbf{H}$ in a linear dielectric medium with electric and magnetic polarisation constants ϵ and μ. You may assume the absence of free charges and currents. Show explicitly that Maxwell's equations admit plane-wave solutions propagating with speed $v = 1/\sqrt{\epsilon\mu}$ and determine the magnetic polarisation vector $\mathbf{B}_0$ in terms of the corresponding electric polarisation vector $\mathbf{E}_0$ and the wave vector $\mathbf{k}$.

(b) Consider two such media, having distinct values $\epsilon_+ > \epsilon_-$ of the electric polarisation constant, filling the regions $x > 0$ and $x < 0$ respectively. The two media are assumed to share a common value of μ. Write down, with brief justification, boundary conditions for the components of the fields tangent and normal to the interface plane $x = 0$. You should state clearly which field components are continuous and which are discontinuous.

(c) Suppose an electromagnetic wave is incident from the region $x < 0$ resulting in a transmitted wave in the region $x > 0$ and also a reflected wave for $x < 0$. The angles of incidence, reflection and transmission are denoted θ_I, θ_R and θ_T respectively. By constructing a corresponding solution of Maxwell's equations, derive the law of reflection $\theta_I = \theta_R$ and Snell's law of refraction, $n_- \sin\theta_I = n_+ \sin\theta_T$, where $n_\pm$ are the indices of refraction of the two media in question.

(d) The incident, reflected and transmitted waves have polarisation vectors $\mathbf{E}_I$, $\mathbf{E}_R$ and $\mathbf{E}_T$ respectively. In the case that these vectors are all *normal* to the plane of incidence (the plane spanned by the incident and reflected wave vectors), determine the ratio $|\mathbf{E}_R|/|\mathbf{E}_I|$ as a function of θ_I and show that it is always non-zero for $0 \leqslant \theta_I \leqslant \pi/2$.

Paper 1, Section II

39D Fluid Dynamics

(a) State the principle of reversibility for Stokes flow.

(b) Consider a rigid cylinder falling downwards at zero Reynolds number near to a rigid, vertical wall. The cylinder's axis is horizontal and parallel to the wall (that is, it is perpendicular to both the direction in which the cylinder falls and the normal of the wall). Use part (a) to argue that the cylinder cannot migrate towards or away from the wall as it falls, but it may rotate around its axis.

(c) Suppose the cylinder in part (b) has radius a, falls with speed V and rotates with angular speed Ω. Its minimum distance from the wall is $h_0 \ll a$.

(i) Use geometrical arguments to show that the horizontal gap $h(x)$ between the wall and the cylinder satisfies $h(x) \approx h_0 \left[1 + x^2/(2ah_0)\right]$ for $|x| \ll a$, where x is the vertical distance above the axis of the cylinder.

(ii) Use lubrication theory to determine the velocity and hence the vertical flux of fluid between the wall and the cylinder in terms of the vertical pressure gradient. Given that the pressure is equal to the uniform ambient pressure ahead of and behind the cylinder, determine the vertical flux in terms of V, Ω, a and h_0.

[*Hint: Use a frame of reference in which the cylinder has no vertical motion.*]

(iii) Given that the forces on the cylinder are dominated by those in the narrow gap between it and the wall, and that there is no torque applied to the cylinder, show that, in fact, the cylinder does not rotate.

$\Bigg[$*Hint: You may quote the following integrals:*

$$\int_{-\infty}^{\infty} \frac{dt}{1+t^2} = \pi, \quad \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^2} = \frac{\pi}{2}, \quad \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^3} = \frac{3\pi}{8}.\Bigg]$$

Paper 2, Section II

39D Fluid Dynamics

Write down the Stokes equations governing the flow of incompressible viscous fluid at zero Reynolds number, and show that the pressure and vorticity are harmonic.

A rigid sphere of radius a moves with velocity $\mathbf{U}$ through fluid of dynamic viscosity μ that is stationary far from the sphere. Write down the boundary conditions that should be applied to the normal and tangential components of the fluid velocity $\mathbf{u}$ on the surface of the sphere, explaining each in physical terms.

The velocity and pressure fields at a point $\mathbf{x}$ in the fluid can be written as

$$\mathbf{u} = \left(\frac{3}{4}\frac{a}{r} + \frac{1}{4}\frac{a^3}{r^3}\right)\mathbf{U} + \left(\frac{3}{4}\frac{a}{r^3} - \frac{3}{4}\frac{a^3}{r^5}\right)(\mathbf{U}\cdot\mathbf{x})\mathbf{x},$$

$$p = \frac{3}{2}\mu a\frac{\mathbf{U}\cdot\mathbf{x}}{r^3},$$

where the origin lies at the centre of the sphere and $r = |\mathbf{x}|$.

Using suffix notation, or otherwise, calculate the velocity gradient $(\boldsymbol{\nabla}\mathbf{u})_{ij} = \partial u_j/\partial x_i$. Hence:

(i) determine an expression for the vorticity;

(ii) calculate $\boldsymbol{\nabla}(1/r)$ and $\nabla^2(1/r)$, and use your answers to argue directly that the pressure and vorticity are harmonic;

(iii) prove that the flow is incompressible;

(iv) determine the stress $(\boldsymbol{\sigma}\cdot\mathbf{n})_i = \sigma_{ij}n_j$ on the surface of the sphere, where $\mathbf{n}$ is the outward unit normal to the sphere;

(v) determine the force exerted by the fluid on the sphere.

Paper 3, Section II

38D Fluid Dynamics

A steady, two-dimensional, laminar plume (narrow, quasi-vertical flow) rising from a point source of buoyancy in an otherwise stationary environment can be modelled using the boundary-layer equations

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}+b(x)\delta(y),$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$

where u and v are vertical and horizontal velocity components, respectively, with respect to Cartesian coordinates x vertical and y horizontal, and where δ is the Dirac delta function, so that $b(x)$ represents a buoyancy force confined to the vertical axis $y=0$. The symbols ρ, μ and p represent the fluid's density, dynamic viscosity and pressure respectively.

(a) Show that

$$\frac{d}{dx}\int_{-\infty}^{\infty}\rho u^2\,dy=b(x).$$

(b) Given that $b(x)=Bx^{-1/5}$, where B is constant, show that the width of the plume Δ and the vertical velocity u scale as

$$\Delta\sim\left(\frac{\mu^2}{\rho B}\right)^{1/3}x^{2/5},\qquad u\sim\left(\frac{B^2}{\rho\mu}\right)^{1/3}x^{1/5}.$$

(c) Introduce a stream function $\psi(x,y)$ and consider a similarity solution $\psi(x,y)=u\Delta f(\eta)$, where f only depends on the similarity variable $\eta=y/\Delta$. Show that $f(\eta)$ satisfies an ordinary differential equation of the form

$$f'''+c_1(f')^2+c_2ff''=\delta(\eta),$$

where primes denote differentiation with respect to η, for some constants c_1 and c_2 that you should determine.

[*Hint: For any constant a,* $\delta(ay)=|a|^{-1}\,\delta(y)$.]

Paper 4, Section II

38D Fluid Dynamics

(a) Consider the incompressible flow of a Newtonian fluid with constant dynamic viscosity μ and density ρ subject to a conservative body force $\mathbf{f} = -\boldsymbol{\nabla}\psi$. Derive the equation for the rate of change of kinetic energy of the fluid in a domain $\mathcal{D}$ with boundary $\partial\mathcal{D}$ in the form

$$\frac{d}{dt}\int_{\mathcal{D}} \tfrac{1}{2}\rho|\mathbf{u}|^2\,dV + \int_{\partial\mathcal{D}} \tfrac{1}{2}\rho|\mathbf{u}|^2\mathbf{u}\cdot\mathbf{n}\,dS = \int_{\mathcal{D}} \mathbf{u}\cdot\mathbf{f}\,dV + \int_{\partial\mathcal{D}} \mathbf{u}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}\,dS - 2\mu\int_{\mathcal{D}} \mathbf{e}:\mathbf{e}\,dV,$$

where $\mathbf{n}$ is the unit normal vector directed out of the boundary $\partial\mathcal{D}$, $\boldsymbol{\sigma}$ is the stress tensor and $\mathbf{e}$ is the rate-of-strain tensor. Give the physical interpretation of each term in this integral equation.

(b) Small-amplitude, free-surface waves on deep water occupying $-\infty < z < \eta(x,t) = A\exp(ikx - i\omega t)$ can be described by a velocity potential

$$\phi = B\exp(kz)\exp(ikx - i\omega t),$$

where A, B and k are constants, g is the gravitational acceleration, $\omega^2 = gk$ and real parts of complex quantities may be assumed. Determine B in terms of A, ω and k.

Assume now that the amplitude A is slowly varying, so A can be treated as constant over one period of oscillation. Determine the mean rate of dissipation averaged over a period of oscillation. Given that the total mean energy is $\frac{1}{2}\rho g|A|^2$, determine the slow rate of decay of the wave amplitude.

[*Hint: The mean over a period of the product of the real part of periodic complex functions F and G is the real part of $\frac{1}{2}FG^*$.*]

Paper 1, Section I

7E Further Complex Methods

Consider the improper integral

$$\int_{-\infty}^{\infty} \frac{1}{x^6 - 1}\,dx\,.$$

Explain what is meant by the *Cauchy principal value* of this integral, and evaluate it.

Paper 2, Section I

7E Further Complex Methods

(a) The function $F(z)$ is defined for all $z \in \mathbb{C} \setminus \{\pm i\}$ by

$$F(z) = \int_0^z \frac{1}{1+t^2}\,dt\,, \qquad (\dagger)$$

where the path taken for the integral is unrestricted except that it does not pass through either of the points $\pm i$. Show that the function $F(z)$ is multivalued. What are the possible values of $F(1)$?

(b) A curve $\mathcal{B}$ joins the points $\pm i$ along the imaginary axis, slightly displaced to the left of 0. Consider the function $F_{\mathcal{B}}(z)$ defined for $z \in \mathbb{C} \setminus \mathcal{B}$ by the integral ($\dagger$), but with the restriction that the path of integration does not cross $\mathcal{B}$. Show that $F_{\mathcal{B}}(z)$ is a single-valued function.

(c) Show that, for large $|z|$, $F_{\mathcal{B}}(z) = \frac{1}{2}\pi + O(|z|^{-1})$. Hence, calculate the integral

$$\lim_{R\to\infty} \oint_{\gamma_R} \frac{F_{\mathcal{B}}(t)}{t}\,dt\,,$$

where the contour γ_R is an anti-clockwise circle of radius R.

Paper 3, Section I

7E Further Complex Methods

The beta function is defined by

$$B(p,q) = \int_0^1 (1-t)^{p-1}\,t^{q-1}\,dt,$$

for $\mathrm{Re}\,(p) > 0$ and $\mathrm{Re}\,(q) > 0$.

(a) By writing $\Gamma(z)^2$ as a double integral, show that for $\mathrm{Re}\,(z) > 0$,

$$\Gamma(z)^2 = B(z,z)\Gamma(2z)\,,$$

where $\Gamma(z)$ denotes the gamma function. [*Hint: You may find the transformation* $(s,t) \to (r,u)$, *given by* $t = ru$, $s = r(1-u)$, *helpful.*]

(b) Deduce that $B(z,z) \sim 2/z$ as $z \to 0$ with $\mathrm{Re}\,(z) > 0$.

Paper 4, Section I

7E Further Complex Methods

The modified Bessel function $I_0(z)$, for $z \in \mathbb{C}$, is the unique solution of the differential equation

$$z\frac{d^2y}{dz^2} + \frac{dy}{dz} - zy = 0, \qquad (\dagger)$$

satisfying $y(0) = 1$.

Explain Laplace's method of seeking a solution $y(z)$ to equation ($\dagger$) of the form

$$y(z) = \int_C e^{zt} f(t)\, dt\,,$$

where the function $f(t)$ and the contour C are suitably chosen. Apply the method to show that

$$I_0(z) = \frac{1}{\pi}\int_{-1}^{1} \frac{e^{zs}}{(1-s^2)^{1/2}}\, ds.$$

Paper 1, Section II

14E Further Complex Methods

The Dirichlet beta function is defined as

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} \qquad (\dagger)$$

for $\operatorname{Re}(s) > 0$ and by analytic continuation to $\mathbb{C}$. The integral representation of equation ($\dagger$) for $\operatorname{Re}(s) > 0$ is given by

$$\beta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t + e^{-t}}\, dt\,,$$

where Γ is the gamma function.

(a) The Hankel representation is defined as

$$\beta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{-\infty}^{(0^+)} \frac{t^{s-1}}{e^t + e^{-t}}\, dt. \qquad (\ddagger)$$

Draw a diagram to show the integration contour implied by the limits of the integral in equation ($\ddagger$). Show that this representation gives an analytic continuation of $\beta(s)$ as defined by equation ($\dagger$) to all $s \in \mathbb{C}$.

[You may assume that $\Gamma(s)\Gamma(1-s) = \pi \operatorname{cosec}(\pi s)$.]

(b) Use equation ($\ddagger$) to evaluate $\beta(0)$ and $\beta(-2)$. Show that if n is a non-negative integer then $\beta(-2n-1) = 0$.

(c) Consider the poles of the integrand of equation ($\ddagger$) on the imaginary axis, except for the pole at $t = 0$, if it exists. For what conditions on s does the sum of the residues at these poles converge? Assume that under these conditions it may be shown that the integral in equation ($\ddagger$) is equal to the sum of the residues multiplied by $-2\pi i$. Deduce the reflection formula

$$\beta(1-s) = \Gamma(s) \left(\frac{\pi}{2}\right)^{-s} \sin\left(\frac{s\pi}{2}\right) \beta(s),$$

explaining carefully why this formula is valid for all $s \in \mathbb{C}$.

Paper 2, Section II

13E Further Complex Methods

(a) Show that under the change of variable $z = \cos x$ the equation

$$\frac{d^2 w}{dx^2} + n^2 w = 0$$

becomes

$$(1-z^2)\frac{d^2 w}{dz^2} - z\frac{dw}{dz} + n^2 w = 0. \qquad (\dagger)$$

(b) Show that equation (†) is a Papperitz equation corresponding to the Papperitz-symbol or P-symbol

$$P\begin{Bmatrix} 1 & -1 & \infty & \\ 0 & 0 & -n & z \\ \frac{1}{2} & \frac{1}{2} & n & \end{Bmatrix},$$

explaining carefully the meaning of the symbol and the different elements appearing in it.

(c) Recall that the notation $F\big(A, B; C; \zeta\big)$ is used to denote the solution of the equation corresponding to the P-symbol

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & A & \zeta \\ 1-C & C-A-B & B & \end{Bmatrix},$$

for which $F\big(A, B; C; 0\big) = 1$.

Show that two linearly independent solutions of equation (†) are

$$w_1(z) = F\left(n, -n; \tfrac{1}{2}; \tfrac{1}{2}(1-z)\right)$$

and

$$w_2(z) = (1-z)^{1/2} F\left(-n+\tfrac{1}{2}, n+\tfrac{1}{2}; \tfrac{3}{2}; \tfrac{1}{2}(1-z)\right),$$

explaining clearly any results on transforming Papperitz equations and P-symbols that you use.

(d) Deduce that $F\left(-\tfrac{1}{2}, \tfrac{3}{2}; \tfrac{3}{2}; u\right) = (1-u)^{1/2}$, clearly justifying your reasoning.

Paper 1, Section II

18J Galois Theory

Let K be a field containing a primitive cube root of unity ω. Consider the cubic polynomial $f(X) = X^3 + aX + b \in K[X]$ with roots $\alpha_1, \alpha_2, \alpha_3$ in a splitting field. Let $g(X) = (X - u^3)(X - v^3)$ where $u = \alpha_1 + \omega\alpha_2 + \omega^2\alpha_3$ and $v = \alpha_1 + \omega^2\alpha_2 + \omega\alpha_3$.

(a) Define the *discriminant* of a monic polynomial, and show that

$$\mathrm{Disc}(g) = -27\,\mathrm{Disc}(f).$$

Write uv and $u^3 + v^3$ as polynomials in a and b. Hence, or otherwise, compute a formula for $\mathrm{Disc}(f)$ in terms of a and b.

(b) Show that there is a formula in terms of radicals for the roots of a cubic polynomial.

(c) Compute the Galois groups of the following polynomials, stating carefully any results from the course that you use:

$$X^3 - 21X - 22, \qquad X^3 - 21X - 28, \qquad X^3 - 21X - 34.$$

Paper 2, Section II

18J Galois Theory

Let L/K be a finite extension of fields. Define what it means for L/K to be *normal*, *separable*, or *Galois*. Let $\overline{K}$ be an algebraic closure of K.

(a) Write $L = K(\alpha_1, \ldots, \alpha_n)$ for some $\alpha_1, \ldots, \alpha_n \in L$. Show by induction on n that $1 \leqslant \#\,\mathrm{Hom}_K(L, \overline{K}) \leqslant [L : K]$ and that the upper bound is an equality if L/K is separable.

(b) Show that $\#\,\mathrm{Aut}(L/K) \leqslant \#\,\mathrm{Hom}_K(L, \overline{K})$ with equality if L/K is normal.

(c) Deduce that if L/K is normal and separable then L/K is Galois.

(d) Find a prime number p such that the extension $\mathbb{F}_p(X)/\mathbb{F}_p(X^5)$ is Galois. [You may assume that all splitting fields are normal.]

Paper 3, Section II

18J Galois Theory

(a) List the transitive subgroups of S_4.

(b) Let L/K be an extension of fields of characteristic not equal to 2. Suppose that $L = K(\sqrt{a}, \sqrt{b})$ for some $a, b \in K^*$ with $a, b, ab \notin (K^*)^2$. Show that L/K is Galois of degree 4, compute its Galois group, and draw the lattice of intermediate fields.

(c) Compute the minimal polynomials of $\alpha = \sqrt{3+\sqrt{3}}$, $\beta = \sqrt{3-\sqrt{3}}$, $\gamma = \sqrt{3+\sqrt{6}}$, and $\delta = \sqrt{3-\sqrt{6}}$ over $\mathbb{Q}$. Show that the hypotheses of part (b) are satisfied by $\mathbb{Q}(\alpha,\beta)/\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\gamma,\delta)/\mathbb{Q}(\sqrt{6})$.

(d) Deduce that $\mathrm{Gal}(\mathbb{Q}(\alpha,\beta)/\mathbb{Q}) \cong D_8$. Draw the lattice of subgroups of D_8, and the lattice of subfields of $\mathbb{Q}(\alpha,\beta)$, writing each field in the form $\mathbb{Q}(x_1,\ldots,x_m)$.

[*Hint: You may use that* $\alpha+\beta = \sqrt{2}\gamma$ *and* $\alpha-\beta = \sqrt{2}\delta$.]

Paper 4, Section II

18J Galois Theory

(a) Define the *nth cyclotomic polynomial* $\Phi_n(X)$. Show that it has coefficients in $\mathbb{Z}$. Show further that if K is a subfield of $\mathbb{C}$ and $\zeta_n \in \mathbb{C}$ is a root of $\Phi_n(X)$ then the extension $K(\zeta_n)/K$ is Galois with abelian Galois group.

(b) What does it mean to say that a subfield $K \subset \mathbb{R}$ is *constructible*? Show that $\mathbb{Q}(\cos(2\pi/17))$ is constructible.

(c) Let K be a field with algebraic closure $\overline{K}$. For each $n \geqslant 1$ let $\zeta_n \in \overline{K}$ be a root of $\Phi_n(X)$. Let $K^{\mathrm{cyc}} = \bigcup_{n\geqslant 1} K(\zeta_n)$. Decide whether $K^{\mathrm{cyc}} = \overline{K}$ in each of the cases $K = \mathbb{R}$, $K = \mathbb{Q}$, and $K = \mathbb{F}_p$. Justify your answers.

Paper 1, Section II
38E General Relativity

The metric for a spherically symmetric static spacetime has line element

$$ds^2 = -\left(1+\frac{r^2}{a^2}\right)dt^2 + \left(1+\frac{r^2}{a^2}\right)^{-1}dr^2 + r^2\left(d\theta^2+\sin^2\theta\, d\phi^2\right),$$

where $-\infty \leqslant t \leqslant \infty$, $r \geqslant 0$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$, a is a positive constant, and units are chosen with $c = 1$.

(a) Consider a time-like geodesic parametrised by proper time τ, with dots denoting differentiation with respect to τ. Find the Euler-Lagrange equation corresponding to the θ coordinate and explain why the geodesic may be assumed to lie in the equatorial plane $\theta = \pi/2$, without loss of generality. For such a geodesic, show that

$$\frac{1}{2}\dot{r}^2 + V(r) = \frac{1}{2}\left(E^2 - 1 - \frac{h^2}{a^2}\right),$$

where $E = \left(1 + r^2/a^2\right)\dot{t}$ and $h = r^2\dot{\phi}$ are constants of the motion and $V(r)$ is a function you should determine.

(b) Show that a massive particle fired from the origin, $r = 0$, attains a maximum value of the radial coordinate, $r = r_{\max}$, before returning to $r = 0$, and find the proper time this journey (from $r = 0$ to $r_{\max}$ and back) takes.

(c) Show that circular orbits with $r = r_0$ are possible for any $r_0 > 0$ and determine whether such orbits are stable. Show further that, on such an orbit, a clock measures coordinate time.

Paper 2, Section II
38E General Relativity

(a) Consider a spacetime with metric $g_{\mu\nu}$. Write down the covariant derivative $\nabla_\alpha g_{\mu\nu}$ in terms of the connection $\Gamma_\alpha{}^\gamma{}_\beta$, which is assumed to satisfy $\Gamma_\alpha{}^\gamma{}_\beta = \Gamma_\beta{}^\gamma{}_\alpha$. Determine the unique choice for the connection that ensures $\nabla_\alpha g_{\mu\nu} = 0$. In the remainder of this question we use this connection.

(b) If $R^\mu{}_{\nu\alpha\beta}$ is the Riemann curvature tensor, then

$$\nabla_\alpha \nabla_\beta U_\mu \, - \, \nabla_\beta \nabla_\alpha U_\mu \; = \; -R^\nu{}_{\mu\alpha\beta} U_\nu \qquad (*)$$

for any covariant vector field U_μ. By setting $U_\mu = \partial_\mu \phi$ in equation $(*)$, where ϕ is a scalar field, show that

$$R^\mu{}_{\alpha\beta\gamma} + R^\mu{}_{\beta\gamma\alpha} + R^\mu{}_{\gamma\alpha\beta} \; = \; 0\,.$$

State clearly any property of a vector field of the form $\partial_\mu \phi$ that your argument depends on.

(c) From equation $(*)$, derive an analogous expression for $\nabla_\alpha \nabla_\beta W_{\mu\nu} - \nabla_\beta \nabla_\alpha W_{\mu\nu}$, where $W_{\mu\nu}$ is a general covariant tensor field of rank 2, stating clearly any assumptions you make. By making a suitable choice for $W_{\mu\nu}$, deduce that

$$R_{\mu\alpha\beta\gamma} = -R_{\alpha\mu\beta\gamma}\,.$$

(d) Define the Ricci tensor $R_{\alpha\beta}$ and show that it is symmetric. For certain spacetimes,

$$R_{\mu\alpha\beta\gamma} = K(\, g_{\mu\beta} g_{\alpha\gamma} - g_{\mu\gamma} g_{\alpha\beta}\,)\,,$$

where K is a scalar. Compute the Ricci tensor and Ricci scalar and deduce that K is constant, assuming that the dimension n of the spacetime is four. How would this conclusion change for other values $n > 1$? [Identities involving covariant derivatives of the Ricci tensor may be used without proof but should be clearly stated.]

Paper 3, Section II
37E General Relativity

The Schwarzschild metric, in units with $G = c = 1$, is given by

$$ds^2 \;=\; -\left(1-\frac{2M}{r}\right)dt^2 \;+\; \left(1-\frac{2M}{r}\right)^{-1}dr^2 \;+\; r^2(\,d\theta^2+\sin^2\theta\,d\phi^2\,)\,. \qquad (*)$$

(a) Show that for a light ray that is radial ($d\theta = d\phi = 0$) and ingoing ($dr/dt < 0$), for $r > 2M$ the quantity

$$v \;=\; t \,+\, r \,+\, 2M\log\left|\,\frac{r}{2M}-1\,\right|$$

is constant.

(b) Express the Schwarzschild metric $(*)$ in terms of coordinates r, v, θ, ϕ, with v defined as above for $r > 0$. What can be deduced about the nature of the metric at $r = 2M$?

(c) Determine all possible radial trajectories for light rays for $r > 0$. For these solutions, find dt_*/dr as a function of r, where $t_* = v - r$, and hence sketch the solutions in the r-t_* plane.

(d) Comment on the contrasting behaviour of light rays in the regions $r > 2M$ and $r < 2M$ and, by considering light cones at representative points, discuss the implications for the motion of massive particles.

(e) An astronaut Alice (A) sends radial light signals at proper time intervals $\Delta\tau_A$ to an observer Bob (B) who receives them at proper time intervals $\Delta\tau_B$. Alice and Bob are at rest in the coordinate system t, r, θ, ϕ with $r_A = 2M + \varepsilon$, where $0 < \varepsilon \ll M$, and $r_B \gg M$. Find an approximate expression for $\Delta\tau_B/\Delta\tau_A$ and comment on the significance of your result.

Paper 4, Section II
37E General Relativity

The metric $g_{\alpha\beta}$ for a four-dimensional spacetime satisfies the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \,=\, \kappa T_{\mu\nu}\,,$$

where $R_{\mu\nu}$ and R are the Ricci tensor and Ricci scalar, $T_{\mu\nu}$ is the energy-momentum tensor, and κ is a constant. We use units where $c = 1$ and assume throughout that $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ where $\eta_{\alpha\beta} = \mathrm{diag}(-1,\,1,\,1,\,1)$ is the Minkowski metric in Cartesian coordinates and $h_{\alpha\beta}$ and its derivatives are small. Then the Riemann tensor is given by

$$R_{\mu\nu\alpha\beta} \,=\, \frac{1}{2}\left(h_{\mu\beta,\,\nu\alpha} \,-\, h_{\nu\beta,\,\mu\alpha} \,-\, h_{\mu\alpha,\,\nu\beta} \,+\, h_{\nu\alpha,\,\mu\beta} \right),$$

to first order in small quantities.

(a) Let $x^0 = t$ and x^i $(i = 1, 2, 3)$ denote the Cartesian coordinates. Assume both that (i) $T_{00} = \rho$ is the mass density, where $\kappa\rho$ is small, and all other components of the energy-momentum tensor are negligible, and (ii) the metric is *almost static*, meaning that derivatives of $h_{\alpha\beta}$ with respect to t are negligible. Find R_{00}, working to first order in all small quantities, and hence show that

$$-\nabla^2 h_{00} \,=\, \kappa\rho \qquad \text{where} \qquad \nabla^2 \,=\, \delta^{ij}\partial_i\partial_j\,. \qquad (\dagger)$$

(b) A massive particle moves non-relativistically in the spacetime of part (a), with $v^i = dx^i/dt$ small. Starting from the geodesic equations, show that

$$\frac{dv^i}{dt} \,=\, \frac{1}{2}\,\delta^{ij}\partial_j h_{00}\,, \qquad (\star)$$

working to first order in both v^i and $h_{\alpha\beta}$. [You may quote the formula for the Levi-Civita connection $\Gamma^{\,\mu}_{\alpha\,\beta}$.]

By comparing equations (†) and (⋆) with the corresponding Newtonian equations, express h_{00} and κ in terms of the Newtonian gravitational potential Φ and Newton's constant G, where $\nabla^2\Phi = 4\pi G\rho$.

(c) Consider a point mass M at the origin $r = 0$, where $r^2 = \delta_{ij}x^i x^j$, in an otherwise vacuum spacetime. Write down the Newtonian potential Φ for this point mass. Suppose that

$$h_{ij} \,=\, f(r)\, x_i x_j\,,$$

for some function $f(r)$, where indices $i, j = 1, 2, 3$ are raised and lowered using δ_{ij}. By considering the Ricci scalar, or otherwise, find a differential equation for $f(r)$ and obtain the general solution for $r > 0$.

[You may use, without proof, the identities

$$\partial_i\partial_i[\, r^2 f(r)\,] \,=\, r^2 f'' + 6rf' + 6f \quad \text{and} \quad \partial_i\partial_j[\, x_i x_j f(r)\,] \,=\, r^2 f'' + 8rf' + 12f\,,$$

where the summation convention applies to repeated indices of type $i, j = 1, 2, 3$.]

Paper 1, Section II

17F Graph Theory

(a) State Menger's theorem for a graph G. Define the *connectivity* $\kappa(G)$ of G. State and deduce the vertex form of Menger's theorem from Menger's theorem.

Let $k \geqslant 2$. Show that every k-connected graph of order at least $2k$ contains a cycle of length at least $2k$.

(b) Suppose G is a graph with $|G| > 1$. Define the *edge connectivity* $\lambda(G)$ of G. Let $\delta(G)$ be the minimum degree of G. Prove that

$$\delta(G) \geqslant \lambda(G) \geqslant \kappa(G).$$

Let d, ℓ and k be any three positive integers with $d \geqslant \ell \geqslant k$. Show that there exists a graph G with $\delta(G) = d$, $\lambda(G) = \ell$ and $\kappa(G) = k$.

Paper 2, Section II

17F Graph Theory

In this question, no form of Menger's theorem or of the max-flow min-cut theorem may be assumed without proof.

(a) Let G be a bipartite graph with vertex classes X and Y. What is a *matching* from X to Y? State and prove Hall's marriage theorem, giving a necessary and sufficient condition for G to contain a matching from X to Y.

We define the *matching number* of a graph G to be the maximum size of a set of independent edges in G.

(i) If G is a k-regular bipartite graph with $|G| = n$ (for some $k > 0$), show that G has matching number $n/2$.

(ii) If G is an arbitrary k-regular graph with $|G| = n$ (for some $k > 0$), show that G has a matching number at least $(\frac{k}{4k-2})n$.

(iii) For $k = 2$, write down an infinite family of graphs G for which equality holds in (ii).

(b) Define the *eigenvalues* of a graph G. Let G be bipartite with vertex classes X and Y. If 0 is not an eigenvalue of G show that G contains a matching from X to Y.

Paper 3, Section II

17F Graph Theory

(a) What does it mean for a graph G to be *Eulerian*? If $|G| \geqslant 3$, state and prove a necessary and sufficient condition for G to be Eulerian.

Define the *line graph* $L(G)$ of G. Show that $L(G)$ is Eulerian if G is regular and connected.

(b) Let G be a connected planar graph with n vertices, e edges and f faces. Prove that $n - e + f = 2$.

The size of a face is the number of edges that form its boundary. Deduce that $e \leqslant g(n-2)/(g-2)$, where g is the smallest size of a face.

(c) Let G be a (not necessarily planar) graph with n vertices and e edges. Suppose that G is drawn in the plane, but with edges allowed to cross. (The edge xy cannot contain any vertex except x or y.) Let $t(G)$ be the number of pairs of edges which cross.

(i) Show that $t(G) \geqslant e - 3n + 6$.

(ii) Suppose now that $e \geqslant 4n$. Show that $t(G) \geqslant e^3/64n^2$. [*Hint: you may wish to consider a random subset of* $V(G)$ *containing each vertex of* G *independently with probability* $4n/e$.]

Paper 4, Section II

17F Graph Theory

(a) For $t \in \mathbb{N}$ define the *Ramsey number* $R(t)$. If $t \geqslant 2$, show that $R(t)$ exists and that $R(t) \leqslant 2^{2t}$.

(b) For any graph G, define $R(G)$ to be the least positive integer n such that in any red–blue colouring of the edges of the complete graph K_n, there must be a monochromatic copy of G. Explain briefly why $R(G)$ exists.

(i) Let $t \in \mathbb{N}$. Show that, whenever the edges of K_{2t} are red–blue coloured, there must be a monochromatic copy of the complete bipartite graph $K_{1,t}$.

(ii) Suppose that t is odd. Show that $R(K_{1,t}) = 2t$. If t is even, what is $R(K_{1,t})$? Justify your answer.

(iii) Let H be the graph on four vertices, obtained by adding an edge to a triangle. Compute $R(H)$, justifying your answer.

Paper 1, Section II
33D Integrable Systems

(a) Let U, V and Φ be $n \times n$ matrices that depend on x and y and satisfy

$$\partial_x \Phi + U\Phi = 0, \quad \partial_y \Phi + V\Phi = 0.$$

Find a compatibility condition for this system of linear partial differential equations that involves only U and V.

(b) Let $n = 3$ and take

$$U = \begin{pmatrix} \partial_x u & 0 & \lambda \\ 1 & -\partial_x u & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & e^{-2u} & 0 \\ 0 & 0 & e^u \\ \lambda^{-1} e^u & 0 & 0 \end{pmatrix},$$

where λ is a constant parameter and $u = u(x, y)$. Show that in this case the compatibility conditions hold if u satisfies a partial differential equation of the form

$$\partial_x \partial_y u = F(u), \tag{$\dagger$}$$

for some function $F(u)$ which should be determined.

(c) Find a one-parameter group of transformations G_α generated by the vector field $x\partial_x - \alpha y \partial_y$, where α is a constant, and determine the value of α for which G_α is a symmetry group of the partial differential equation (†).

(d) For this value of α, find an ordinary differential equation characterising solutions to equation (†) that are invariant under G_α.

Paper 2, Section II
34D Integrable Systems

(a) Define a *completely integrable system* on a $2n$-dimensional phase space M, and state the Arnold–Liouville theorem.

(b) Consider $M = \mathbb{R}^{2n}$ with coordinates $(p_i, q_i), i = 1, \ldots, n$, and the standard Poisson structure. Let

$$H = \frac{1}{2}\Big({p_1}^2 + \cdots + {p_n}^2 + {W_1}^2 {q_1}^2 + \cdots + {W_n}^2 {q_n}^2 + a_1 q_1 + \cdots + a_n q_n \Big),$$

where $W_1, \ldots, W_n, a_1, \ldots, a_n$ are constants with $W_k \neq 0$ for all k.

(i) Find n independent functions $F_1, F_2, \ldots, F_n$ in involution with $\sum_{i=1}^n F_i = H$, and demonstrate that Hamilton's equations with Hamiltonian H are completely integrable.

(ii) Find the action variables.

Paper 3, Section II

32D Integrable Systems

(a) Use the Gelfand–Levitan–Marchenko equation

$$K(x,y)+F(x+y)+\int_x^\infty K(x,z)F(z+y)dz=0,$$

with $F(x)=\beta_0\exp\left(8\chi^3 t-\chi x\right)$ to find the one-soliton solution

$$u(x,t)=-\frac{2\chi^2}{\cosh^2\left[\chi(x-4\chi^2 t-\phi)\right]}$$

to the KdV equation

$$u_t-6uu_x+u_{xxx}=0,$$

where $u=u(x,t)$. Here β_0 and χ are constants, and ϕ is another constant that you should determine.

[*You may use any facts about the inverse scattering transform without proof.*]

(b) By considering the operators $A^\dagger A$ and $AA^\dagger$ where $A=\partial_x+\chi\tanh\left(\chi x\right)$, show that the Schrödinger operator $-\partial_x^2+U$ with a potential

$$U(x)=u(x,t=0,\phi=0)$$

admits only one bound state. Find the corresponding energy level.

Paper 1, Section II

22I Linear Analysis

(a) Let X be a Banach space.

(i) Define the *dual space* X^* of X, and show that it is a Banach space.

(ii) Find, with proof, the dual of l_p for each $1 < p < \infty$.

(iii) Describe, without proof, the duals of l_1 and c_0.

(b) Let c denote the space of convergent real sequences, with the supremum norm. Is c isomorphic to c_0? Justify your answer.

Paper 2, Section II

22I Linear Analysis

(a) State the inversion theorem. State and prove the closed graph theorem.

(b) Now let X and Y be Banach spaces and let $S \in L(X, Y)$ be injective. We write S^{-1} for the inverse map to S, so that S^{-1} is a linear map from the image of S to X.

(i) Give an example to show that S^{-1} need not be continuous.

(ii) If $T \in L(X, Y)$ has the property that the image of T is contained in the image of S, show that $S^{-1} \circ T$ is continuous.

(iii) Give a counterexample to show that (ii) need not remain true if we drop the assumption that X is complete.

Paper 3, Section II

21I Linear Analysis

(a) State and prove the Ascoli-Arzelà theorem.

(b) Consider a sequence of differentiable functions $f_n : \mathbb{R} \to \mathbb{R}$ with

$$\sup_{n \geqslant 0} \sup_{x \in \mathbb{R}} (|f_n(x)| + |f_n'(x)|) < +\infty.$$

Show that there exist a subsequence $f_{\phi(n)}$ (where $\phi : \mathbb{N} \to \mathbb{N}$ is strictly increasing) and a continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\forall R > 0, \quad \lim_{n \to \infty} \sup_{|x| \leqslant R} |f_{\phi(n)}(x) - f(x)| = 0.$$

Can we conclude that $\lim_{n \to \infty} \sup_{x \in \mathbb{R}} |f_{\phi(n)}(x) - f(x)| = 0$? Justify your answer.

Paper 4, Section II

22I Linear Analysis

(a) State the Riesz representation theorem. For a Hilbert space H and an operator $T \in L(H)$, define the *adjoint* T^* of T, proving that it exists and that $T^* \in L(H)$. If H has an orthonormal basis, describe (without proof) the relation between the matrices of T and T^* with respect to this basis.

(b) State the spectral theorem for compact Hermitian operators on l_2. Explain why it follows from this that every compact Hermitian operator on l_2 is a limit of finite rank Hermitian operators.

(c) Prove that every compact operator on l_2 is a limit of finite rank operators.

Paper 1, Section II

16H Logic and Set Theory

In this question, an ordinal is a transitive set well-ordered by $\in$.

(a) Explain briefly why for every well-ordered set (a, r) there is a unique ordinal isomorphic to (a, r), the *order-type of* (a, r).

(b) Let $\alpha < \beta$ be ordinals, and let γ be the order-type of the interval

$$\beta \backslash \alpha = \{\delta \in \mathrm{ON} : \alpha \leqslant \delta < \beta\}.$$

Explain briefly why $\alpha + \gamma = \beta$.

(c) What is the order-type of the interval $\omega_1 \backslash \omega$? Justify your answer.

(d) Let α be a non-zero ordinal. Show that $\alpha = \omega^\delta \cdot n + \eta$ for ordinals δ, n and η such that $1 \leqslant n < \omega$ and $\eta < \omega^\delta$.

(e) Let δ be an ordinal, and assume that $\omega^\delta = X \cup Y$. Show that at least one of X and Y has order-type ω^δ.

(f) Let $\alpha = X \cup Y$ be a non-zero ordinal with both X and Y having order-type β. Using (d) and (e) or otherwise, show that $\alpha < \beta + \beta + \beta$. Is it always true that $\alpha < \beta + \beta$? Give a proof or counterexample.

Paper 2, Section II

16H Logic and Set Theory

Let L be a first-order language.

If M is an L-structure, we say that $\vartheta\colon M \to M$ is an automorphism of M if it is a bijection, and for all $m_1, \ldots, m_n \in M$ and all operation symbols ω of L (with arity n)

$$\vartheta(\omega_M(m_1, \ldots, m_n)) = \omega_M(\vartheta(m_1), \ldots, \vartheta(m_n)),$$

and for every predicate symbol φ of L (with arity n)

$$\big\{(\vartheta(m_1), \ldots, \vartheta(m_n)) : (m_1, \ldots, m_n) \in \varphi_M\big\} = \varphi_M.$$

We write $\mathrm{Aut}(M)$ for the set of automorphism of M. The set $\mathrm{Aut}(M)$ forms a group under composition. [*You do not need to prove this.*]

(a) Define the following notions:

(i) φ is a *sentence* in L;

(ii) T is a *theory* in L;

(iii) M is a *model* of T;

(iv) T is *consistent*.

(b) By appealing to a suitable theorem from the lectures, show that T is consistent if and only if it has a model.

(c) State the compactness theorem of first-order predicate logic and prove it using part (b) or otherwise.

For the remainder of this question, fix a consistent theory T in L. Expand L to a new language L_f obtained from L by adding a unary operation symbol f. Note that any L_f-structure can be thought of as a pair (M, θ), where M is an L-structure and θ is the interpretation in M of the additional operation symbol f.

(d) Specify a consistent theory T_f in L_f such that $T_f \supset T$ and an L_f-structure (M, ϑ) is a model of T_f if and only if M is a model of T and $\vartheta \in \mathrm{Aut}(M)$. Justify your claim.

Let X be any set and let L_X be the expansion of L with the additional operation symbols $\{f_x : x \in X\}$. Fix any group G and consider the expansion L_G of L and form the following theory in L_G: use the theories T_{f_g} from (d) and let

$$T_G := \bigcup_{g \in G} T_{f_g} \cup \Big\{(\exists\, x)\neg(f_g x = f_h x) \,:\, g, h \in G,\, g \neq h\Big\} \\ \cup \Big\{(\forall\, x)(f_g f_h x = f_k x) \,:\, g, h, k \in G,\, gh = k\Big\}.$$

We call a group G T-*good* if there is a model M of T such that $\mathrm{Aut}(M)$ contains an isomorphic copy of G and T-*bad* if it is not T-good.

[QUESTION CONTINUES ON THE NEXT PAGE]

(e) Let G be a group. Show that the following are equivalent:

(i) G is T-bad;

(ii) T_G is inconsistent;

(iii) there is a finite $X \subseteq G$ such that $T_G \cap L_X$ is inconsistent.

(f) Show that there is a theory T^* in the language of groups such that for any group G, G is a model of T^* if and only if G is T-good.

Paper 3, Section II
16H Logic and Set Theory

Let x be a set. A choice function for subsets of x is a function $f\colon \mathbb{P}x \setminus \{\varnothing\} \to x$ such that $f(y) \in y$ for all non-empty subsets y of x.

(a) Let x be a set and f be a choice function for subsets of x. Use f and recursion to show that there is an ordinal α and a bijection between α and x. Use this to show that the axiom of choice implies the well-ordering principle.

(b) Show that the statement "for any two sets x and y, either there is an injection from x to y or an injection from y to x" implies the axiom of choice.

[You may use Hartogs's lemma without proof.]

(c) Define the notion of *initial ordinal* and define $\aleph_\alpha$. Show that an ordinal is an infinite initial ordinal if and only if it has cardinality $\aleph_\alpha$ for some ordinal α.

Working in ZFC, we write $\mathrm{card}(x)$ for the least ordinal α that is in bijection with x. Let I be a set and $\{\kappa_i : i \in I\}$ be initial ordinals. We define

$$\sum_{i\in I} \kappa_i := \mathrm{card}\left(\bigsqcup_{i\in I} \kappa_i\right) \text{ and } \prod_{i\in I} \kappa_i := \mathrm{card}\left(\prod_{i\in I} \kappa_i\right).$$

(d) Assume that $\{\kappa_i : i \in I\}$ and $\{\lambda_i : i \in I\}$ are initial ordinals such that for every $i \in I$, we have $\kappa_i < \lambda_i$. Show that $\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i$.

[*Hint: Construct an injection from the disjoint union to the product, and show that there is no such surjection.*]

(e) Using part (d) or otherwise, show that $\aleph_\omega < \aleph_\omega^{\aleph_0}$. Deduce that $2^{\aleph_0} \neq \aleph_\omega$.

[You may use standard properties of cardinal arithmetic without proof.]

Paper 4, Section II

16H Logic and Set Theory

In this question, let V be a model of ZF set theory. Set-theoretic notation such as $\varnothing$, $\{x, y\}$, $\bigcup x$, and "x is finite" refers to the operations and properties in V. Let φ be a formula with one free variable. The φ-instance of the axiom-scheme of separation is the formula

$$(\forall x)(\exists s)(\forall z)(z \in s \Leftrightarrow (z \in x \wedge \varphi(z))).$$

For a set x, the set $\{z \in x : \varphi\}$ exists by the validity of the φ-instance of the axiom-scheme of separation in V.

(a) Define what it means to be a *class* and a *proper class* in V.

A class M is called transitive if whenever $x \in y$ and y is in M, then x is in M. A class M is called φ-closed if for all x in M, the set $\{z \in x : \varphi\}$ is also in M.

(b) Show that the collection of all finite sets in V is a proper class. Is it transitive? Is it φ-closed? Justify your claims.

(c) A set is called hereditarily finite if it is contained in a transitive and finite set. Is the collection of all hereditarity finite sets in V a proper class? Is it transitive? Is it φ-closed? Justify your claims.

[You are allowed to use properties of the von Neumann hierarchy and its rank function as proved in the lectures, provided that you state them correctly and precisely.]

(d) Let M be a transitive class in V such that for all x, y in M, we have that $\varnothing$, $\{x, y\}$, and $\bigcup x$ are in M. Show that the axioms of extensionality, empty set, pair-set and union are satisfied in M.

(e) Let M be a transitive class. Explain briefly why being φ-closed is not sufficient to prove the φ-instance of the axiom-scheme of separation in M.

[You do not have to provide an example of a class M where this fails.]

(f) Provide a map (without proof) $\varphi \mapsto \varphi^*$, where φ^* is also a formula, such that every transitive class M that is φ^*-closed satisfies the φ-instance of the axiom-scheme of separation. Note that this map can depend on M.

Paper 1, Section I

6A Mathematical Biology

A large group of people, of fixed number, are debating a proposition. The group consists of those in favour of the proposition, $Y(t)$, those opposed, $N(t)$, and those who are undecided, $U(t)$. Debates can change people's minds. The outcome of a continuous debate taking place over time is modelled by the equations

$$\frac{dY}{dt} = -\beta YN \ , \quad \frac{dN}{dt} = (\beta - \alpha)YN + \zeta U \ , \quad \frac{dU}{dt} = \alpha YN - \zeta U \ .$$

The constants α, β and ζ are positive.

(a) Briefly give an interpretation of α, β, and ζ.

(b) Show that those in favour of the proposition and those opposed cannot exist in equilibrium. Determine the stability of any equilibria.

(c) Using part (b), determine if there are values of α, β and ζ for which everyone eventually favours the proposition, irrespective of the initial conditions.

Paper 2, Section I

6A Mathematical Biology

A population of healthy foxes, $S(\mathbf{x}, t)$, is territorial and tends not to move. In contrast, rabid infected foxes, with population $I(\mathbf{x}, t)$, change their behaviour and migrate. The dynamics of foxes is captured by the following, non-dimensionalised, equations:

$$\frac{dS}{dt} = -IS \ , \quad \frac{dI}{dt} = \nabla^2 I + IS - \mu I \ , \quad \frac{dR}{dt} = \mu I \ ,$$

where μ is a constant.

(a) Give a biological explanation for each term on the right-hand side of these equations. What is the meaning of the population $R(\mathbf{x}, t)$?

(b) Consider the spatially homogeneous case. Define the *reproductive ratio* for this system. Explain the reasoning behind the statement that rabies spreads among foxes only if the reproductive ratio is greater than 1.

(c) Suppose foxes move only in one spatial dimension. By writing $S(x, t) = S(\xi)$ and $I(x, t) = I(\xi)$, where $\xi = x - ct$ and $c > 0$, write down the equations for $S(\xi)$ and $I(\xi)$ that govern a travelling wave in this system.

(d) Consider the situation in which $S(\xi) \to 1$ and $I(\xi) \to 0$ as $\xi \to \infty$. By linearising about the leading edge of a wavefront, determine the minimum velocity at which a wave of infection spreads.

Paper 3, Section I

6A Mathematical Biology

A population $n(t)$ is modelled by the Malthusian delay differential equation

$$\frac{dn}{dt}(t) = r\, n(t-\tau)\,,$$

where r and $\tau > 0$ are constants.

(a) Give a biological interpretation of the delay time τ.

(b) Suppose that $n(t) = 1$ for $t \in [-\tau, 0]$. Show that $n(t) = rt + 1$ for $t \in [0, \tau]$. Determine $n(t)$ for $t \in [\tau, 2\tau]$.

(c) Show that the delay differential equation admits a periodic solution when $r\tau = -\pi/2$. Why is this solution not appropriate for describing a population?

Paper 4, Section I

6A Mathematical Biology

Let x_n be the number of plants in season n. Each year, a proportion $r < 1$ of plants survives to season $n+1$. In addition, the number of seeds produced per plant that become plants the following year is $ke^{-\lambda x_n}$ with constants $k > 0$ and $\lambda > 0$.

(a) What values of k permit a non-vanishing equilibrium population?

(b) What additional requirement on k is needed for this equilibrium to be stable?

Paper 3, Section II

13A Mathematical Biology

A discrete population $n = 0, 1, 2, \ldots$ undergoes a stochastic birth-death process, with birth rate $\alpha + \beta n$ and death rate $\gamma n(n-1)$, where α, β, and γ are all positive constants. Let $P_n(t)$ be the probability that the population is n at time t.

(a) Write down the master equation for $P_n(t)$.

(b) Determine a differential equation for the expectation value $\langle n(t) \rangle$.

(c) Assume that the probability distribution can be approximated by the Poisson distribution $P_n(t) = e^{-\lambda(t)}\lambda(t)^n/n!$ for some $\lambda(t)$. Compute $\langle n \rangle$ and $\langle n(n-1) \rangle$ in terms of $\lambda(t)$. Write down a differential equation for $\lambda(t)$ and determine the value of $\langle n(t) \rangle$ as $t \to \infty$.

(d) Treating $n(t)$ as continuous, and renaming it $x(t)$, the Fokker-Planck equation for the probability $P(x,t)$ takes the form

$$\frac{\partial P}{\partial t} = -\frac{\partial(uP)}{\partial x} + \frac{\partial^2(DP)}{\partial x^2}\ .$$

What are the functions $u(x)$ and $D(x)$ for the birth-death process above? Show that $d\langle x(t) \rangle/dt$ is determined by the expectation value of $u(x)$ and that the resulting differential equation coincides with the equation for $\langle n(t) \rangle$ computed in part (b) above.

Paper 4, Section II

14A Mathematical Biology

Consider the reaction-diffusion system in one dimension with $x \in \mathbb{R}$,

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \alpha - (\beta+1)u + u^2 v,$$
$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \beta u - u^2 v,$$

where the constants D, α, and β obey $D > 0$, $\beta > 1$, and $\alpha^2 > \beta - 1$. The variables u and v are both positive.

(a) First consider spatially homogeneous solutions. Determine the fixed point and sketch the nearby trajectories in the system's phase space.

(b) Now consider inhomogeneous solutions. Without calculation, explain why the system is stable for $D = 1$. Find the condition relating D, α, and β for the system to be unstable.

(c) Suppose we vary the diffusivity from $D = 1$ to the value at which the system first becomes unstable. What is the critical wavenumber $k_\star$ at which the instability first occurs?

Paper 1, Section II

31L Mathematics of Machine Learning

Consider i.i.d. random variables $(X_1, Y_1), \ldots, (X_n, Y_n)$ taking values in $\mathcal{X} \times \{-1, 1\}$ and a convex surrogate loss $(x, y) \mapsto \phi(yh(x))$.

(a) Define the *empirical Rademacher complexity*, $\hat{\mathcal{R}}(\mathcal{H}(x_{1:n}))$, and the *Rademacher complexity*, $\mathcal{R}_n(\mathcal{H})$, for a class of functions $\mathcal{H}$ mapping $\mathcal{X}$ to $\mathbb{R}$. State the contraction lemma for the Rademacher complexity.

(b) Fix $s > 0$. Let $S \subseteq \mathbb{R}^{d \times d}$ be the set of symmetric, positive semidefinite matrices with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_d$ satisfying $\sum_{i=1}^{d} \lambda_i \leqslant s$. Show that S is a convex set.

(c) Suppose that $\mathcal{X} = \{x \in \mathbb{R}^d : \|x\|_2 \leqslant C\}$, and let $\mathcal{H} = \{x \mapsto x^T M x : M \in S\}$. Prove that

$$\mathcal{R}_n(\mathcal{H}) \leqslant \frac{C^2 s}{\sqrt{n}}.$$

[*Hint: If $A \in \mathbb{R}^{d \times d}$ is a symmetric matrix with eigenvalues $\alpha = (\alpha_1, \ldots, \alpha_d)$, then $Tr(A^T A) = \|\alpha\|_2^2$.*]

(d) Let $\hat{h}$ minimise the empirical risk $\hat{R}_\phi(h)$ over $h \in \mathcal{H}$, where ϕ is the hinge loss. Let h^* be the minimiser of the risk $R_\phi(h)$ over $h \in \mathcal{H}$. Quoting any necessary result from the course, deduce that

$$\mathbb{E} R_\phi(\hat{h}) - R_\phi(h^*) \leqslant \frac{K}{\sqrt{n}},$$

for a constant K which you must specify.

Paper 2, Section II

31L Mathematics of Machine Learning

(a) Define the *shattering coefficient* $s(\mathcal{H}, n)$ and the *VC dimension* $VC(\mathcal{H})$ for a hypothesis class $\mathcal{H}$. State the Sauer-Shelah lemma.

(b) In each of the following cases, find $VC(\mathcal{H}_i)$,

(i) $\mathcal{H}_1 = \{\mathbb{1}_{[a,b]} : a, b \in \mathbb{R}\}$.

(ii) $\mathcal{H}_2 = \{\delta(2\mathbb{1}_{[a,b]} - 1) : a, b \in \mathbb{R}, \delta \in \{-1, 1\}\}$.

(c) Let $\mathcal{H}_3 = \{x \mapsto \operatorname{sgn}(x^T M x) : M \in \mathbb{R}^{d \times d}\}$. Show that

$$VC(\mathcal{H}_3) \leqslant \binom{d+1}{2}.$$

[You may use any theorems from lectures if they are precisely stated.]

(d) Consider

$$\mathcal{F} = \{\sum_{j=1}^{J} \beta_j h_j(x) : J < \infty, h_j \in \mathcal{H}_3, \beta_j > 0 \text{ for } j = 1, \ldots, J, \|\beta\|_1 \leqslant 1\}.$$

Prove that

$$\hat{\mathcal{R}}(\mathcal{F}(x_{1:n})) \leqslant \sqrt{\frac{(d^2 + d)\log(n+1)}{n}}.$$

[You may use any theorems from lectures about convex analysis and sub-Gaussian random variables if they are precisely stated.]

Paper 4, Section II

30L Mathematics of Machine Learning

Fix $s > 0$. Let $S \subseteq \mathbb{R}^{d\times d}$ be the convex set of symmetric, positive semidefinite matrices with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_d \geqslant 0$ satisfying $\sum_{i=1}^d \lambda_i \leqslant s$. For any symmetric matrix $M \in \mathbb{R}^{d\times d}$, the projection $\pi(M)$ of M onto S is defined as the minimiser of $\mathrm{Tr}((M-Z)^T(M-Z))$ over $Z \in S$.

(a) Let $M \in \mathbb{R}^{d\times d}$ be symmetric. Show that if $\Pi \in \mathbb{R}^{d\times d}$ satisfies

$$\mathrm{Tr}((M-\Pi)^T(Z-\Pi)) \leqslant 0 \text{ for all } Z \in S,$$

then Π is the projection $\pi(M)$ of M onto S. [*Hint: The function* $(A,B) \mapsto Tr(A^TB) = \sum_{i,j} A_{ij}B_{ij}$ *is an inner-product.*]

(b) Now suppose that $M \notin S$ is positive semidefinite, with eigenvalues and eigenvectors (μ_i, v_i) for $i = 1,\ldots,d$. Using part (a), or otherwise, show that

$$\pi(M) = \sum_{i=1}^d \max(0, \mu_i - \rho) v_i v_i^T,$$

where $\rho > 0$ is such that $\sum_{i=1}^d \max(0,\mu_i - \rho) = s$. [*Hint: By von Neumann's trace inequality, if* $A, B \in \mathbb{R}^{d\times d}$ *are symmetric with eigenvalues* $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_d \geqslant 0$ *and* $\beta_1 \geqslant \beta_2 \geqslant \ldots, \geqslant \beta_d \geqslant 0$*, respectively, then* $|Tr(AB)| \leqslant \sum_{i=1}^d \alpha_i\beta_i$*.*]

(c) Consider i.i.d. random variables $(X_1,Y_1),\ldots,(X_n,Y_n)$ taking values in $\{x \in \mathbb{R}^d : \|x\|_2 \leqslant C\} \times \{-1,1\}$. Let $M^{(1)} = 0 \in \mathbb{R}^{d\times d}$, and iteratively define, for a step size $\eta > 0$ and iteration $i = 1,\ldots,k-1$,

$$g_i = -\frac{1}{n}\sum_{j=1}^n Y_j X_j X_j^T \frac{\exp(-Y_j X_j^T M^{(i)} X_j)}{1+\exp(-Y_j X_j^T M^{(i)} X_j)},$$
$$M^{(i+1)} = \pi(M^{(i)} - \eta g_i).$$

Let $\bar{M} = \frac{1}{k}\sum_{i=1}^k M^{(i)}$. The function $\bar{h} : x \mapsto x^T\bar{M}x$ approximates an empirical risk minimiser $\hat{h}$ over a certain hypothesis class $\mathcal{H}$ with a certain loss function ϕ. Give explicit forms for $\mathcal{H}$ and ϕ.

Carefully quoting any necessary result from the course, show that, for a choice of step size η which you must specify,

$$\hat{R}_\phi(\bar{h}) - \hat{R}_\phi(\hat{h}) \leqslant \frac{2sC^2}{\sqrt{k}}.$$

Paper 1, Section II

20G Number Fields

(a) Let $f(X)$ be a monic polynomial with algebraic integer coefficients. Prove that the roots of f are algebraic integers. [You may use without proof the characterization of algebraic integers in terms of finitely generated modules, provided you state the result precisely.]

(b) Determine the ring of integers $\mathcal{O}_K$ in the field $K = \mathbb{Q}(\sqrt{17})$. Justify your answer.

(c) Let $\alpha = \sqrt{4 + \sqrt{17}}$. By computing $\mathrm{N}_{K|\mathbb{Q}}(\alpha^2)$, or otherwise, show that $\alpha \notin K$.

(d) With α as in part (c), let $L = \mathbb{Q}(\alpha)$. Show for $\beta \in L$ that $\beta \in \mathcal{O}_L$ if and only if $\mathrm{N}_{L|K}(\beta) \in \mathcal{O}_K$ and $\mathrm{Tr}_{L|K}(\beta) \in \mathcal{O}_K$.

(e) Show that, if $a + b\alpha \in \mathcal{O}_L$ for some $a, b \in K$, then $2a \in \mathcal{O}_K$ and $2b \in \mathcal{O}_K$.

Paper 2, Section II

20G Number Fields

(a) Define the *class group* of a number field. [You do not need to prove that it is a group.]

(b) Prove that the class group of a number field is finite. [You may use without proof the fact that, for every number field K, there is a constant C such that every ideal $I \subset \mathcal{O}_K$ contains a non-zero element α with $|\mathrm{N}(\alpha)| \leqslant C\mathrm{N}(I)$.]

(c) Let K be a number field and $I \subset \mathcal{O}_K$ an ideal. Prove that there is a positive integer n such that I^n is a principal ideal.

(d) A proper ideal $I \subset \mathcal{O}_K$ is called a primary ideal, if for all $\alpha, \beta \in \mathcal{O}_K$ such that $\alpha\beta \in I$ but $\alpha \notin I$, there is a positive integer k such that $\beta^k \in I$. Prove that an ideal in $\mathcal{O}_K$ is primary if and only if it is a power of a prime ideal.

Paper 4, Section II

20G Number Fields

(a) State Dirichlet's unit theorem.

(b) Define the *logarithmic embedding* (Log) and prove that its kernel contains only roots of unity.

(c) What can you say about the image of a fundamental system of units $u_1, \ldots, u_m$ under the logarithmic embedding? Here m is the rank of the unit group. [You do not need to prove your answer.]

(d) Let K be a number field with r real embeddings and s pairs of complex conjugate embeddings. Let $I = \langle \beta \rangle \subset \mathcal{O}_K$ be a principal ideal. Show that

$$\mathrm{Log}(\beta) = t(1, \ldots, 1, 2, \ldots, 2) + \sum_{i=1}^{m} \lambda_i \mathrm{Log}(u_i)$$

where $t, \lambda_1, \ldots, \lambda_m \in \mathbb{R}$ and the vector $(1, \ldots, 1, 2, \ldots, 2)$ has r 1's and s 2's. Compute t in terms of $\mathrm{N}(I)$ and $[K : \mathbb{Q}]$. [*Hint: Relate the sum of the coordinates of* $\mathrm{Log}(\beta)$ *to* $\mathrm{N}(\beta)$*.*] [Standard facts about norms may be quoted without proof.]

(e) Prove that, for every number field K, there is a constant $C < \infty$ such that, for every principal ideal $I \subset \mathcal{O}_K$, there is an element $\alpha \in I$ such that $I = \langle \alpha \rangle$ and $|\sigma(\alpha)| < C\mathrm{N}(I)^{1/[K:\mathbb{Q}]}$ for all embeddings $\sigma : K \to \mathbb{C}$.

Paper 1, Section I

1G Number Theory

(a) Using Fermat factorisation, find a non-trivial factorisation of $N = 14351$.

(b) Let $N \geqslant 1$ be an odd, composite integer that is not a square, and let $k \geqslant 1$. We say that Fermat factorisation for N succeeds after k steps if the first value $r \geqslant \sqrt{N}$ such that $r^2 - N$ is a square is $r = \lfloor \sqrt{N} \rfloor + k$.

Suppose that $N = 3p$, where $p > 3$ is a prime number. Find the value of k such that Fermat factorisation for N succeeds after k steps.

Paper 2, Section I

1G Number Theory

Let $N = 136$, and let $G = (\mathbb{Z}/N\mathbb{Z})^{\times}$ denote the multiplicative group of units modulo N.

(a) Compute the order of G.

(b) Compute the least integer $m \geqslant 1$ such that for any $g \in G$, $g^m \equiv 1 \bmod N$.

(c) Write down an element $g \in G$ of order m.

Paper 3, Section I

1G Number Theory

(a) Let $f(x, y)$, $g(x, y)$ be binary quadratic forms. Define what it means for f and g to be *equivalent*.

(b) A binary quadratic form $f(x, y) = ax^2 + bxy + cy^2$ is said to be primitive if $\gcd(a, b, c) = 1$ (i.e. there is no prime number p dividing each of a, b, and c). Show that if f and g are equivalent, then f is primitive if and only if g is primitive.

(c) Compute the number of equivalence classes of primitive, positive definite binary quadratic forms of discriminant -80.

Paper 4, Section I

1G Number Theory

Let p be an odd prime.

(a) Define the Legendre symbol $\left(\frac{a}{p}\right)$, and state the law of quadratic reciprocity that it satisfies.

(b) Give necessary and sufficient conditions on p mod 3 for the equation

$$X^2 + 11X + 31 = 0$$

to have a solution in $\mathbb{Z}/p\mathbb{Z}$.

(c) Give necessary and sufficient conditions on p mod 3 for the equation

$$X^3 = -1$$

to have (i) a solution in $\mathbb{Z}/p\mathbb{Z}$; and (ii) a unique solution in $\mathbb{Z}/p\mathbb{Z}$.

Paper 3, Section II

11G Number Theory

(a) Let $\theta \in \mathbb{R}$ be an irrational number with continued fraction expansion $\theta = [a_0, a_1, a_2, \ldots]$. Define the *convergents* p_n, q_n of θ. Show that if $\gamma > 0$, then there is a formula for each $n \geqslant 1$:

$$[a_0, a_1, a_2, \ldots, a_n, \gamma] = \frac{p_n\gamma + p_{n-1}}{q_n\gamma + q_{n-1}}.$$

(b) Compute the continued fraction expansion of $\theta = \sqrt{11}$.

(c) Let p_n, q_n be the convergents of $\theta = \sqrt{11}$. Show that if $n \geqslant 2$ is even, then

$$p_n^2 - 11q_n^2 = -2.$$

Paper 4, Section II

11G Number Theory

(a) Let p be a prime number, and let $N \in \mathbb{N}$. Define the *p-adic valuation* $v_p(N)$ of N.

(b) Show that if $k \in \mathbb{N}$, and p is a prime number such that $k+2 \leqslant p \leqslant 2k+1$, then $v_p(\binom{2k+1}{k+1}) = 1$.

(c) If $X \geqslant 1$ is a real number, define $P(X) = \prod_{p \leqslant X} p$, where the product is over prime numbers p less than or equal to X. Show that $P(X) \leqslant 4^X$ for all $X \geqslant 1$.

(d) If $X \geqslant 1$ is a real number, let $\pi(X)$ denote the number of prime numbers less than or equal to X. By considering $\pi(X) - \pi(\sqrt{X})$, or otherwise, show that there is a constant $c > 0$ such that $\pi(X) \leqslant cX/\log X$ for all $X \geqslant 2$.

Paper 1, Section II

41D Numerical Analysis

(a) Show that if $A \in \mathbb{R}^{n\times n}$ is symmetric, there exists a symmetric and tridiagonal matrix $H \in \mathbb{R}^{n\times n}$ that has the same eigenvalues as A, and that can be computed in finitely many arithmetic operations from the matrix elements of A.

(b) The standard QR algorithm (without shifts) is applied to a symmetric and tridiagonal matrix H. For $k = 0, 1, 2, ...$, let H_k be the k^{th} iteration of the QR algorithm and recall that $H_{k+1} = \overline{Q}_k^T H \overline{Q}_k$, where $\overline{Q}_k$ is orthogonal and $\overline{Q}_k \overline{R}_k$ is the QR factorization of H^{k+1} (that is, the $(k+1)^{\text{th}}$ power of H).

Suppose that the eigenvalues λ_i $(i = 1, ..., n)$ of H satisfy $|\lambda_1| < |\lambda_2| < \cdots < |\lambda_{n-1}| = |\lambda_n|$, and let the corresponding normalised eigenvectors of H be $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$. Suppose also that the first two canonical basis vectors, $\mathbf{e}_1$ and $\mathbf{e}_2$, can be written as $\mathbf{e}_1 = \sum_{i=1}^n b_i \mathbf{w}_i$ and $\mathbf{e}_2 = \sum_{i=1}^n c_i \mathbf{w}_i$ where b_i and c_i $(i = 1, \ldots, n)$ are non-zero constants.

(i) Show that if $(H/\lambda_n)^k \mathbf{e}_1 \to \mathbf{v}_1$ and $(H/\lambda_n)^k \mathbf{e}_2 \to \mathbf{v}_2$ as $k \to \infty$, then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linear combinations of $\mathbf{w}_{n-1}$ and $\mathbf{w}_n$.

(ii) Let $h_{3,2}^{(k)}$ be the matrix element of H_k at the 3rd row and 2nd column. Show that $h_{3,2}^{(k)} \to 0$ as $k \to \infty$.

Paper 2, Section II

41D Numerical Analysis

Let N be an integer power of 2. The discrete Fourier transform (DFT) $\mathcal{F}_{2N} : \mathbb{C}^{2N} \to \mathbb{C}^{2N}$ is defined by

$$\mathbf{Y} = \mathcal{F}_{2N}\mathbf{y}, \text{ where } Y_k = \sum_{n=0}^{2N-1} y_n \exp\left(-\frac{\pi i}{N} nk\right), \quad 0 \leqslant k \leqslant 2N-1, \qquad (\dagger)$$

while the discrete cosine transform (DCT) $\mathcal{C}_N : \mathbb{R}^N \to \mathbb{R}^N$ and the discrete sine transform (DST) $\mathcal{S}_N : \mathbb{R}^N \to \mathbb{R}^N$ are defined by

$$\mathbf{Z} = \mathcal{C}_N\mathbf{x}, \text{ where } Z_k = \sum_{n=0}^{N-1} x_n \cos\left[\frac{\pi}{N}\left(n + \frac{1}{2}\right)k\right], \quad 0 \leqslant k \leqslant N-1,$$

$$\tilde{\mathbf{Z}} = \mathcal{S}_N\mathbf{x}, \text{ where } \tilde{Z}_k = \sum_{n=0}^{N-1} x_n \sin\left[\frac{\pi}{N}\left(n + \frac{1}{2}\right)(k+1)\right], \quad 0 \leqslant k \leqslant N-1,$$

for N even.

(a) Show that there exists an algorithm that computes the DFT of a vector of length $2N$ for which the number of multiplications required is $\mathcal{O}(N \log N)$.

(b) Let $\mathbf{x} \in \mathbb{R}^N$ and $\mathbf{y} \in \mathbb{R}^{2N}$ be related by $y_n = x_n$ for $0 \leqslant n \leqslant N-1$ and $y_n = x_{2N-n-1}$ for $N \leqslant n \leqslant 2N-1$. With $\mathbf{Y}$ defined as in equation ($\dagger$), show that

$$\frac{1}{2}\exp\left(-\frac{\pi i}{2N}k\right) Y_k = \sum_{n=0}^{N-1} x_n \cos\left[\frac{\pi}{N}\left(n + \frac{1}{2}\right)k\right], \quad 0 \leqslant k \leqslant 2N-1.$$

(c) Use parts (a) and (b) to show that there exists an algorithm to compute the DCT of a vector of length N using $\mathcal{O}(N \log N)$ multiplications.

(d) Let $\mathbf{x} \in \mathbb{R}^N$ and $\boldsymbol{\xi} \in \mathbb{R}^N$ be related by $\xi_n = (-1)^n x_n$ for $0 \leqslant n \leqslant N-1$. By considering the DCT of $\boldsymbol{\xi}$, or otherwise, show that there exists an algorithm to compute the DST of a vector of length N using $\mathcal{O}(N \log N)$ multiplications.

Paper 3, Section II

40D Numerical Analysis

Consider the following two Cauchy problems: the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,, \qquad 0 \leqslant t \leqslant 1, \quad x \in \mathbb{R}, \tag{$\dagger$}$$

with initial condition $u(x,0) = u_0(x)$; and the wave equation

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2}\,, \qquad 0 \leqslant t \leqslant 1, \quad x \in \mathbb{R},$$

with initial conditions $v(x,0) = v_0(x)$ and $\frac{\partial v}{\partial t}(x,0) = v_1(x)$. Further consider the discretisation of the diffusion equation,

$$u_m^{n+1} - \frac{1}{2}\mu\left(u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}\right) = u_m^n + \frac{1}{2}\mu\left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right), \tag{$\star$}$$

and the discretisation of the wave equation,

$$v_m^{n+1} - 2\rho v_m^n + v_m^{n-1} = \mu\left(v_{m+1}^n - 2v_m^n + v_{m-1}^n\right), \tag{$\star\star$}$$

where $m \in \mathbb{Z}$, $n = 1, \ldots, N$, $\mu > 0$ is the Courant number, and $\rho \in [1,2]$ is a constant parameter. The notation f_m^n here denotes the function f evaluated at the n^{th} time step and located at a spatial grid point labelled by index m. In all parts of the question below regarding stability, consider the 2-norm $\|\cdot\|_2$.

(a) Derive an expression for the amplification factor in a Fourier analysis of stability applied to a finite-difference discretisation of a linear partial differential equation.

(b) Determine the values of μ that make the method in equation $(\star)$ stable for the diffusion equation as described above.

(c) Determine the values of μ, as a function of ρ, that make the method in equation $(\star\star)$ stable for the wave equation as described above.

(d) Suppose we replace the Cauchy problem $(x \in \mathbb{R})$ in equation $(\dagger)$ with the finite domain $0 \leqslant x \leqslant 1$, on which we apply Dirichlet boundary conditions $u(0,t) = u(1,t) = 0$. Determine the values of μ that make the method in equation $(\star)$ stable for this problem.

[*You may use basic spectral properties of Toeplitz symmetric tridiagonal (TST) matrices without proof.*]

Paper 4, Section II

40D Numerical Analysis

(a) State and prove the Householder-John theorem.

(b) Define the *Jacobi method* for solving a system $A\mathbf{x} = \mathbf{b}$, with $A \in \mathbb{R}^{n\times n}$ and $\mathbf{b} \in \mathbb{R}^n$. Show that if A is a symmetric, positive-definite, tridiagonal matrix,

$$A = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & b_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{n-1} \\ 0 & \cdots & 0 & b_{n-1} & a_n \end{bmatrix},$$

then the Jacobi method converges.

[*You may use without proof general convergence results of iterative methods for linear systems based on the spectral radius, provided they are clearly stated.*]

Paper 1, Section II

34C Principles of Quantum Mechanics

(a) A Fermi oscillator has Hilbert space $\mathcal{H} = \mathbb{C}^2$ and Hamiltonian $H = B^\dagger B$, where $B^2 = 0$ and $B^\dagger B + BB^\dagger = 1$.

(i) Find the eigenvalues of H.

(ii) If $|1\rangle$ is a state obeying $H|1\rangle = |1\rangle$ and $\langle 1|1\rangle = 1$, find $B|1\rangle$ and $B^\dagger|1\rangle$.

(iii) Obtain a matrix representation of the operators B, $B^\dagger$ and H.

(b) Now consider a composite system comprised of two decoupled Fermi oscillators, with $H_a = B_a^\dagger B_a$ for $a = 1, 2$. The Hamiltonian of the composite system is $H_{\text{tot}} = E_1 B_1^\dagger B_1 + E_2 B_2^\dagger B_2$ where $E_{1,2}$ are non-negative real numbers.

(i) Determine the exact eigenvectors and eigenvalues of H_{tot}.

(ii) Write H_{tot}, $B_{1,2}$ and $B_{1,2}^\dagger$ as matrices in the basis of energy eigenvectors.

(iii) Assuming $E_1 \ll E_2$ and treating $E_1 B_1^\dagger B_1$ as a small perturbation to the unperturbed Hamiltonian $H_{\text{tot}}^{(0)} = E_2 B_2^\dagger B_2$, determine the eigenvectors and eigenvalues of H_{tot} to first order in perturbation theory. Discuss how your derivation relates to degenerate perturbation theory.

(c) Finally assume $E_1 = 0$ and $E_2 > 0$, and consider the new Hamiltonian

$$\tilde{H}_{\text{tot}} = E_2 B_2^\dagger B_2 + g(B_1 + B_1^\dagger)$$

where g is a real constant.

(i) Determine the exact eigenvectors and eigenvalues of $\tilde{H}_{\text{tot}}$.

(ii) Treating $g(B_1 + B_1^\dagger)$ as a small perturbation to the unperturbed Hamiltonian $\tilde{H}_{\text{tot}}^{(0)} = E_2 B_2^\dagger B_2$, determine the eigenvectors and eigenvalues of $\tilde{H}_{\text{tot}}$ to first order in perturbation theory. Discuss how your derivation relates to degenerate perturbation theory.

Paper 2, Section II

35C Principles of Quantum Mechanics

(a) State the commutation relations for the spin operator $\mathbf{S}$ and describe the associated irreducible representations $\{|s,\sigma\rangle\}$, where s and σ are quantum numbers you should specify. Determine the Hermitian conjugate and the trace of $\mathbf{S}$.

(b) Henceforth consider only the Hilbert space of a particle of spin 3/2 and use the basis $\{|\sigma\rangle\}$ of eigenvectors of S_z. Using the relation

$$S_{\pm}|\sigma\rangle = \sqrt{s(s+1)-\sigma(\sigma\pm 1)}\hbar|\sigma\pm 1\rangle\,, \tag{1}$$

write down S_x and S_y as matrices.

(c) Let $\mathbf{n}=(\cos\varphi,\sin\varphi,0)$ be a vector in $\mathbb{R}^3$, using the standard basis. Derive the states $|\mathbf{n},3/2\rangle$ for which the spin along the direction $\mathbf{n}$ is always measured to be $\frac{3}{2}\hbar$.

(d) Let $H=-\gamma\mathbf{B}\cdot\mathbf{S}$ be the Hamiltonian of the system, where γ is a constant and $\mathbf{B}=B\hat{\mathbf{z}}$ is an external magnetic field. Compute the state of the system at time t assuming that it started at time $t=0$ from (i) $|\hat{\mathbf{z}},3/2\rangle$, and (ii) $|\hat{\mathbf{x}},3/2\rangle$. Briefly interpret the results.

Paper 3, Section II

33C Principles of Quantum Mechanics

(a) Write down the Hamiltonian for a two-dimensional quantum harmonic oscillator with unit mass and frequency. Determine the energy eigenvectors $|E\rangle$ and eigenvalues E and discuss their degeneracy.

(b) Define the angular momentum operator L and explicitly compute its commutation relations with H. Hence constrain the form of $\langle E|L|E'\rangle$.

(c) Consider the operators $T_{ij}=a_i^{\dagger}a_j$ where $i,j\in\{x,y\}$, and a_x and a_y are the annihilation operators in the x and y directions respectively. Compute the commutator $[T_{ij},T_{kl}]$ and hence $[T_{ij},H]$.

(d) Now consider $T^a=\frac{1}{2}\sigma^a_{ij}T_{ij}$ for $a=1,2,3$ where σ^a are the Pauli matrices satisfying $[\sigma^a,\sigma^b]=2i\epsilon_{abc}\sigma^c$. Compute the commutators $[T^a,T^b]$. Relate L to the T^a. Using these results and what you know about the representations of the group SU(2), determine the possible angular momentum eigenvalues $L|E,\ell\rangle=\ell|E,\ell\rangle$ that an energy eigenvector $|E\rangle$ can have. Compare your result to the degeneracy discussed in part (a).

$$\left[\textit{Hint: the Pauli matrices are } \sigma^1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \sigma^2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\ \sigma^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.\ \right]$$

Paper 4, Section II

33C Principles of Quantum Mechanics

(a) Consider the commutators $[L_i, X_j]$ and $[L_i, P_j]$ between orbital angular momentum and the position and momentum operators. Write $\mathbf{L}$ in terms of $\mathbf{X}$ and $\mathbf{P}$ and use the canonical commutation relations between position and momentum to determine these commutators.

(b) Express $\mathbf{L} \cdot \mathbf{L}$ in the form $c_1 \mathbf{X} \cdot \mathbf{P} + c_2 (\mathbf{X} \cdot \mathbf{X})(\mathbf{P} \cdot \mathbf{P}) + c_3 (\mathbf{X} \cdot \mathbf{P})(\mathbf{X} \cdot \mathbf{P})$ where c_1, c_2, and c_3 are constants you should determine. Hence, by expressing $\mathbf{X}$ and $\mathbf{P}$ as operators acting on wavefunctions, determine the relation between $\mathbf{L} \cdot \mathbf{L}$ and the spherical Laplacian $\nabla^2_{S^2}$. [*Hint: recall that* $\nabla^2 = \partial_r^2 + (2/r)\partial_r + r^{-2}\nabla^2_{S^2}$.]

(c) Consider the hydrogen atom and neglect the spin of the electron. Give a basis that spans the degenerate energy subspace corresponding to the first excited state ($n = 2$ in the usual labelling). Focussing exclusively on this subspace, consider the following two scenarios.

(i) The Hamiltonian is perturbed by $\Delta H = g \mathbf{L} \cdot \mathbf{L}$ where g is a constant. Compute the new energy eigenstates and eigenvalues exactly.

(ii) The Hamiltonian is perturbed by $\Delta H = E(t) X_3 + B(t) P_3$, where $E(t)$ and $B(t)$ are time-dependent functions. Determine which matrix elements of ΔH must vanish using the result for the commutators $[L_3, X_3]$ and $[L_3, P_3]$ and the transformation of X_3 and P_3 under parity. Now define

$$z(t) := E(t)\langle 2,0,0|X_3|2,1,0\rangle + B(t)\langle 2,0,0|P_3|2,1,0\rangle \in \mathbb{C}$$

and assume $z(t) = c\, e^{i\omega t/\hbar}$ for some real constants c and ω. Find an exact solution of the Schrödinger equation given an initial state of the form $A_0|2,0,0\rangle + A_1|2,1,0\rangle$ where $A_0, A_1 \in \mathbb{C}$.

Paper 1, Section II

29L Principles of Statistics

(a) Suppose real-valued random variables $\widehat{\psi}_1, \widehat{\psi}_2, \ldots$ satisfy $\sqrt{n}(\widehat{\psi}_n - \psi) \xrightarrow{d} N(0, v)$ as $n \to \infty$ for $v > 0$ and deterministic $\psi \in \mathbb{R}$. Suppose $\phi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable at ψ. Write down the asymptotic distribution of $\sqrt{n}(\phi(\widehat{\psi}_n) - \phi(\psi))$. [No proof is necessary.]

(b) Suppose we have data $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Exp}(\theta)$ with rate $\theta > 0$.

(i) Find the maximum likelihood estimator (MLE) $\widehat{\theta}_n$ for the rate θ.

(ii) Without appealing to the general theory for MLEs, obtain, with justification, an asymptotic confidence interval $\widehat{C}_n$ for θ centred on $\widehat{\theta}_n$ that satisfies $\mathbb{P}_\theta(\theta \in \widehat{C}_n) \to 1 - \alpha$ as $n \to \infty$ for a given $\alpha \in (0, 1)$.

(c) Now suppose that rather than observing the random variables X_i as in part (b), we instead only observe data $Y_1, \ldots, Y_n$ where $Y_i = \lfloor X_i \rfloor$ is the greatest integer less than or equal to X_i.

(i) Show that the MLE $\widetilde{\theta}_n$ for θ based on the data $Y_1, \ldots, Y_n$ is given by

$$\widetilde{\theta}_n = \log\left(\frac{1 + \bar{Y}}{\bar{Y}}\right).$$

(ii) Without appealing to the general theory for MLEs, obtain, with justification, the asymptotic distribution of

$$\sqrt{n}(\widetilde{\theta}_n - \theta).$$

[*Hint: Recall that if random variable Z follows a geometric distribution supported on $\{0, 1, 2, \ldots\}$ with success probability p, then $\mathbb{E}Z = (1-p)/p$ and $\text{Var}(Z) = (1-p)/p^2$.*]

Paper 2, Section II

29L Principles of Statistics

(a) Consider a statistical model $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} f(\cdot, \theta)$, $\theta \in \Theta \subseteq \mathbb{R}^p$, that satisfies the usual regularity conditions.

(i) Define the *score function* $S_n(\theta)$ and *Fisher information matrix* $I_n(\theta)$.

(ii) Show that $\mathbb{E}_\theta(S_1(\theta)) = 0$. [You may interchange integration and differentiation without justification.]

(iii) Recall that the score test statistic T_n for the null hypothesis $H_0 : \theta \in \Theta_0$ is given by

$$T_n := \frac{1}{n} S_n(\widetilde{\theta})^\top I_1(\widetilde{\theta})^{-1} S_n(\widetilde{\theta}),$$

where $\widetilde{\theta}$ maximises the log-likelihood over $\theta \in \Theta_0$. Show that in the case of a simple null $H_0 : \theta = \theta_0$, we have $T_n \xrightarrow{d} \chi^2_p$ as $n \to \infty$.

(b) Now consider the model $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} N(\mu, \sigma^2)$ where $\mu \in \mathbb{R}$ and $\sigma^2 > 0$.

(i) Consider the composite null $H_0 : \mu \in \mathbb{R}$, $\sigma^2 = 1$. Show that the score test statistic T_n for H_0 is given by

$$T_n = \left(\frac{1}{\sqrt{2n}} \sum_{i=1}^n \{(X_i - \bar{X})^2 - 1\} \right)^2,$$

where $\bar{X}$ is the sample mean.

(ii) Determine, with proof, the asymptotic distribution of T_n as $n \to \infty$ under the null hypothesis H_0. [*Hint: If* $Z \sim \chi^2_1$ *then* $\mathrm{Var}(Z) = 2$.]

Paper 3, Section II

28L Principles of Statistics

(a) Consider a Bayesian model $X \mid \theta \sim \text{Pois}(\theta)$ where the parameter $\theta \in (0, \infty)$ has prior distribution π given by $\theta \sim \text{Gamma}(\alpha, \lambda)$ where $\alpha, \lambda > 0$. Show that the posterior distribution $\theta \mid X$ has a $\text{Gamma}(\alpha + X, \lambda + 1)$ distribution. [*Hint: A* $\text{Gamma}(\alpha, \lambda)$ *distribution has density function* $f(y) = \lambda^\alpha y^{\alpha-1} e^{-\lambda y} / \Gamma(\alpha)$ *for* $y > 0$.]

(b) Consider now a decision problem involving a statistical model $\{P_\theta : \theta \in \Theta\}$ and loss function $L : \Theta \times \Theta \to [0, \infty)$.

(i) What is meant by the *risk* of a decision rule $\delta : \mathcal{X} \to \Theta$? Given a prior distribution π on Θ, what does it mean for δ to be a π-*Bayes estimator*? What does it mean for δ to be *minimax*?

(ii) Suppose a decision rule δ has constant risk r, and there is a sequence $\pi_1, \pi_2, \ldots$ of priors on θ where, writing $r_j < \infty$ for the π_j-Bayes risk of the π_j-Bayes estimator, we have that $r = \lim_{j \to \infty} r_j$. Show that δ is minimax.

(iii) Suppose now that $\mathcal{X} = \mathbb{R}$ and $\Theta = (0, \infty)$. Consider the weighted quadratic loss $L(\delta(x), \theta) = \theta^{-1}(\theta - \delta(x))^2$ and a prior π for θ. Show that a π-Bayes rule δ_π is given by $\delta_\pi(x) = \big(\mathbb{E}(\theta^{-1} \mid X = x)\big)^{-1}$.

(c) Finally show that in the model $X \sim \text{Pois}(\theta)$ where $\theta \in \Theta = (0, \infty)$, the decision rule $\delta(X) = X$ is minimax under the loss given in part (b) (iii). [*Hint: If* $Y \sim \text{Gamma}(\alpha, \lambda)$ *for* $\alpha > 1$ *and* $\lambda > 0$, *then* $\mathbb{E}(Y^{-1}) = \lambda/(\alpha - 1)$.] [You may interchange expectations and limits, that is apply the dominated convergence theorem, in your answer without justification.]

Paper 4, Section II

28L Principles of Statistics

(a) Given a distribution function $F : \mathbb{R} \to [0,1]$, let $F^{-1} : [0,1] \to \mathbb{R}$ be the quantile function given by

$$F^{-1}(p) := \inf\{t : F(t) \geqslant p\}.$$

Show that if $U \sim U[0,1]$ then $F^{-1}(U) \sim F$. [*Hint: F is always right continuous, that is, $F(t+a_n) \downarrow F(t)$ for all $a_n \downarrow 0$.*]

(b) Describe the steps taken by the *Gibbs sampler* to generate approximate samples from a bivariate density $f_{XY} : \mathbb{R}^2 \to [0,\infty)$. Writing $(Y_1, X_1), (Y_2, X_2), \ldots$ for the Markov chain generated by the algorithm, show that f_{XY} is stationary for its transition kernel.

(c) Let n be an even number. Consider a Bayesian model

$$Z_1, \ldots, Z_n \mid \mu, \omega \overset{\text{i.i.d.}}{\sim} N(\mu, \omega^{-1}),$$

with improper prior density $\pi(\mu,\omega) = \lambda e^{-\lambda\omega}, \omega > 0$, i.e. an $\text{Exp}(\lambda)$ density for ω and a flat prior on μ. Explain how you can generate approximate samples from the posterior distribution $\Pi(\mu, \omega \mid Z_1, \ldots, Z_n)$ if you have ways of generating independent samples from $U[0,1]$ and $N(0,1)$. [You may assume $\Pi(\mu, \omega \mid Z_1, \ldots, Z_n)$ has a well-defined density.]

[*Hint: Recall that a* $\text{Gamma}(m,\lambda)$ *distribution has density* $f(y) \propto y^{m-1}e^{-\lambda y}$*. Moreover if* $m \in \mathbb{N}$ *and* $Z_1, \ldots, Z_m \overset{\text{i.i.d.}}{\sim} \text{Exp}(\lambda)$*, then* $\sum_{i=1}^m Z_i \sim \text{Gamma}(m,\lambda)$.]

Paper 1, Section II

27H Probability and Measure

Let $(X_j)_{j\geqslant 1}$ be a sequence of independent real random variables with uniform density $p_{X_j} = \frac{1}{2j}\mathbf{1}_{[-j,j]}$. Let $S_n = X_1 + \cdots + X_n$.

(a) State Lévy's theorem from the lectures.

(b) Show that the characteristic function of $n^{-3/2}S_n$ satisfies

$$\Phi_{n^{-3/2}S_n}(\xi) = \frac{n^{\frac{3n}{2}}}{\xi^n n!}\prod_{j=1}^{n}\sin\left(\frac{j\xi}{n^{3/2}}\right).$$

(c) Show that $n^{-3/2}S_n$ converges in law to a limit to be determined.

[*Hint: You can use the formula* $\sum_{j=1}^n j^2 = n(n+1)(2n+1)/6$.]

(d) Let σ_j^2 be the variance of X_j. Show that $\left(\sum_{j=1}^n \sigma_j^2\right)^{-1/2} S_n$ converges in law to a limit to be determined.

Paper 2, Section II

27H Probability and Measure

Given a function $f : \mathbb{R} \to \mathbb{R}$, its Fourier transform is $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\,dx$ for $\xi \in \mathbb{R}$.

(a) State and prove the monotone convergence theorem.

(b) Let $\theta_n(x) = \left(1 - \frac{|x|}{n}\right)_+$ where $x_+ = \max\{x, 0\}$. Compute $\hat{\theta}_n$.

(c) Prove that there exists a universal constant $\alpha > 0$ such that: for any $f \in L^1 \cap L^\infty$ whose Fourier transform satisfies $\hat{f}(\xi) \geqslant 0$ for all $\xi \in \mathbb{R}$, one has

$$\|\hat{f}\|_{L^1} \leqslant \alpha \|f\|_{L^\infty}.$$

Paper 3, Section II

26H Probability and Measure

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. We consider in this question real valued random variables. We recall $a \wedge b = \min\{a, b\}$.

(a) Show that $X_n \xrightarrow[n\to\infty]{(P)} X \iff \lim_{n\to\infty} \mathbb{E}(|X_n - X| \wedge 1) = 0$ and that convergence in probability implies almost sure convergence along a subsequence.

(b) Show that $X_n \xrightarrow[n\to\infty]{(P)} X$ does not imply almost sure convergence $X_n \xrightarrow[n\to\infty]{a.s.} X$ by considering a sequence $(X_n)_{n\geqslant 1}$ of independent real random variables with $\mathbb{P}(X_n = 0) = 1 - \frac{1}{n}$ and $\mathbb{P}(X_n = 1) = \frac{1}{n}$.

(c) Let $X_n \xrightarrow[n\to\infty]{(P)} X$ and suppose that for some $1 < r < +\infty$, $(X_n)_{n\geqslant 1}$ is bounded in L^r. Show that $\forall 1 \leqslant p < r$, $X_n \xrightarrow[n\to\infty]{L^p} X$.

Paper 4, Section II

26H Probability and Measure

We consider a rope broken into two strands of lengths Y and Z. We let $X = Y + Z$ be the random length of the rope. We assume $\mathbb{E}[X^2] < +\infty$ and $Y = XU$ where U is a random variable independent of X with uniform law on $[0, 1]$ (explicitly $p_U(u) = \mathbf{1}_{[0,1]}(u)$).

(a) Compute $\mathbb{E}[Y]$ and $\mathrm{Var}[Y]$ in terms of $\mathbb{E}[X]$ and $\mathrm{Var}[X]$.

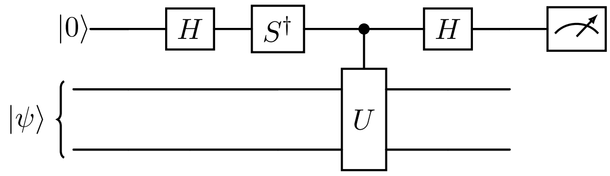
(b) From now on, we assume that X has a continuous density $g \geqslant 0$, and that $h(x) = \int_x^{+\infty} \frac{g(t)}{t} dt$ is well defined and C^1 on $\mathbb{R}_+$. Compute the densities of (X, Y), (Y, Z), Y and Z.

(c) Give a necessary and sufficient condition on h for Y and Z to be independent, and compute the law of Y and Z in this case.

Paper 1, Section I

10C Quantum Information and Computation

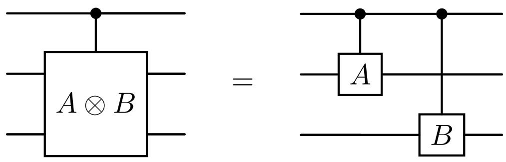
(a) Consider the following quantum circuit acting on a state $|0\rangle\,|\psi\rangle$



where $H = \dfrac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$ and $S = \begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}$. Show that on measuring the first qubit in the computational basis, the probability of outcome 1 is

$$p(1) = \frac{1}{2}(1 - \mathrm{Im}\,\langle\psi|U|\psi\rangle)\,.$$

(b) Verify the following identity, where A and B are unitary matrices:



(c) Consider a 2-qubit initial state $|\psi\rangle$ and a matrix description $W = Z\otimes Z + X\otimes I$. By modifying any of the above circuits, draw new circuits to obtain $\langle\psi|W|\psi\rangle$ in terms of outcome probabilities of 1-qubit measurements. [*Hint: note that W is not a unitary matrix but is a linear combination of orthogonal matrices.*]

Paper 2, Section I

10C Quantum Information and Computation

Let $|\psi\rangle$ denote a 2-qubit state. Let $\mathcal{A} = \{|a_0\rangle, |a_1\rangle\}$, $\mathcal{B} = \{|b_0\rangle, |b_1\rangle\}$ and $\mathcal{C} = \{|c_0\rangle, |c_1\rangle\}$ be three orthonormal bases of $\mathbb{C}^2$, where,

$$|a_0\rangle = |0\rangle\,, \qquad |b_0\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\,, \qquad |c_0\rangle = \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle\,,$$
$$|a_1\rangle = |1\rangle\,, \qquad |b_1\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\,, \qquad |c_1\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle\,.$$

Suppose the first qubit of the state $|\psi\rangle$ is measured in the basis $\mathcal{A}$ and the second qubit is measured in the basis $\mathcal{B}$. Let $P_\psi(\mathcal{A},\mathcal{B})$ denote the probability that these two measurements either yield the outcome (a_0, b_0) or the outcome (a_1, b_1). Probabilities $P_\psi(\mathcal{B},\mathcal{C})$ and $P_\psi(\mathcal{C},\mathcal{A})$ are defined analogously.

(a) Give expressions for $P_\psi(\mathcal{A},\mathcal{B})$, $P_\psi(\mathcal{B},\mathcal{C})$ and $P_\psi(\mathcal{C},\mathcal{A})$.

(b) What transformations relate (i) the basis $\mathcal{A}$ to the basis $\mathcal{B}$, and (ii) the basis $\mathcal{A}$ to the basis $\mathcal{C}$? Show that

$$\frac{|a_0a_0\rangle + |a_1a_1\rangle}{\sqrt{2}} = \frac{|b_0b_0\rangle + |b_1b_1\rangle}{\sqrt{2}} = \frac{|c_0c_0\rangle + |c_1c_1\rangle}{\sqrt{2}}\,. \qquad (*)$$

(c) For $|\psi\rangle = |0\rangle \otimes |0\rangle$, show that

$$P_\psi(\mathcal{A},\mathcal{B}) + P_\psi(\mathcal{B},\mathcal{C}) + P_\psi(\mathcal{C},\mathcal{A}) \geqslant 1\,.$$

(d) Denote the state in equation $(*)$ by $|\phi^+\rangle$. For $|\psi\rangle = |\phi^+\rangle$, determine $P_\psi(\mathcal{A},\mathcal{B}) + P_\psi(\mathcal{B},\mathcal{C}) + P_\psi(\mathcal{C},\mathcal{A})$.

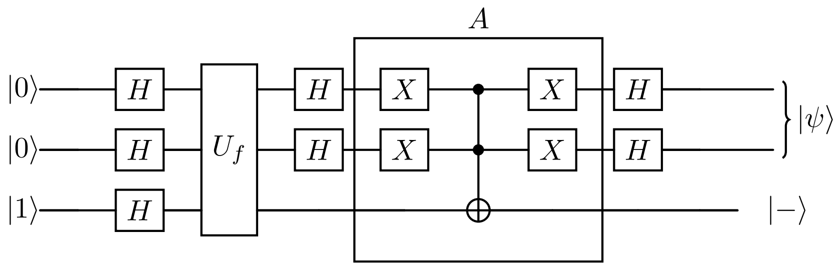
Paper 3, Section I

10C Quantum Information and Computation

Suppose you have a search space of dimension 4, with its elements encoded in binary $\{00, 01, 10, 11\}$. You are searching for the element $x_0 = 11$.

(a) Construct the circuit implementing the quantum oracle $U_f : |x\rangle |y\rangle \rightarrow |x\rangle |y \oplus f(x)\rangle$, for the function f, where $f(x) = 1$ if $x = x_0$, otherwise $f(x) = 0$.

(b) Consider the following quantum circuit:



(i) Prove that the boxed part, A, of the circuit implements the operator $I_0 = I - 2|00\rangle\langle 00|$, by showing that $A\,|x_1 x_2\rangle\,|-\rangle = I_0\,|x_1 x_2\rangle\,|-\rangle$ for $x_1, x_2 \in \{0, 1\}$. Hence, justify that the entire circuit implements the initial Hadamard transformations and a single Grover iteration $-Q = H^{\otimes 2} I_0 H^{\otimes 2} I_{x_0}$, where $I_{x_0} = I - 2\,|x_0\rangle\,\langle x_0|$.

(ii) Compute the output state $|\psi\rangle$.

(iii) What happens when we measure $|\psi\rangle$ in the computational basis?

(iv) How many times do we have to repeat $-Q$ to obtain x_0 in this example?

Paper 4, Section I

10C Quantum Information and Computation

(a) A *GHZ state* between three parties (Alice, Bob, and Charlie) is defined as

$$|\psi\rangle_{ABC} := \frac{1}{\sqrt{2}}(|000\rangle_{ABC} + |111\rangle_{ABC}) .$$

Write down a quantum circuit that generates this state from the initial state $|000\rangle_{ABC}$, justifying your answer.

(b) The aim below is to design a multi-party super-dense coding protocol between Alice, Bob, and Charlie, who are spatially separated and share the GHZ state $|\psi\rangle_{ABC}$ before communication begins.

Suppose that Alice wants to send two (classical) bits to Charlie and Bob wants to send one (classical) bit to Charlie. What operations should Alice and Bob perform on their respective qubits of $|\psi\rangle_{ABC}$ before sending their qubits to Charlie (over ideal qubit channels), and how does Charlie infer the classical bits sent by Alice and Bob? Be sure to consider all cases.

[The following orthonormal basis of 3 qubits will be useful,

$$\begin{aligned}
\left|\chi_1^{\pm}\right\rangle &:= \frac{1}{\sqrt{2}}(|000\rangle_{ABC} \pm |111\rangle_{ABC}) \\
\left|\chi_2^{\pm}\right\rangle &:= \frac{1}{\sqrt{2}}(|010\rangle_{ABC} \pm |101\rangle_{ABC}) \\
\left|\chi_3^{\pm}\right\rangle &:= \frac{1}{\sqrt{2}}(|001\rangle_{ABC} \pm |110\rangle_{ABC}) \\
\left|\chi_4^{\pm}\right\rangle &:= \frac{1}{\sqrt{2}}(|011\rangle_{ABC} \pm |100\rangle_{ABC}) .]
\end{aligned}$$

Paper 2, Section II

15C Quantum Information and Computation

Let N be an odd integer that is not equal to the power of a prime number. Let a be an integer coprime to N with $1 < a < N$.

(a) Define the *order* of $a \bmod N$.

(b) Let $f : \mathbb{Z} \to \mathbb{Z}_N$ be the modular exponentiation function that has period r equal to the order of $a \bmod N$. Write down an explicit form for f and show that it is one-to-one within each period.

(c) Suppose r from part (b) is even and $(a^{r/2} + 1)$ is not divisible by N. How can one use Euclid's algorithm to obtain a factor of N? Justify your answer.

(d) Continuing from part (c), let m be the smallest integer for which $2^m > N^2$, and let B and b be integers such that $2^m = Br + b$ with $B = \lfloor \frac{2^m}{r} \rfloor$. Consider the state

$$|\varphi_1\rangle = \frac{1}{\sqrt{A}} \sum_{j=0}^{A-1} |x_0 + jr\rangle \,,$$

where $x_0 \in \{0, 1, \ldots, 2^m - 1\}$ and $A = \begin{cases} B+1 & \text{if } x_0 \leqslant b \\ B & \text{if } x_0 > b \end{cases}$.

(i) For a positive integer M, let QFT_M denote the quantum Fourier transform modulo M. Give the action of QFT_M on the state $|x\rangle$, where $x \in \mathbb{Z}_M$.

(ii) Show that

$$|\varphi_2\rangle := \mathrm{QFT}_{2^m} |\varphi_1\rangle = \sum_{u=0}^{2^m-1} g(u) \, |u\rangle \,,$$

and give a closed-form expression for $g(u)$.

(iii) Suppose $|\varphi_2\rangle$ is measured in the basis $\{|u\rangle\}_{u=0}^{2^m-1}$ to obtain a value of c satisfying

$$\left| c - k\frac{2^m}{r} \right| < \frac{1}{2},$$

for some $k \in \{0, 1, 2, \ldots, r-1\}$ that is coprime to r. Prove that there is at most one fraction k/r with a denominator $r < N$ satisfying

$$\left| \frac{c}{2^m} - \frac{k}{r} \right| < \frac{1}{2N^2} \,. \qquad (*)$$

(iv) Suppose $N = 21$, $a = 10$, and you get the measured outcome $c = 427$. Using equation $(*)$ and a suitable continued fraction expansion, find r.

[*Hint:* $\left|\frac{427}{512} - \frac{5}{6}\right| < \frac{1}{2(21)^2}$]

Paper 3, Section II

15C Quantum Information and Computation

(a) Given two equally likely states $|\alpha_0\rangle$ and $|\alpha_1\rangle$, show that the probability P_s of correctly distinguishing between the states using a quantum measurement is bounded as follows,

$$P_s \leqslant \frac{1}{2}\left(1 + \sqrt{1 - |\langle\alpha_0|\alpha_1\rangle|^2}\right),$$

and that the bound is tight. Consequently, show that $|\alpha_0\rangle$ and $|\alpha_1\rangle$ can be perfectly distinguished, that is, $P_s = 1$, if and only if they are orthogonal.

(b) Consider the task of distinguishing between two equally likely unitary gates U_1 and U_2. This is accomplished by choosing some state $|\psi\rangle$ and then distinguishing between the outputs $U_1|\psi\rangle$ and $U_2|\psi\rangle$ as in part (a). Let us define the numerical range of a unitary U as the following subset of the complex plane

$$N(U) := \{\langle\psi|U|\psi\rangle \ : \ ||\psi|| = 1\} \subseteq \mathbb{C}.$$

Show that U_1 and U_2 can be perfectly distinguished if and only if $0 \in N(U_2^\dagger U_1)$.

(c) Denote the spectrum (the set of all eigenvalues) of a unitary U by $\operatorname{spec} U$. Show that the spectrum of any unitary matrix U is contained in the unit circle in the complex plane: $\operatorname{spec} U \subseteq \{\lambda \in \mathbb{C} \ : \ |\lambda| = 1\}$.

(d) The numerical range of a unitary U is equal to the convex hull of its eigenvalues; that is, if $\operatorname{spec} U = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, we have

$$N(U) = \operatorname{conv}(\operatorname{spec} U) = \left\{\sum_{i=1}^{n} p_i \lambda_i \ : \ p_i \geqslant 0, \sum_i p_i = 1\right\}.$$

Use this to draw a sketch of the numerical range of the unitary matrix

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\pi/4} & 0 \\ 0 & 0 & e^{i3\pi/4} \end{bmatrix}.$$

(e) The spectral arc length $\theta(U) \in [0, 2\pi)$ of a unitary U is the length of the smallest arc (in radians) that contains all the eigenvalues of U on the unit circle. Show by means of two figures that for the unitary phase gate

$$U_\gamma = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\gamma} \end{bmatrix},$$

where $\gamma \in [0, 2\pi)$, the spectral arc length is given by

$$\theta(U_\gamma) = \begin{cases} \gamma, & \text{if } \gamma < \pi \\ 2\pi - \gamma, & \text{if } \gamma \geqslant \pi\,. \end{cases}$$

(f) Using parts (b)–(e), justify that two equally likely unitary gates U_1 and U_2 can be perfectly distinguished if and only if $\theta(U_2^\dagger U_1) \geqslant \pi$.

Paper 1, Section II

19J Representation Theory

Let G be the (infinite) group generated by two elements r and t such that $trt^{-1} = r^{-1}$, $t^2 = 1$ and with all other relations a consequence of these.

(a) Let V be a finite dimensional complex representation of G. Show that if V is irreducible, then $\dim V \leqslant 2$.

(b) Find all one dimensional complex representations of G, and find all irreducible two dimensional complex representations of G up to isomorphism.

(c) For every positive integer $n \geqslant 3$, there is a surjective homomorphism $G \to D_{2n}$ to the dihedral group of order $2n$. Using this, we can regard a representation of D_{2n} as a representation of G. Which of the irreducible finite dimensional representations of G do *not* arise in this way?

(d) Show that the following 2×2 matrices can be used to construct a two dimensional representation of G containing a one dimensional subrepresentation that has no G-invariant complement:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Paper 2, Section II

19J Representation Theory

(a) Let $\rho : G \to GL_n(\mathbb{C})$ be a representation of a finite group G. Prove that ρ is isomorphic to a representation $\rho' : G \to GL_n(\mathbb{C})$ with

$$\rho'(G) \leqslant U_n = \{A \in \mathrm{Mat}_n(\mathbb{C}) \mid A\overline{A^T} = I\}.$$

(b) Let V be a finite dimensional complex representation of a group G. A bilinear form

$$(-,-) : V \times V \to \mathbb{C}$$

on V is G-invariant if $(v,w) = (gv,gw)$ for all $v,w \in V$ and $g \in G$.

Suppose now that V is an irreducible representation of G.

(i) Show that any G-invariant bilinear form on V is either non-degenerate or zero, and that any two G-invariant bilinear forms are proportional.

(ii) Show that any non-zero G-invariant bilinear form satisfies $(w,v) = \lambda(v,w)$ for all $v,w \in V$, where $\lambda \in \{\pm 1\}$ does not depend on v and w.

(c) Let $H \leqslant G$ be a subgroup of a finite group G, and V a finite dimensional complex representation of H. Define the *induced representation* $\mathrm{Ind}_H^G(V)$, and compute its character in terms of the character of V.

Show that if W is a finite dimensional complex representation of G, then the representations $\mathrm{Ind}_H^G(W \otimes V)$ and $W \otimes \mathrm{Ind}_H^G(V)$ are isomorphic.

Paper 3, Section II

19J Representation Theory

Let V_n be the vector space of homogeneous polynomials in x, y of degree n over the complex numbers.

(a) Define the *standard action* of $G = SU_2$ on V_n.

Write down the matrix by which an element of G acts on V_3, with respect to the standard basis $\{x^i y^j : i \geqslant 0, j \geqslant 0, i + j = 3\}$ of V_3.

Define the *character* of a finite dimensional complex representation V of G and write down the character of V_n.

(b) Show every finite dimensional complex representation V of G is isomorphic to V^*.

(c) Show that for every irreducible finite dimensional complex representation V of G the action of G on $V \otimes V$ factors through $G/\{\pm I\}$. Is this true for complex representations which are not irreducible?

(d) Decompose $V_n \otimes V_n$ into irreducibles.

(e) For any finite dimensional complex representation V of G, compute the character of $\bigwedge^2 V$ in terms of the character of V.

(f) Decompose $\bigwedge^2 V_n$ into irreducibles.

[*You must justify or prove your answers. You may use any results from lectures, but you must quote them carefully. In part (d) you may not just quote the Clebsch–Gordon formula.*]

Paper 4, Section II

19J Representation Theory

Let p be an odd prime, $G = SL_2(\mathbb{F}_p)$ be the special linear group over the field with p elements,

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \, a, b \in \mathbb{F}_p, a \neq 0 \right\} < G$$

be the subgroup of upper triangular matrices and

$$U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \, b \in \mathbb{F}_p \right\} < B$$

be the subgroup of uni-triangular matrices.

(a) Suppose that $\theta, \varphi \colon B \to \mathbb{C}^*$ are 1-dimensional complex representations of B.

(i) State Mackey's restriction formula and explain carefully what it says for $\mathrm{Res}_B^G \mathrm{Ind}_B^G \theta$.

(ii) Determine $\langle \mathrm{Ind}_B^G \theta, \mathrm{Ind}_B^G \varphi \rangle_G$ for all possible choices of θ and φ.

(b) Let $\chi : U \to \mathbb{C}^*$ be a non-trivial one dimensional representation of U.

(i) If $v \in \mathbb{F}_p$ show that

$$\chi(v \cdot) : \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto \chi\left(\begin{pmatrix} 1 & vx \\ 0 & 1 \end{pmatrix} \right)$$

is also a one dimensional representation of U.

(ii) Now consider any representation V of B. Show that if $\langle \mathrm{Res}_U^B V, \chi \rangle_U \neq 0$, then $\langle \mathrm{Res}_U^B V, \chi(v \cdot) \rangle_U \neq 0$ for at least $\frac{p-1}{2}$ elements v in $\mathbb{F}_p^*$.

(iii) Let T be the subgroup of B consisting of diagonal matrices and let θ be a one dimensional representation of T. Show that $\mathrm{Ind}_T^B \theta$ is a sum of three pairwise non-isomorphic irreducible representations of B of dimensions 1, $\frac{p-1}{2}$ and $\frac{p-1}{2}$.

Paper 1, Section II
24G Riemann Surfaces

Give the definition of a *Riemann surface.*

If R is a Riemann surface, show that any open connected subset of R is also a Riemann surface. Show also that if $z_1, \ldots, z_p \in R$ then $R \setminus \{z_1, \ldots, z_p\}$ is a Riemann surface. Can it happen that a Riemann surface R with a countably infinite set of points removed is still a Riemann surface?

Which of the following topological spaces can be given the structure of a Riemann surface? Justify your answers.

(i) The unit sphere $S^2 = \{(x,y,z) \,|\, x^2+y^2+z^2=1\}$ in $\mathbb{R}^3$.

(ii) The set X of points $\{(x, y, x/\sqrt{x^2+y^2}, y/\sqrt{x^2+y^2})\}$ in $\mathbb{R}^4$ where x and y are not both zero.

(iii) The set Y of points $\{(z,w) \in \mathbb{C} \times \mathbb{C} \,|\, zw - 2iw - iz - 2 = 0\}$.

Paper 2, Section II
24G Riemann Surfaces

State and prove the identity theorem for Riemann surfaces.

Define what it means for $h : U \to \mathbb{R}$ to be a *harmonic function*, where U is a non-empty open connected subset of $\mathbb{R}^2$. Show that $h \in C^\infty(U)$.

Define also a *harmonic function* $H : R \to \mathbb{R}$, where R is a Riemann surface. Show that this is independent of the atlas chosen for R.

Suppose we have two functions $f, g : \mathbb{C} \to \mathbb{C}$ such that the product $f \cdot g$, defined pointwise by $(f \cdot g)(z) = f(z)g(z)$, is identically zero on $\mathbb{C}$. Must one of f or g be identically zero on $\mathbb{C}$ if:

(i) Both f and g are continuous on $\mathbb{C}$?

(ii) Both f and g are continuous on $\mathbb{C}$ and never simultaneously zero?

Now suppose we have two functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ such that $f \cdot g$ is identically zero on $\mathbb{R}^2$. Must one of f or g be identically zero on $\mathbb{R}^2$ if:

(iii) Both f and g are in $C^\infty(\mathbb{R}^2)$?

(iv) Both f and g are harmonic?

Paper 3, Section II

23G Riemann Surfaces

Given a Riemann surface R and a covering map $\pi : S \to R$, where S is a connected Hausdorff topological space, explain how S can be given the structure of a Riemann surface such that π is an analytic map.

What does it mean to say that S is *simply connected*? State the uniformisation theorem and write down the group of analytic automorphisms $\mathrm{Aut}(R)$ for each simply connected Riemann surface R.

If X is a topological space and G is a group of homeomorphisms of X, define what it means to say that this action of G on X is a *covering space action*.

If R is the Riemann surface $\mathbb{C}_\infty$ and H is a subgroup of $\mathrm{Aut}(R)$ whose action on R is a covering space action, show that the quotient R/H is a Hausdorff space.

Give an example of a Riemann surface R and a group G of homeomorphisms of R whose action is a covering space action but such that the quotient space R/G is not Hausdorff. Must R/G be Hausdorff if R is simply connected?

Paper 1, Section I

5K Statistical Modelling

A random variable $Y > 0$ is said to follow the Weibull distribution with parameters $\lambda > 0$ and $k > 0$ if $X = (Y/\lambda)^k$ follows the exponential distribution with rate parameter 1 (so the probability density function of X is e^{-x}, $x > 0$). Let $Y_1, \ldots, Y_n$ be an independent and identically distributed sample from this Weibull distribution.

Show that the probability density function of Y is given by

$$f_\lambda(y) = \frac{k}{\lambda}\left(\frac{y}{\lambda}\right)^{k-1} e^{-(y/\lambda)^k},\ y > 0.$$

For the rest of this question, suppose k is fixed. Show that $\{f_\lambda\ ;\ \lambda > 0\}$ is a one-parameter exponential family, and write down its natural parameter and sufficient statistic.

Show that $\mathbb{E}(Y^k) = \lambda^k$, then find the maximum likelihood estimator of λ from $Y_1, \ldots, Y_n$.

Paper 2, Section I
5K Statistical Modelling

The `potato` dataset contains the crop yields of 60 equally divided plots in a farm, each randomly planted with one of three genotypes of potato. There are two possible alleles (`A`, `a`) and three possible genotypes (`aa`, `Aa`, `AA`). The `count` column counts the number of `A` alleles in `genotype`. Consider the following `R` code with truncated output.

```
> potato[c(1, 20, 21, 40, 41, 60), ]
   genotype count     yield
1        aa     0  9.038067
20       aa     0 10.199812
21       Aa     1 10.421516
40       Aa     1 11.793761
41       AA     2 12.786507
60       AA     2 11.214858

> summary(model1 <- lm(yield ~ genotype - 1, potato))
Coefficients:
           Estimate Std. Error t value Pr(>|t|)
genotypeaa   9.8328     0.2062   47.68   <2e-16 ***
genotypeAa  11.0535     0.2062   53.60   <2e-16 ***
genotypeAA  11.8059     0.2062   57.24   <2e-16 ***

> summary(model2 <- lm(yield ~ I(count >= 1), potato))
Coefficients:
                  Estimate Std. Error t value Pr(>|t|)
(Intercept)         9.8328     0.2161  45.511  < 2e-16 ***
I(count >= 1)TRUE   1.5969     0.2646   6.035 1.19e-07 ***

> anova(model2, model1)
Analysis of Variance Table
Model 1: yield ~ I(count >= 1)
Model 2: yield ~ genotype - 1
  Res.Df    RSS Df Sum of Sq      F  Pr(>F)
1     58 54.148
2     57 48.487  1    5.6612 6.6551 0.01249 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

After introducing necessary mathematical notation, write down the statistical models fitted above with all assumptions that are used to calculate the p-values. Which hypothesis below is tested in the analysis of variance, and what can you conclude about it? Write down the `R` code to test the other hypothesis using analysis of variance.

1. Full dominance: the effect of genotype on crop yield only depends on whether the genotype contains `A` allele or not.
2. No dominance: the effect of genotype on crop yield only is linear in the number of `A` alleles.

Paper 3, Section I

5K Statistical Modelling

Define the *generalized linear model* in its most general form as introduced in the lectures, which should include a link function, a dispersion parameter, and known weights on the data points. Your answer should clearly describe the mathematical assumptions on different components of the model. Write down the log-likelihood function of this model (up to an additive constant).

What is the canonical link function for Poisson generalized linear models? Justify your answer.

Paper 4, Section I

5K Statistical Modelling

Define *Akaike's Information Criterion* (AIC) for a general statistical model.

Consider the normal linear model $Y \sim \mathrm{N}(X\beta, \sigma^2 I_n)$ where $\beta \in \mathbb{R}^p$ is unknown, $\sigma^2 > 0$ is known, and $X \in \mathbb{R}^{n\times p}$ is non-random with full column rank. Show that the AIC in this model is equal to Mallows' C_p (up to constants)

$$C_p = \|Y - \hat{\mu}\|^2 + 2p\sigma^2,$$

where $\hat{\mu}$ is the fitted value of Y using ordinary least squares.

The mean squared prediction error (MSPE) of $\hat{\mu}$ is defined as

$$\mathrm{MSPE} = \mathbb{E}(\|Y^* - \hat{\mu}\|^2),$$

where Y^* is an independent and identically distributed copy of Y. Show that C_p is an unbiased estimator of the MSPE.

Paper 1, Section II

13K Statistical Modelling

Suppose $(X_1, Y_1), \ldots, (X_n, Y_n) \in \mathbb{R}^p \times \mathbb{R}$ are independent and identically distributed and the conditional distribution of Y_i given X_i is given by

$$Y_i \mid X_i \sim \mathrm{N}(X_i^T \beta, \sigma^2 v(X_i)),$$

where $v(x) > 0$ is a known function. We would like to use a sample of $(X_1, Y_1), \ldots, (X_n, Y_n)$ to estimate the unknown parameters $\beta \in \mathbb{R}^p$ and $\sigma^2 > 0$. Let $X \in \mathbb{R}^{n \times p}$ denote the matrix with the ith row being X_i and let $Y = (Y_1, \ldots, Y_n)^T \in \mathbb{R}^n$.

(a) Show that the maximum likelihood estimator of β is given by

$$\hat{\beta}(\Sigma) = (X^T \Sigma^{-1} X)^{-1} X^T \Sigma^{-1} Y,$$

where $\Sigma \in \mathbb{R}^{n \times n}$ is a diagonal matrix with the ith diagonal entry given by $v(X_i)$.

(b) Explain why the `R` code below returns the estimator in (a), where `X`, `Y`, and `Sigma` in the `R` environment store the value of X, Y, and Σ in the above model.

```
Y.tilde <- Y / sqrt(diag(Sigma))
X.tilde <- X / sqrt(diag(Sigma))
fit1 <- lm(Y.tilde ~ X.tilde - 1)
fit1$coefficients
```

Write down `R` code that returns the estimator $\hat{\beta}(I_n)$, where $I_n \in \mathbb{R}^{n \times n}$ denote the identity matrix.

(c) Sketch an argument that shows both $\hat{\beta}(\Sigma)$ and $\hat{\beta}(I_n)$ are consistent for estimating β and are asymptotically normal when $n \to \infty$. You may assume that the law of large numbers and the central limit theorem can be used for this model.

(d) Suppose $p = 1$. Find an expression for

$$\rho = \lim_{n \to \infty} \frac{\mathrm{Var}(\hat{\beta}(\Sigma))}{\mathrm{Var}(\hat{\beta}(I_n))}.$$

Your answer should depend on the distribution of X_1.

(e) Do you expect ρ to be $\geqslant 1$ or $\leqslant 1$? Explain why. Prove it using your expression of ρ in (d).

Paper 4, Section II

13K Statistical Modelling

A three-year study was conducted at three sites on the survival status of patients suffering from cancer. The dataset also records whether or not the initial tumour was malignant. The data are tabulated in `R` as follows:

```
> cancer
  site malignant survive die total
1    A        no      40   7    47
2    A       yes      36  17    53
3    B        no      24   3    27
4    B       yes      35   6    41
5    C        no      15   4    19
6    C       yes       5   5    10
```

(a) Write down the mathematical model that is being fitted by the following `R` commands.

```
> fit1 <- glm(survive/total ~ site + malignant, family = binomial,
+              data = cancer, weights = total)
```

(b) In words or using mathematical equations, explain the (slightly abbreviated) output from the code below and describe how the numbers in the `Coefficients` table are computed. What are your conclusions based on the hypothesis tests in this table?

```
> summary(fit1)
Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept)    1.6855     0.3431   4.913 8.98e-07 ***
siteB          0.8096     0.4344   1.864   0.0624 .
siteC         -0.5423     0.4825  -1.124   0.2610
malignantyes  -0.9048     0.3809  -2.375   0.0175 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
    Null deviance: 11.69300  on 5  degrees of freedom
Residual deviance:  0.85048  on 2  degrees of freedom
AIC: 29.003
```

[QUESTION CONTINUES ON THE NEXT PAGE]

(c) Consider a slightly different model fitted by the next `R` commands and the corresponding (abbreviated) `summary` output below. Explain why some of the p-values (under the column `Pr(>|z|)`) are the same in these two tables and others are different. Are you surprised that the p-value for `siteC` is significant in `summary(fit1)` (at level 0.05) but not significant in `summary(fit2)`? Explain your answer.

```
> cancer$site <- factor(cancer$site, levels = c("B", "A", "C"))
> fit2 <- glm(survive/total ~ site + malignant, family = binomial,
+             data = cancer, weights = total)
> summary(fit2)
Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept)    2.4951     0.4610   5.413 6.21e-08 ***
siteA         -0.8096     0.4344  -1.864   0.0624 .
siteC         -1.3520     0.5613  -2.409   0.0160 *
malignantyes  -0.9048     0.3809  -2.375   0.0175 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

(d) Describe the hypothesis test performed in the `R` code below and your conclusion based on the results.

```
> fit3 <- glm(survive/total ~ malignant, family = binomial,
            data = cancer, weights = total)
> anova(fit3, fit1)
Analysis of Deviance Table

Model 1: survive/total ~ malignant
Model 2: survive/total ~ site + malignant
  Resid. Df Resid. Dev Df Deviance
1         4     7.4923
2         2     0.8505  2   6.6418

> qchisq(c(0.01, 0.05, 0.1, 0.9, 0.95, 0.99), 4)
[1]  0.2971095  0.7107230  1.0636232  7.7794403  9.4877290 13.2767041
> qchisq(c(0.01, 0.05, 0.1, 0.9, 0.95, 0.99), 2)
[1] 0.02010067 0.10258659 0.21072103 4.60517019 5.99146455 9.21034037
```

Paper 1, Section II

36C Statistical Physics

(a) What is the definition of a *partition function*? Explain why this quantity is useful to calculate.

(b) A spherical, hard planet with surface area A has an ideal-gas atmosphere consisting of N atoms, each with mass m and no internal degrees of freedom. The Hamiltonian for each atom is

$$H = \frac{p^2}{2m} + mgz,$$

where $z > 0$ is the height above the surface. Assume that the gravitational acceleration g is a constant and that the density of the gas becomes negligible at a height which is still small compared to the radius of the planet. The gas is in thermal equilibrium at temperature T. Calculate the following quantities for the atmosphere:

(i) The expected total energy $\langle E \rangle$ and the fractional fluctuations $\Delta E / \langle E \rangle$.

(ii) The average height $\langle z \rangle$ of an atom and the atmospheric pressure $p(z)$ considered as a function of height.

(iii) The entropy S, using the approximate form of Stirling's formula,

$$\ln N! \approx N \ln N - N.$$

Express your final answer in terms of the thermal wavelength $\lambda = \sqrt{\dfrac{2\pi\hbar^2}{mk_BT}}$, as well as the variables A, N, and $\langle z \rangle$. Comment on its relation to the Sackur-Tetrode equation for a gas in a box of volume V,

$$S = Nk \left(\ln \frac{V}{\lambda^3 N} + \frac{5}{2} \right).$$

Paper 2, Section II

37C Statistical Physics

(a) Starting with the first law of thermodynamics for the energy E, derive a formula for the variation of the enthalpy H. Define the temperature T and volume V as derivatives of H. From this, deduce the Maxwell relation

$$\left.\frac{\partial T}{\partial p}\right|_S = \left.\frac{\partial V}{\partial S}\right|_p ,$$

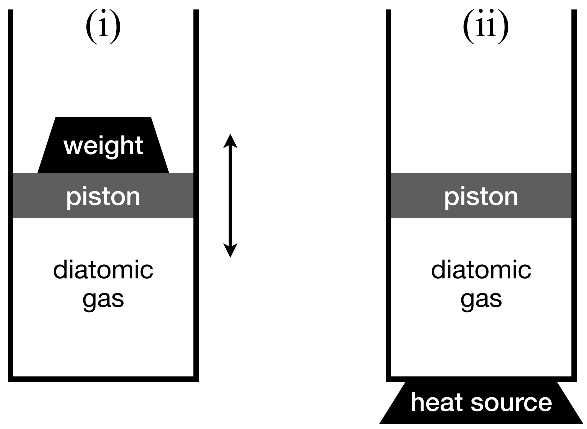
where p is the pressure and S is the entropy.

(b) Determine the enthalpy H of a diatomic ideal gas in terms of N and T, where the temperature T lies in a range for which vibrations of the molecule freeze out but rotations do not.

(c) A freely moving piston is inserted in a cylindrical container. The chamber below the piston contains the diatomic gas from part (b), which is at an initial temperature T_0 and thermally insulated from the outside environment. This piston is initially at a pressure p_0 due to the outside atmosphere. Consider the following two processes and calculate the change of temperature, ΔT, in each case. [See figures depicting the two processes below.]

(i) A weight is placed onto the piston, thereby increasing the pressure to p_1 on the piston. [You may assume this process is adiabatic.]

(ii) The pressure is held fixed while the base of the cylinder is heated such that an amount of heat Q is gradually added to the gas in the cylinder. As a result, the gas does some amount of work W on the piston, pushing it upward. [*Hint: use enthalpy.*]



Paper 3, Section II

35C Statistical Physics

This question concerns a Fermi gas of non-relativistic, non-interacting electrons confined to a two-dimensional planar sheet with total area A, at chemical potential $\mu > 0$. Note that the electron has $g_s = 2$ spin states.

(a) Calculate the density of states $g(E)$ of the 1-particle system (including the numerical coefficient). Write down the Fermi–Dirac distribution.

(b) Suppose the system is at absolute zero ($T = 0$). Show that the total energy of the system is $E_{\text{tot}} = \frac{1}{2} N E_F$, where E_F is the Fermi energy and N is the number of electrons. Calculate the degeneracy pressure p.

(c) Show that at finite temperature $T > 0$, the change in the number of electrons N relative to the $T = 0$ state is given by

$$\Delta N = X \int_{-E_F}^{\infty} dE' \left[\operatorname{sgn}(\beta E'/2) - \tanh(\beta E'/2) \right],$$

where $E' = E - E_F$, $\operatorname{sgn}(x) = x/|x|$ is the sign function, and X is a coefficient which you should determine. Explain why, at low temperatures, $\Delta N \approx 0$ to a very good approximation. Work out the corresponding formula for the change of the total energy ΔE_{tot}, and use its scaling with respect to β to show that

$$\Delta E_{\text{tot}} \propto T^2.$$

[You need not work out the constant of proportionality.]

Paper 4, Section II

35C Statistical Physics

(a) Define the *latent heat* L between the gas and liquid phases of a substance. Starting with the Gibbs free energy, derive the Clausius–Clapeyron relation for the first-order phase transition in the (T, p) plane

$$\frac{dp}{dT} = \frac{L}{T(V_{\text{gas}} - V_{\text{liq}})} .$$

What happens to L at the critical point?

(b) Consider a chain of N spin-1 atoms in an external magnetic field B, each with Hamiltonian

$$H = -\mu B s_z ,$$

where μ is a constant, and $s_z \in \{-1, 0, 1\}$. Suppose that the spin chain is in a canonical ensemble with inverse temperature β. Calculate the free energy F and the heat capacity C. What is the high-temperature limit of $-\beta F$ and C?

(c) Consider the same system as in part (b), but now suppose that the sign of the external magnetic field B is instantly reversed by an experimenter. What happens to each of β, F, and C?

Explain why the resulting system cannot be in thermal equilibrium with any gas. If the system is coupled to a gas, in which direction will heat flow? Your answers should make reference to the appropriate law(s) of thermodynamics.

Paper 1, Section II

30K Stochastic Financial Models

Let Z be a square-integrable random vector in $\mathbb{R}^n$ with $\mathbb{E}(Z) = b$ and $\mathrm{Cov}(Z) = V$. Assume that the $n \times n$ matrix V is positive definite.

(a) Find the vector $\theta \in \mathbb{R}^n$ to maximise $\mathbb{E}(X) - \frac{1}{2}\mathrm{Var}(X)$ subject to $X = \theta^\top Z$.

(b) Let θ_{M} be the maximiser from part (a). Consider the problem of maximising $F(\mathbb{E}(X), \mathrm{Var}(X))$ subject to $X = \theta^\top Z$, where $F(\cdot, \sigma^2)$ is strictly increasing for all σ^2 and $F(m, \cdot)$ is strictly decreasing for all m. Assuming that a maximiser θ^* exists, show that it is of the form $\theta^* = \lambda\, \theta_{\mathrm{M}}$ where λ is a non-negative real number.

(c) Set $X_{\mathrm{M}} = \theta_{\mathrm{M}}^\top Z$. For real constants α and β and for vectors $\varphi \in \mathbb{R}^n$, consider the expression

$$\mathbb{E}[(\alpha + \beta X_{\mathrm{M}} - Y)^2]$$

where $Y = \varphi^\top Z$. For any fixed φ, find the values α and β that minimise this expression, and show that the minimum equals $\varphi^\top Q \varphi$ for a symmetric matrix Q that you should identify.

Paper 2, Section II

30K Stochastic Financial Models

Let $(Z_n)_{0 \leqslant n \leqslant N}$ be a real-valued process adapted to the filtration $(\mathcal{F}_n)_{0 \leqslant n \leqslant N}$ where $\mathcal{F}_0$ is trivial and $N < \infty$ is not random. Suppose $\mathbb{E}(|Z_n|) < \infty$ for all $0 \leqslant n \leqslant N$. Let $(V_n)_{0 \leqslant n \leqslant N}$ be a supermartingale such that $V_n \geqslant Z_n$ almost surely for all $0 \leqslant n \leqslant N$.

(a) Show that $\mathbb{E}(Z_\tau) \leqslant V_0$ for any stopping time τ. [You may use any result from the course if carefully stated.]

(b) Let

$$A_n = \sum_{k=0}^{n-1} \left[V_k - \mathbb{E}(V_{k+1} | \mathcal{F}_k) \right]$$

for $1 \leqslant n \leqslant N$. Show that $(A_n)_{1 \leqslant n \leqslant N}$ is previsible and non-decreasing.

(c) Set $A_0 = 0$ and let $M_n = V_n + A_n$ for $0 \leqslant n \leqslant N$. Show that $(M_n)_{0 \leqslant n \leqslant N}$ is a martingale.

(d) Now assume $V_N = Z_N$ and

$$V_n = \max\{Z_n, \mathbb{E}(V_{n+1} | \mathcal{F}_n)\},$$

and set $A_{N+1} = \infty$. Show that $\min\{V_n - Z_n, A_{n+1} - A_n\} = 0$ for all $0 \leqslant n \leqslant N$.

(e) Let $\tau^* = \min\{0 \leqslant n \leqslant N : A_{n+1} > 0\}$. Show that τ^* is a stopping time such that $\mathbb{E}(Z_{\tau^*}) = V_0$.

Paper 3, Section II

29K Stochastic Financial Models

Consider the a discrete-time market model with interest rate r and one stock with time-n price S_n. Suppose $S_0 > 0$ is given and that $S_n = S_{n-1}\xi_n$ for all $n \geqslant 1$, where the stochastic process $(\xi_n)_{n\geqslant 1}$ generates the filtration. Suppose that there are constants $-1 < a < r < b$ such that the random variable ξ_n takes values in $\{1+a, 1+b\}$ and that $0 < \mathbb{P}(\xi_n = 1+b) < 1$ for every $n \geqslant 1$.

(a) Introduce a European call of maturity N and strike K. Use the fundamental theorem of asset pricing to show that the call has a unique time-0 no-arbitrage price $\mathrm{EC}(N,K)$ of the form

$$\mathrm{EC}(N,K) = \sum_{n=0}^{N} w(n,N)\Big(S_0(1+a)^n(1+b)^{N-n} - K\Big)^+$$

where the positive numbers $w(n,N)$ are to be determined in terms of the given constants.

(b) Let $\mathrm{EP}(N,K)$ be the unique time-0 no-arbitrage price of a European put of maturity N and strike K. Show that

$$\mathrm{EP}(N,K) = (1+r)^{-N}K - S_0 + \mathrm{EC}(N,K).$$

(c) Find positive numbers u and v such that

$$\mathrm{EC}(N+1,K) = u\ \mathrm{EC}\left(N, \frac{K}{1+a}\right) + v\ \mathrm{EC}\left(N, \frac{K}{1+b}\right)$$

for all N and K.

(d) A forward start call option is the right, but not the obligation, to buy one share of the stock at time N for the price λS_M, where $0 \leqslant M \leqslant N$ and λ are given constants. Let $\mathrm{FSC}(M,N,\lambda)$ be its unique time-0 no-arbitrage price. Determine a strike K such that the following holds:

$$\mathrm{FSC}(M,N,\lambda) = \mathrm{EC}(N-M,K).$$

Paper 4, Section II

29K Stochastic Financial Models

Let f be a smooth function that grows slowly enough so that all the integrands in this question are integrable.

(a) Let $Z \sim N(0,1)$. Show that

$$\frac{1}{2}\int_0^t \mathbb{E}[f''(\sqrt{s}Z)]ds = \mathbb{E}[f(\sqrt{t}Z)] - f(0)$$

for all $t \geqslant 0$.

(b) What does it mean to say a stochastic process is a *Brownian motion*?

(c) Let $(W_t)_{t\geqslant 0}$ be a Brownian motion and f be a function satisfying the assumptions of part (a). Define a process $(M_t)_{t\geqslant 0}$ by

$$M_t = f(W_t) - \frac{1}{2}\int_0^t f''(W_s)ds.$$

Show that $(M_t)_{t\geqslant 0}$ is a martingale with respect to the filtration generated by $(W_t)_{t\geqslant 0}$.

(d) Let $(W_t)_{t\geqslant 0}$ be a continuous process with $W_0 = 0$ such that for every $c \in \mathbb{R}$ the process $(M_t)_{\geqslant 0}$ is a martingale with respect to the filtration generated by $(W_t)_{t\geqslant 0}$, where

$$M_t = e^{cW_t} - \frac{c^2}{2}\int_0^t e^{cW_s}ds.$$

Show that $(W_t)_{t\geqslant 0}$ is a Brownian motion. [*Hint: You may wish to compute the conditional moment generating function of the increment* $W_t - W_s$.]

Paper 1, Section I

2I Topics in Analysis

State Liouville's theorem on approximation of algebraic numbers by rationals.

Prove that the number $\displaystyle\sum_{n=0}^{\infty} \frac{1}{10^{n^n}}$ is transcendental.

Deduce that there are uncountably many transcendental numbers.

Paper 2, Section I

2I Topics in Analysis

State Chebyshev's equal ripple criterion.

Let T_n be the Chebyshev polynomial of degree n satisfying $T_n(\cos\theta) = \cos(n\theta)$ for all $\theta \in \mathbb{R}$. Determine in terms of T_n a minimizer for $\sup_{-1 \leqslant t \leqslant 1} |t^n - q(t)|$ among all the polynomials q of degree less than n.
[You may assume without proof that the coefficient of $T_n(t)$ at t^n is 2^{n-1}.]

Let f be a polynomial of degree at most n and such that $|f(t)| < 1$ for $-1 \leqslant t \leqslant 1$. By considering the roots of $T_n - f$, or otherwise, show that

$$|f(t)| < \max\{1, |T_n(t)|\}, \text{ for all } t \in \mathbb{R}.$$

Paper 3, Section I

2I Topics in Analysis

State Runge's theorem about the uniform approximation of holomorphic functions by polynomials.

Explain how to explicitly construct a sequence of polynomials converging uniformly to $1/z$ on the semicircle $\{z : |z| = 1,\ \operatorname{Re} z \leqslant 0\}$.

Show that there exists a sequence of polynomials $P_n(z)$ such that

$$P_n(z) \to \begin{cases} 1 & \text{if } |z| < 1 \text{ and } \operatorname{Re} z > 0, \\ 0 & \text{if } |z| < 1 \text{ and } \operatorname{Re} z = 0, \\ -1 & \text{if } |z| < 1 \text{ and } \operatorname{Re} z < 0 \end{cases}$$

pointwise as $n \to \infty$.

Paper 4, Section I

2I Topics in Analysis

Explain how to obtain a *continued fraction* expansion of a real number $x > 0$. Prove that the continued fraction for x terminates if and only if x is rational.

Determine the continued fraction of $\sqrt{3}$.

Paper 2, Section II

11I Topics in Analysis

(a) Let $T \subset \mathbb{R}^2$ be a triangle with $I,\ J,\ K$ the three sides of T and $\partial T = I \cup J \cup K$. Prove that the following two statements are equivalent:

(i) If $A,\ B,\ C$ are closed subsets of $\mathbb{R}^2$ such that $I \subset A$, $J \subset B$, $K \subset C$ and $T \subset A \cup B \cup C$, then $A \cap B \cap C \neq \emptyset$.

(ii) There does not exist a continuous map $f : T \to \partial T$ such that $f(I) \subset I$, $f(J) \subset J$ and $f(K) \subset K$.

(b) State Brouwer's fixed point theorem in the plane. Prove, using Brouwer's fixed point theorem, that there exists a complex number z with $|z| \leqslant 1$ such that $z^6 - 2z^5 + 4z^2 + 9z + 2 = 0$.

(c) Let $I = [-1, 1]$ and let $\beta, \gamma : I \to I \times I$ be continuous paths such that $\beta(-1) = (a, -1)$, $\beta(1) = (b, 1)$ and $\gamma(-1) = (-1, c)$, $\gamma(1) = (1, d)$ with $a, b, c, d \in I$. By considering a suitable continuous map $I \times I \to I \times I$ prove that the paths β and γ intersect.

[If you use the Jordan curve theorem, you must prove it.]

Paper 4, Section II

12I Topics in Analysis

What is a *nowhere dense* set in a metric space? State and prove a version of the Baire category theorem. Deduce the following:

(i) There exists a continuous function $f : [0, 1] \to \mathbb{R}$ that is not monotone on any interval of positive length. [You may assume that the space of continuous real valued functions on $[0, 1]$ with the uniform norm is complete.]

(ii) If $F : \mathbb{R} \to \mathbb{R}$ is an infinitely differentiable function such that for each x there is an n (depending on x) such that $F^{(n)}(x) = 0$, then F is a polynomial.

Paper 1, Section II

40A Waves

Consider small and smooth perturbations of a compressible and homentropic fluid with reference density ρ_0, pressure p_0, and sound speed c_0.

(a) Using the linearized mass and momentum conservation equations, show that the velocity potential ϕ satisfies the wave equation.

(b) Hence derive the energy equation

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{I} = 0,$$

and give expressions for the acoustic energy density E and the acoustic energy flux $\mathbf{I}$.

(c) The fluid occupies the half space $z > 0$, and is bounded by a flexible membrane of negligible thickness and mass at an undisturbed position $z = 0$. Small, smooth acoustic perturbations in the fluid with velocity potential $\phi(x, z, t)$ deflect the membrane to $z = \eta(z, t)$. The membrane is supported by springs that, in the deflected state, exert a restoring force $\mu\eta$ per unit area on the membrane, where μ is a constant.

(i) Show that waves proportional to $\exp[ik(x-ct)]$ and propagating freely along the membrane possess the dispersion relation

$$A^2 \left(\frac{c}{c_0}\right)^4 + \left(\frac{c}{c_0}\right)^2 - 1 = 0,$$

where A is a dimensionless parameter that you must determine.

(ii) Show that the wave's time-averaged acoustic energy flux perpendicular to the membrane $\langle I_z \rangle$ is zero, where you must carefully define the average $\langle \cdot \rangle$.

(iii) Derive approximate expressions for the phase speed c in the two limits $A \ll 1$ and $A \gg 1$, and briefly interpret the two limits.

Paper 2, Section II

40A Waves

Consider the linearized Cauchy momentum equation, which governs small and smooth displacements $\mathbf{u}(\mathbf{x}, t)$ in a uniform, linear, isotropic and elastic solid of density ρ,

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u}, \qquad (\dagger)$$

where λ and μ are the Lamé moduli.

(a) Show that this equation supports two distinct classes of wave motion: P-waves for the dilatation $\theta = \boldsymbol{\nabla} \cdot \mathbf{u}$ with phase speed c_P; and S-waves for the rotation $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}$ with phase speed c_S. You should express c_P and c_S explicitly in terms of the Lamé moduli.

(b) Consider plane-wave solutions to equation (†) of the form $\mathbf{u} = \mathbf{f}(\widehat{\mathbf{k}} \cdot \mathbf{x} - ct)$, where $\widehat{\mathbf{k}}$ is a unit vector. By direct substitution into equation (†), determine the form that $\mathbf{f}$ must take for P-waves and for S-waves, and express the dilatation and rotation in terms of these forms for each class of waves.

[*Hint: You may find the vector identity* $\nabla^2 \mathbf{q} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{q}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{q})$ *useful.*]

(c) A planar interface at $z = 0$ separates two elastic solids of different densities and elastic moduli. A harmonic P-wave with wavevector $\mathbf{k}$ lying in the (x, z) plane is incident from $z < 0$ at an oblique angle. Show in a diagram the directions of all the reflected and transmitted waves, labelled with their polarisations, assuming that none of these waves are evanescent. State the boundary conditions on the components of the displacement and the stress that would, in principle, determine the amplitudes.

(d) Now consider a harmonic P-wave of unit amplitude with $\mathbf{k} = k(\sin\phi, 0, \cos\phi)$ incident on the planar interface $z = 0$ between two elastic and inviscid liquids with wave speed c_P and modulus λ in $z < 0$ and wave speed $\hat{c}_P = 2c_P$ and modulus $\hat{\lambda}$ in $z > 0$. Obtain solutions for the reflected and transmitted waves. Show that the magnitudes of these two waves are equal if

$$\sin^2 \phi = \frac{3Z^2 - 4Z\hat{Z} + \hat{Z}^2}{\hat{Z}(\hat{Z} - 4Z)},$$

where $Z = \lambda / c_P$ and $\hat{Z} = \hat{\lambda} / \hat{c}_P$.

Paper 3, Section II

39A Waves

Let $\phi(x,t)$ be a real-valued function that satisfies the equation

$$\frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial x^2} + A^2c^2\phi = 0,$$

where both A and c are positive constants.

(a) Consider wave solutions of frequency ω and wavenumber k.

(i) Find the dispersion relation for such waves.

(ii) Sketch both the phase velocity c_p and the group velocity c_g as functions of k.

(iii) Do wave crests move faster or slower than a wave packet?

(b) Suppose that $\phi(x,0)$ is real and that

$$\phi(x,0) = \int_{-\infty}^{\infty} a(k)e^{ikx}dk, \quad \frac{\partial}{\partial t}\phi(x,0) = 0,$$

where $a(k)$ is a given function.

(i) Use the method of stationary phase to obtain an approximation for $\phi(Vt,t)$ for fixed $0 \leqslant V < c$ and large t.

[*Hint: You will need the result* $\int_{-\infty}^{\infty} e^{-au^2}\,du = \sqrt{\pi/a}$ *for* $\mathrm{Re}(a) \geqslant 0, a \neq 0$.]

(ii) Now suppose the initial condition is even, so that $\phi(x,0) = \phi(-x,0)$. Consider the limit of large t and deduce an approximation for the sequence of times at which $F(t) = \phi(Vt,t)$ satisfies both $F(t) = 0$ and $F'(t) > 0$.

Paper 4, Section II

39A Waves

A perfect (but unusual) gas occupies a tube that lies parallel to the x-axis. The gas is initially at rest, with density ρ_0, pressure p_0, and specific heat ratio $\gamma = 3$, and occupies the region $x > 0$. At times $t > 0$, a piston, initially at $x = 0$, is pushed into the gas at a constant speed u_1. A shock wave then propagates at a constant speed V into the undisturbed gas ahead of the piston. Downstream of the shock, i.e. in the region between the piston and the shock, the density is $\rho_1 > \rho_0$ and the pressure is $p_1 > p_0$.

(a) Transform into a frame where the shock is at rest and write down the appropriate expressions for conservation of mass, momentum and energy across the shock.

(b) Determine the ratio ρ_1/ρ_0 when the shock moves three times as fast as the piston, i.e. when $u_1 = V/3$.

(c) Determine the corresponding ratio p_1/p_0 when $u_1 = V/3$.

(d) Express V in terms of p_0 and ρ_0 when $u_1 = V/3$.

[*You may assume that the internal energy per unit mass of perfect gas is* $p/[\rho(\gamma - 1)]$.]

END OF PAPER