UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB 2025

List of Courses

Paper 2, Section I

2G Analysis and Topology

For each of the following sequences of functions $f_n : \mathbb{R} \to \mathbb{R}$, determine whether $(f_n)_{n=1}^{\infty}$ converges uniformly, justifying your answer:

i) $f_n(x) = \tanh(nx)$,

ii) $f_n(x) = \sin\left(x + \frac{1}{n}\right)$,

iii) $f_n(x) = \dfrac{\exp\left(x + \frac{1}{n}\right)}{\cosh(ax)}$ (your answer may depend on the value of $a \in \mathbb{R}$),

iv) $f_n(x) = 2n\left[h\left(x + \dfrac{1}{n}\right) - h\left(x - \dfrac{1}{n}\right)\right]$,
where $h : \mathbb{R} \to \mathbb{R}$ is differentiable with h' uniformly continuous.

Paper 4, Section I

2G Analysis and Topology

Let X be a complete, non-empty, metric space and $T : X \to X$. What does it mean to say that T is a contraction? State and prove the contraction mapping theorem.

Suppose that the n^{th} iterate of T (i.e. T applied repeatedly n times), T^n, is a contraction for some $n > 0$. Must T have a fixed point? If so, must it be unique?

Paper 1, Section II

10G Analysis and Topology

Given a metric space (X, d) state what it means for a function $f : X \to \mathbb{R}$ to be uniformly continuous. Let $C_{b,u}(X)$ be the space of bounded, uniformly continuous, functions $f : X \to \mathbb{R}$ equipped with the metric

$$d'(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

Show that $(C_{b,u}(X), d')$ is complete.

Assume $\mathbb{R}^n$ carries the Euclidean metric and let $C_0(\mathbb{R}^n)$ be the space of continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ satisfying $f(x) \to 0$ as $|x| \to \infty$. Show that $C_0(\mathbb{R}^n)$ is a subset of $C_{b,u}(\mathbb{R}^n)$. Is $C_0(\mathbb{R}^n)$ a closed subset of $C_{b,u}(\mathbb{R}^n)$? Is it compact? Justify your answer in each case.

Paper 2, Section II

10G Analysis and Topology

Let X, Y be topological spaces. Briefly describe the product topology on the space $X \times Y$ and show that the maps

$$\begin{array}{rcl} \Pi_X : X \times Y & \to & X \\ (x, y) & \mapsto & x \end{array}, \qquad \text{and} \qquad \begin{array}{rcl} \Pi_Y : X \times Y & \to & Y \\ (x, y) & \mapsto & y \end{array}$$

are continuous.

Let $\Delta = \{(y_1, y_2) \in Y \times Y : y_1 = y_2\}$ be the diagonal in $Y \times Y$. Show that Δ is closed if and only if Y is Hausdorff.

Suppose that $f : X \to Y$ is continuous and Y is Hausdorff. Show that the graph of f

$$\Gamma_f = \{(x, y) \in X \times Y : y = f(x)\}$$

is closed.

Let $f : X \to Y$ where X and Y are both compact Hausdorff spaces and suppose that Γ_f is closed. By considering $\Pi_X(\Gamma_f \cap (X \times C))$ for C a closed set in Y, or otherwise, show that f is continuous.

Give an example of a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ whose graph is closed to show that the previous result need not hold if X and Y are only assumed to be Hausdorff.

[*You may use results from lectures provided they are clearly stated.*]

Paper 3, Section II

11G Analysis and Topology

Let X, Y be topological spaces.

Define what it means for X to be *connected*. Show that if X is connected and $f : X \to Y$ is continuous then $f(X)$ is connected, where $f(X)$ inherits the subspace topology from Y.

Show that X is connected if and only if every continuous map $g : X \to \{0, 1\}$ is constant, where $\{0, 1\}$ carries the discrete topology.

Show that $\mathbb{R}$ with the topology induced by the Euclidean metric is connected. You may assume the intermediate value theorem.

Let $\sim$ be the equivalence relation on $\mathbb{R}$ given by $x \sim y$ if $x - y \in \mathbb{Q}$. Is $\mathbb{R}/\sim$ connected in the quotient topology? Justify your answer.

For $A \subset X$ define

$$\mathrm{Cl}(A) = \bigcap_{E \text{ closed}; A \subset E} E.$$

Suppose A is connected and $A \subset B \subset \mathrm{Cl}(A)$. Show that B is connected.

Paper 4, Section II

10G Analysis and Topology

State what it means for a function $f : \mathbb{R}^n \to \mathbb{R}^m$ to be differentiable at $x \in \mathbb{R}^n$, and define the differential $Df|_x$. You need not establish the uniqueness of the differential.

State the inverse function theorem.

Let $\mathcal{M} = \mathrm{Mat}(n \times n)$ be the space of real square matrices with n rows and n columns, which can be identified with $\mathbb{R}^{n^2}$. Consider the function $F : \mathcal{M} \to \mathcal{M}$ given by

$$F(A) = A^T A.$$

Briefly explain why F is differentiable at A for all $A \in \mathcal{M}$ and determine $DF|_A$. What is $\mathrm{Ker}\, DF|_I$?

Let $\mathcal{O} = \{R \in \mathcal{M} : R^T R = I\}$ and $T = \{B \in \mathcal{M} : B^T + B = 0\}$, each inheriting their topology as a subspace of $\mathcal{M}$.

By considering the map $A \mapsto F(A) + A - A^T - I$, or otherwise, show that there exist open sets $U, V \subset \mathcal{M}$ with $I \in U$ and $0 \in V$, together with a continuously differentiable bijection $\Phi : U \to V$, with continuously differentiable inverse, satisfying $\Phi(U \cap \mathcal{O}) = V \cap T$.

Deduce that every point in $\mathcal{O}$ has an open neighbourhood which is homeomorphic to an open set in T.

Paper 4, Section I

3E Complex Analysis

State and prove the local maximum modulus principle. You may assume the mean value property for holomorphic functions provided it is clearly stated.

Let $D = \{z \in \mathbb{C} : |z| < 1\}$, and suppose $f : D \to D$ is a holomorphic function satisfying $f(0) = 0$. Show that if $\operatorname{Re} f(z) \leqslant \operatorname{Im} f(z)$ for all $z \in D$ then f must be constant. [*You may find it helpful to consider $e^{af(z)}$, where a is a constant to be chosen.*]

Paper 3, Section II

13E Complex Analysis

State and prove Liouville's theorem. You may assume Cauchy's integral formula provided it is clearly stated.

For $R > 0$ let $A = \{z \in \mathbb{C} : |z| > R\}$. Suppose $f : A \to \mathbb{C}$ is holomorphic and satisfies $f(z) \to a$ as $|z| \to \infty$ for some $a \in \mathbb{C}$. Show that $z^2 f'(z) \to b$ as $|z| \to \infty$ for some $b \in \mathbb{C}$.

Let g be holomorphic on $\mathbb{C}$ except at $z \in \{p_1, \ldots, p_n\}$ where g has a simple pole. Assume that g has simple zeros at $z \in \{q_1, \ldots, q_m\}$ and no other zeros, and that $g(z) \to 1$ as $z \to \infty$. Show that $m = n$ and hence determine g.

Paper 1, Section I

3 Complex Analysis OR Complex Methods

This is the joint question for Complex Analysis/Complex Methods. Attempt only ONE of the sub-questions. On your answer sheet, specify the question number as either "3.1G" or "3.2A".

(3.1G) Complex Analysis

State and prove Jordan's Lemma.

Find

$$\int_{-\infty}^{\infty} \frac{x \sin x}{1+x^2} dx.$$

(3.2A) Complex Methods

(a) State the Residue Theorem.

(b) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos(nx)}{x^4+1} dx, \quad n \in \mathbb{N}.$$

Paper 1, Section II

12 Complex Analysis OR Complex Methods

This is the joint question for Complex Analysis/Complex Methods. Attempt only ONE of the sub-questions. On your answer sheet, specify the question number as either "12.1G" or "12.2A".

(12.1G) Complex Analysis

Let $A = \{z \in \mathbb{C} : r < |z| < R\}$ and suppose $f : A \to \mathbb{C}$ is holomorphic. Show that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

with the sum converging locally uniformly, where you should give an expression for the coefficients $a_n \in \mathbb{C}$ in terms of a contour integral involving f.

Let $D^*(R) = \{z \in \mathbb{C} : 0 < |z| < R\}$ and suppose $f : D^*(R) \to \mathbb{C}$ is holomorphic. What does it mean in terms of the a_n for f to have a (*i*) *removable singularity*; (*ii*) *pole of order* $k \geqslant 1$; (*iii*) *essential singularity* at $z = 0$.

For each of the following holomorphic functions $f_i : D^*(1) \to \mathbb{C}$, determine the type of the singularity at $z = 0$:

(i) $f_1(z) = \dfrac{1}{z^2} - \dfrac{1}{\sin^2 z}$;

(ii) $f_2(z) = \displaystyle\int_{-1}^{1} e^{-t^2/z^2}\, dt$.

(12.2A) Complex Methods

(a) Let f be an analytic function on an open disc D whose centre is the point $z_0 \in \mathbb{C}$. Assume that $|f'(z) - f'(z_0)| < |f'(z_0)|$ on D. Prove that f is one-to-one on D.

(b) What does it mean for a function $g : \mathbb{R}^2 \to \mathbb{R}$ to be harmonic?

(i) Suppose that $\tilde{u}$ is a positive ($\tilde{u} \geqslant 0$) harmonic function on $\mathbb{R}^2$. Show that $\tilde{u}$ is constant.

(ii) Let u be a real valued harmonic function in the complex plane (we identify $\mathbb{C}$ with $\mathbb{R}^2$) such that

$$u(z) \leqslant a\big|\log(|z|)\big| + b$$

for all $z \in \mathbb{C}$, where a and b are positive constants. Prove that u is constant.

Paper 2, Section II

12 Complex Analysis OR Complex Methods

This is the joint question for Complex Analysis/Complex Methods. Attempt only ONE of the sub-questions. On your answer sheet, specify the question number as either "12.1G" or "12.2A".

(12.1G) Complex Analysis

Use the residue theorem to give a proof of Cauchy's derivative formula: if f is holomorphic on $D(a,R)=\{z\in\mathbb{C}:|z-a|<R\}$ and $|w-a|<r<R$ then

$$f^{(n)}(w)=\frac{n!}{2\pi i}\int_{|z-a|=r}\frac{f(z)}{(z-w)^{n+1}}dz.$$

Let (g_k) be a sequence of holomorphic functions $g_k:D(a,R)\to\mathbb{C}$ which converges locally uniformly to a holomorphic function g.

Show that $\left(g_k^{(n)}\right)$ converges locally uniformly to $g^{(n)}$ for all $n=0,1,\ldots$.

Suppose further that g has a zero of order $m\geqslant 1$ at a and vanishes nowhere else in $D(a,R)$. Show that for any $0<\epsilon<R$ there exists $K\in\mathbb{N}$ such that for all $k\geqslant K$, g_k has exactly m zeros in $D(a,\epsilon)$, counting with multiplicity.

(12.2A) Complex Methods

(a) Let f be a holomorphic function in the complex plane, except for potentially $n\in\mathbb{N}$ points that are poles. Suppose also that $\int_\gamma p(z)^2 f(z)\,dz=0$ for all complex polynomials p and every closed contour γ avoiding the potential poles of f. Show that f is entire.

(b) Suppose that h is entire, of the form $h(z)=h(x+iy)=u(x,y)+iv(x,y)$, is real on the real axis, and has positive imaginary part in the upper half-plane (that is $v(x,y)>0$ when $y>0$).

(i) Show that $h'(x)\geqslant 0$ when x is real.

(ii) Show that if $h(0)=0$, then $h'(0)\neq 0$.

Paper 3, Section I

3A Complex Methods

(a) Consider the complex function f analytic in the open disc D centred at zero, except with a singularity at $z = 0$, and its Laurent series around zero,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n \, z^n = \cdots + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \cdots .$$

Show that if f is even, that is $f(z) = f(-z)$ for $z \in D \setminus 0$, then $a_n = 0$ when n is odd.

(b) Evaluate the integrals

$$I_n = \oint_{C_n} \frac{1}{z^3 \sin(z)} dz, \quad n = 0, 1, 2, \ldots,$$

where C_n is the circle $\{z \in \mathbb{C} : |z| = (n + 1/2)\pi\}$, with the counterclockwise orientation.

Paper 4, Section II

12A Complex Methods

Recall the Heaviside function

$$H(t) := \begin{cases} 1, & t \geqslant 0 \\ 0, & t < 0 \end{cases}, \quad t \in \mathbb{R},$$

and recall that a function $h : \mathbb{R}_+ = [0, \infty) \to \mathbb{R}$ is said to be $T > 0$ periodic if $h(t+T) = h(t)$ for all $t \in \mathbb{R}_+$.

(a) Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a bounded continuous function. Show that for any real number $\alpha \geqslant 0$,

$$\mathcal{L}\{f(t-\alpha)H(t-\alpha)\}(s) = e^{-\alpha s}F(s), \quad s > 0,$$

where $\mathcal{L}\{f(t)\}(s) = F(s)$ and $\mathcal{L}$ is the Laplace transform.

(b) Let $g : \mathbb{R}_+ \to \mathbb{R}$ be continuous and $T > 0$ periodic, and define

$$g_T(t) = \begin{cases} g(t), & 0 \leqslant t \leqslant T \\ 0, & t > T \end{cases}.$$

Show that

$$\mathcal{L}\{g(t)\}(s) = \frac{\mathcal{L}\{g_T(t)\}(s)}{1 - e^{-sT}}, \quad s > 0.$$

(c) Find the Laplace transform of the periodic function $h : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$h(t) = \begin{cases} \sin(t), & 0 \leqslant t < \pi \\ 0, & \pi \leqslant t \leqslant 2\pi \end{cases}, \qquad h(t+2\pi) = h(t), \quad t \geqslant 0.$$

Paper 2, Section I

4B Electromagnetism

State Maxwell's equations in the presence of charge density ρ and current density $\mathbf{J}$. Derive the continuity equation that ensures the conservation of charge,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0\,.$$

Suppose that all non-zero ρ and $\mathbf{J}$ are confined to a finite time-independent volume D (vanishing on the boundary ∂D). Show that the total charge Q in the region D remains constant. In addition, prove the following relation:

$$\frac{d}{dt}\int_D \mathbf{x}\,\rho\, d^3x = \int_D \mathbf{J}\, d^3x\,.$$

Paper 4, Section I

5B Electromagnetism

Beginning with the Maxwell equations in vacuum, derive a wave equation for the electric field $\mathbf{E}$ and show the plane wave of the following form is a solution:

$$\mathbf{E}(\mathbf{x},t) = Re\left(\mathbf{E}_0\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right)\,, \quad \text{with} \;\; \mathbf{k}\cdot\mathbf{E}_0 = 0\,,$$

where $\mathbf{k}$ and $\mathbf{E}_0$ are constant vectors. Give an expression relating ω and $\mathbf{k}$. Find the corresponding plane wave solution for the magnetic field $\mathbf{B}$.

Consider the specific solution

$$\mathbf{E} = E_0\left(0,\, \frac{1}{\sqrt{2}},\, \frac{1}{\sqrt{2}}\right)\cos(kx-\omega t)\,,$$

for which you should state the wavevector direction and the polarisation vector. Calculate the corresponding Poynting vector $\mathbf{S} = (1/\mu_0)\mathbf{E}\times\mathbf{B}$ and its time-average. Briefly explain its meaning.

Paper 1, Section II

15B Electromagnetism

In a volume V, an electrostatic charge density $\rho(\mathbf{x})$ induces an electric field $\mathbf{E}(\mathbf{x})$ with electrostatic potential $\phi(\mathbf{x})$ which vanishes on the boundary. Use Maxwell's equations, to show that the electrostatic energy ,

$$U = \frac{1}{2}\int_V d^3x\,\rho(\mathbf{x})\,\phi(\mathbf{x})\,,$$

can be expressed in terms of the electric field $\mathbf{E}(\mathbf{x})$.

Consider three concentric spherical shells with uniformly distributed surface charges $Q_1 = q$, $Q_2 = -2q$, $Q_3 = q$, placed around the origin at radii $r_1 = R$, $r_2 = 2R$, $r_3 = 3R$, respectively. Use Gauss's Law to find the electric field $\mathbf{E}(\mathbf{x})$ at all points in space. Likewise determine the potential $\phi(\mathbf{x})$ everywhere. Calculate the total electrostatic energy U, both by using the displayed equation above and using the electric-field formulation, and verify that they agree.

Paper 2, Section II

16B Electromagnetism

Consider two different vector potential fields $\mathbf{A}_1(x, y, z) = b_1\,(-y,\ x,\ 0)$ and $\mathbf{A}_2(x, y, z) = b_2\,(-y\,z,\ x\,z,\ 0)$, where b_1, b_2 are constants and the vertical direction is aligned with the z-axis. Calculate the associated static magnetic fields $\mathbf{B}_1(x, y, z)$ and $\mathbf{B}_2(x, y, z)$ and show they satisfy the vacuum Maxwell equations.

Consider a circular conducting loop of radius r and resistance R that is constrained to lie horizontally and centred along the z-axis. For the two magnetic fields $\mathbf{B}_1$ and $\mathbf{B}_2$, determine the respective magnetic fluxes through the loop at a vertical position z. Suppose a current I flows around the loop in a clockwise direction and calculate the force $\mathbf{F}$ from the magnetic field acting on the loop in each case.

Suppose the loop has a mass m and is allowed to fall from rest at position z_0 under the influence of gravity (with no initial current). What is the induced current that results from its motion $\dot{z}$ for the two different magnetic fields $\mathbf{B}_1$ and $\mathbf{B}_2$? Hence calculate any resistive forces that emerge. Write down an equation of motion for the vertical position in each case and identify the asymptotic behaviour of the trajectories. In one case, the trajectory approaches a terminal velocity at which you should compare the power loss from the current with the change in the gravitational potential energy. Briefly comment on the behaviour of a superconducting loop with $R = 0$ if it is dropped in the same manner.

Paper 3, Section II

15B Electromagnetism

Give the electromagnetic tensor $F_{\mu\nu}$ explicitly in terms of the components of the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$. Show that the dual electromagnetic tensor defined as $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}$, is given by

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix}.$$

Calculate the Lorentz scalar $F^{\mu\nu}F_{\mu\nu}$ and express in terms of the fields $\mathbf{E}$ and $\mathbf{B}$. State what the remaining Lorentz scalars $F^{\mu\nu}\tilde{F}_{\mu\nu}$ and $\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu}$ are in terms of $\mathbf{E}$ and $\mathbf{B}$.

In a particular reference frame S, the components of uniform electric and magnetic fields are restricted to the y-z plane, taking the form $\mathbf{E} = (0, E_y, E_z)$ and $\mathbf{B} = (0, B_y, B_z)$. Consider another inertial frame S' related by the Lorentz transformation,

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where v is the velocity of S' in S along the x-axis with $\gamma = (1 - v^2/c^2)^{-1/2}$. Determine the components of the fields $\mathbf{E}'$ and $\mathbf{B}'$ in the new frame S'.

Now suppose that $\mathbf{E} = E_0(0, 1, 0)$ lies parallel to the y-axis, while $\mathbf{B}$ has magnitude $B_0 = E_0/c$ and lies in the y-z plane at an angle θ to the y-axis with $0 \leqslant \theta \leqslant \pi/2$. Determine the velocity of a reference frame S' in which the two fields align to become parallel. Briefly discuss the two limits when $\theta \ll 1$ and when $\theta \to \pi/2$.

Paper 2, Section I

5D Fluid Dynamics

Write down the Euler equations governing the inviscid flow $\mathbf{u}$ of an incompressible fluid with no body force. Derive the corresponding vorticity equation.

At some initial time, the velocity

$$\mathbf{u} = (A \sin z, B \sin x + A \cos z, B \cos x)$$

in Cartesian coordinates (x, y, z), where A and B are constants. Show that the vorticity is parallel to $\mathbf{u}$. Hence show that the vorticity is constant, independent of time.

Use the Euler equation to show that $H \equiv \frac{1}{2}\rho|\mathbf{u}|^2 + p$ is uniform in space, where p is the fluid pressure and ρ is its density.

[*Hint: You may use the vector identities* $\mathbf{u} \times (\nabla \times \mathbf{u}) = \nabla(\frac{1}{2}|\mathbf{u}|^2) - \mathbf{u} \cdot \nabla \mathbf{u}$ *and* $\nabla \times (\mathbf{a} \times \mathbf{b}) = (\nabla \cdot \mathbf{b})\mathbf{a} + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\nabla \cdot \mathbf{a})\mathbf{b} - (\mathbf{a} \cdot \nabla)\mathbf{b}$.]

Paper 3, Section I

7D Fluid Dynamics

A layer of viscous fluid of density ρ, dynamic viscosity μ, and uniform thickness h flows down a rigid, vertical wall, adjacent to stationary, inviscid, ambient fluid of density ρ_a. The ambient fluid exerts no shear stress on the viscous fluid layer.

What is the hydrostatic pressure in the ambient fluid? Write down the dynamic boundary conditions at the interface between the two fluids and the boundary condition to be applied at the vertical wall.

Write down the equations governing steady, parallel flow of the viscous fluid, and solve them to determine its pressure and velocity fields.

Paper 1, Section II

16D Fluid Dynamics

State Bernoulli's equation for steady flow.

Starting from Euler's equations governing steady, inviscid, flow $\mathbf{u}$ of an incompressible fluid of density ρ subject to a conservative body force $\mathbf{f} = -\nabla\chi$, derive the integral momentum equation

$$\int_{\partial V} (\rho\mathbf{u}\cdot\mathbf{n}\,\mathbf{u} + p\mathbf{n} + \chi\mathbf{n})\; dS = 0,$$

where p is the fluid pressure and $\mathbf{n}$ is the unit normal to the surface ∂V of a closed domain V.

A large circular blood vessel of cross-sectional area A bifurcates symmetrically with respect to its axis into two smaller circular blood vessels, each of cross-sectional area a and each inclined at angle α to the axis of the larger blood vessel. The constant volume flux through the system is q.

(i) Determine the pressure drop between a location in the larger vessel upstream of the junction and a location in one of the smaller blood vessels downstream of the junction.

(ii) Given that the pressure inside the smaller vessels far downstream of the junction is equal to the uniform pressure of the body tissue surrounding the vessels, determine the force on the junction in terms of the parameters given above.

Paper 3, Section II

16D Fluid Dynamics

Incompressible fluid is contained between rigid plates at $\theta = \pm\alpha$ hinged together at $r = 0$ in plane polar coordinates (r,θ). The fluid was initially at rest but is set into motion by rotating the plates towards each other, each with angular speed Ω. The subsequent instantaneous fluid flow $\mathbf{u}(r,\theta)$ can be treated as being inviscid.

Explain why the flow can be written in terms of a velocity potential ϕ that satisfies Laplace's equation $\nabla^2\phi = 0$. What boundary conditions are satisfied by ϕ?

Find the velocity potential in the form $\phi = r^2 f(\theta)$, determining the function $f(\theta)$ explicitly.

Determine the velocity field and thence determine a streamfunction for the flow. Describe and sketch the streamlines.

Calculate the flux of fluid across the radial arc $r = R$, $-\alpha < \theta < \alpha$.

Paper 4, Section II

16D Fluid Dynamics

An infinite range of hills has elevation $y = \eta(x) \equiv h\cos kx$ in Cartesian coordinates (x, y), where h and k are constants. High above the hills, the wind has uniform velocity $\mathbf{U} = (U, 0)$. Assume that the air flow above the hills is a laminar, potential flow $\mathbf{u} = \mathbf{U} + \nabla\phi$.

Without approximation, write down the equation and boundary conditions satisfied by ϕ. [*Note that the vector* $\mathbf{n} = (-\eta_x, 1)$ *is normal to the surface.*]

Now assume that both $hk \ll 1$ and $|\nabla\phi| \ll U$. Describe these approximations in physical terms.

Derive the linearised equation and boundary conditions satisfied by ϕ given these approximations, taking care to explain all the approximations that you make.

Solve the linearised equations for ϕ, and use your solution to determine the difference in pressure between the crests and troughs of the hills, assuming that the air has uniform density ρ. What is the dominant physical reason for the pressure difference in the limits (i) $kU^2/g \ll 1$ and (ii) $kU^2/g \gg 1$?

Paper 1, Section I

2F Geometry

Let $\sigma : U \to \Sigma \subset \mathbb{R}^3$ be a smooth parametrization of an embedded surface in $\mathbb{R}^3$ and let $\gamma : [a, b] \to \Sigma$ be a smooth curve on Σ. Define the *energy* of γ. Deduce from the Euler–Lagrange equations of a stationary curve for the energy function the ordinary differential equations on U defining the geodesics on $\sigma(U) \subset \Sigma$.

Suppose that a plane $P \subset \mathbb{R}^3$ contains the unit normal vector for Σ at each point of the intersection $\Sigma \cap P$. If a curve η is parametrized with constant speed and is contained in $\Sigma \cap P$, show that η is a geodesic on Σ.

Paper 3, Section I

2E Geometry

Define what it means for an element $g \in SL_2(\mathbb{R})$, $g \neq \pm I$ to be *elliptic, parabolic* or *hyperbolic* in terms of the action of g on the hyperbolic plane $\mathfrak{h}$ and its boundary.

Give an example of each.

Prove that every element $g \neq \pm I$ in $SL_2(\mathbb{R})$ is precisely one of the three.

Let $g \in SL_2(\mathbb{R})$ be an element with $g^n = I$, $g \neq \pm I$, $n > 1$. Determine when g is elliptic, parabolic, or hyperbolic.

Paper 1, Section II

11F Geometry

Define an (allowable) parametrization of an embedded smooth surface $S \subset \mathbb{R}^3$.

Suppose that S is a surface of revolution, meaning for each real θ the rotation

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

defines a diffeomorphism of S onto itself. Stating any result(s) that you use, show that S admits, around each point which is not on the z-axis, a local parametrization of the form

$$\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u)) \quad \text{with } (f'(u), g'(u)) \neq (0,0),$$

where $|v| < \pi$, $|u| < \varepsilon$ for some $\varepsilon > 0$.

We say that an embedded smooth $\Sigma \subset \mathbb{R}^3$ is a *ruled surface* if Σ admits a parametrization of the form

$$\psi(s,t) = a(s) + tb(s), \qquad (s,t) \in I \times \mathbb{R},$$

where $I \subset \mathbb{R}$ is an interval, $a, b : I \to \mathbb{R}^3$ are embedded smooth curves and $b(s)$ is a unit vector for all $s \in I$. Explain why we must have $a'(s) \times b(s) + tb'(s) \times b(s) \neq 0$ for all s, t.

Suppose that a path-connected, ruled surface Σ is also a surface of revolution. Suppose also that for some s_0 the affine line $a(s_0) + tb(s_0)$, $t \in \mathbb{R}$, in $\mathbb{R}^3$ neither meets the z-axis, nor is parallel to the z-axis. Prove that then Σ is diffeomorphic to a one-sheet hyperboloid $x^2 + y^2 = 1 + z^2$.

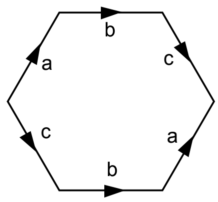
[*You may assume if you wish that a hyperboloid $x^2 + y^2 = 1 + z^2$ is a complete surface, not contained as a proper subset in any embedded smooth connected surface.*]

Paper 2, Section II

11F Geometry

(a) What is a *topological surface*?

Show that a regular hexagon Σ with opposite sides identified as shown is a compact topological surface.



Explain briefly why Σ is homeomorphic to a torus.

Show that a cone $C = \{(x, y, z) : x^2 + y^2 - z^2 = 0\}$ in $\mathbb{R}^3$ is not a topological surface but a half-cone $C_+ = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 0,\ z > 0\}$ is.

(b) By considering parametrizations, construct a map $\pi : \mathbb{R}^2 \setminus \{(0, 0)\} \to C_+$ which is a local isometry. Show that if f is an isometry of C_+ onto itself and $f(p) = p$ for some point $p \in C_+$, then f is either a restriction to C_+ of a reflection in a plane in $\mathbb{R}^3$ or the identity map.

[*The expression for the first fundamental form on $\mathbb{R}^2$ in polar coordinates can be assumed without proof. You may assume that local isometries map geodesics to geodesics.*]

Paper 3, Section II

12F Geometry

What is the *Euler characteristic* of a closed topological surface? State the Gauss–Bonnet theorem for geodesic polygons and for closed smooth surfaces.

Let $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that $f(x,y) = 0$ when $x^2 + y^2 > 1$ and let $S \subset \mathbb{R}^3$ be a non-compact embedded surface parametrized by $\varphi(x,y) = (x, y, f(x,y))$ for $(x,y) \in \mathbb{R}^2$. Prove that if the Gaussian curvature K of S is everywhere non-negative, then K is everywhere zero.

Let $T \subset \mathbb{R}^3$ be the torus given by

$$\sigma(u,v) = \big((2+\cos u)\cos v,\ (2+\cos u)\sin v,\ \sin u\big), \quad 0 \leqslant u, v < 2\pi.$$

Determine the regions T_+ and T_- of T where the Gaussian curvature is positive and negative, respectively. Show, by considering an appropriate closed surface but without explicitly integrating K, that

$$\int_{T_+} K dA = -\int_{T_-} K dA = 4\pi.$$

[*You may assume that the Gauss–Bonnet theorem holds when a closed topological surface in* $\mathbb{R}^3$ *is a union of finitely many smooth surfaces joined at their boundaries.*]

Paper 4, Section II

11E Geometry

(a) Define the *disc model* (D, g_{disc}) and the *upper half-plane model* $(\mathfrak{h}, g_{\mathfrak{h}})$ for the hyperbolic plane, and show that they are isometric.

If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, then g induces an isometry of $\mathfrak{h}$. What is the matrix of the corresponding isometry of D?

(b) Define what is meant by a *hyperbolic triangle* in the hyperbolic plane, its *vertices*, and *ideal vertices*.

Let Δ be a hyperbolic triangle with only ideal vertices. What are its internal angles? Compute the area of Δ.

Compute the area of a hyperbolic triangle with internal angles α, β, γ.

For fixed $\alpha,\ \beta,\ \gamma < \pi$, show that $SL_2(\mathbb{R})$ acts transitively on the set of triangles with these internal angles.

Paper 2, Section I

1E Groups, Rings and Modules

(a) Let R be a ring. Define what it means for R to be i) an *integral domain*, ii) *Noetherian*, and iii) a *Principal Ideal Domain (PID)*.

(b) Let $R = \{(f(x), g(y)) \in \mathbb{C}[x] \times \mathbb{C}[y] \mid f(0) = g(0)\}$. Verify that this is a subring of $\mathbb{C}[x] \times \mathbb{C}[y]$.

Let $p : R \to \mathbb{C}[x], (f, g) \mapsto f$. Determine the kernel of p.

Is R an integral domain? A PID?

Paper 3, Section I

1E Groups, Rings and Modules

Let G be a finite group. Show that there exists subgroups

$$G = H_1 \rhd H_2 \rhd \cdots \rhd H_n = \{1\}$$

such that H_{i+1} is a normal subgroup of H_i, and the quotient H_i/H_{i+1} is simple.

Write such a series for $G = S_4$, the symmetric group on 4 letters, and determine whether each H_i in your series is normal in G.

Paper 1, Section II

9E Groups, Rings and Modules

(a) Let H be a proper subgroup of a finite, non-abelian group G. Prove that if G is simple, $|G/H| \geqslant 5$.

(b) Let S be a Sylow p-subgroup of a finite group G. Suppose that $gSg^{-1} \cap S = \{1\}$ for all $g \in G \setminus N_G(S)$.

Show that the number of Sylow p-subgroups is congruent to 1 mod $|S|$.

(c) Let G be a simple group of order 168.

(i) Compute the number of Sylow 7-subgroups of G. Compute the number of elements of G of order exactly 7.

(ii) Let S be a Sylow 2-subgroup of G. Show that there exists a Sylow 2-subgroup S' of G, $S \neq S'$, with $S \cap S' \neq \{1\}$. Show $S \cap S'$ contains an element of order 2.

Paper 2, Section II

9E Groups, Rings and Modules

Define what it means for a ring to be a *Euclidean domain*.

Let R be a Euclidean domain.

(a) Prove that R is a principal ideal domain (PID).

(b) Let p be a prime in R. Fix $e \geqslant 1$, and let $\varphi : R/(p^e) \to R/(p)$ be the natural ring homomorphism. Let $a \in R/(p^e)$.

Describe in terms of $\varphi(a)$ (i) when a is a unit, and (ii) when a is nilpotent. You must justify your answer.

(c) Let $I = (p_1^{e_1} p_2^{e_2} \ldots p_n^{e_n})$ be an ideal in R, where $p_1, \ldots, p_n$ are irreducible elements, pairwise non-associated, and $e_1, \ldots, e_n \geqslant 1$ are integers. Describe the units in R/I.

(d) Let $d = p_1 p_2 \cdots p_n$, where $p_1, \ldots, p_n$ are as defined in (c). Prove that the homomorphism $R/I \to R/(d)$ induces a surjective map on units, $(R/I)^* \to (R/(d))^*$. Here we write S^* for the set of invertible elements in a ring S.

Paper 3, Section II

10E Groups, Rings and Modules

(a) Let R be a unique factorisation domain, and $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x]$.

Define the content $c(f)$ of f, and what it means for f to be *primitive*.

(i) Prove that if $f, g \in R[x]$ are primitive, then so is fg. Deduce that $c(fg) = c(f)c(g)u$, for some unit $u \in R^*$.

(ii) Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$ be a monic polynomial with integer coefficients. Suppose $f(\lambda) = 0$ for some $\lambda \in \mathbb{Q}$. Deduce from (i) that $\lambda \in \mathbb{Z}$.

(b) Show $x^3y^3 + x^3y + x^2y^2 + 1 - x - y - y^2$ is irreducible in $\mathbb{C}[x, y]$.

Paper 4, Section II

9E Groups, Rings and Modules

(a) A module M for a ring R is called *irreducible* if the only submodules $N \subseteq M$ are 0 and M.

(i) Show that a module M is irreducible if and only if for all $m \in M$, $m \neq 0$, the map $R \to M$, $r \mapsto rm$ is surjective.

(ii) Let $I = \{r \in R \mid rm = 0 \text{ for all } m \in M\}$. Show that if M is irreducible, R/I is a field.

(b) Let V be a finite dimensional vector space over a field k, and $\varphi : V \to V$ a k-linear map.

A subspace $W \leqslant V$ is *indecomposable* if $\varphi(W) \subseteq W$, and W can not be written as a direct sum $W' \oplus W''$, with $W' \neq 0, W'' \neq 0$, $\varphi(W') \subseteq W'$, $\varphi(W'') \subseteq W''$

(i) State the primary decomposition theorem for modules over a Euclidean domain, and explain how it gives a decomposition

$$V = \oplus V_\alpha,$$

where each summand is indecomposable.

Describe the *minimal polynomial* and the *characteristic polynomial* of φ in terms of this decomposition.

(ii) Now suppose $k = \mathbb{R}$. List the prime ideals in $\mathbb{R}[x]$.

For each prime ideal I in $\mathbb{R}[x]$ and $n > 0$ write an explicit matrix A that represents the action of x on $\mathbb{R}[x]/I^n$.

(iii) Let $B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \end{pmatrix}$.

Give the explicit normal form for $B : \mathbb{R}^6 \to \mathbb{R}^6$ that you have described in part (ii).

Paper 1, Section I

1F Linear Algebra

Define the *determinant* of an $n \times n$ matrix A. Define the *adjugate matrix* $\operatorname{adj}(A)$. Express $\det A$ in terms of $\operatorname{adj}(A)$ and A.

For each $n \geqslant 2$ let A_n be the $n \times n$ matrix defined by

$$(A_n)_{ij} = \begin{cases} 2 & i = j, \\ -1 & |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

What is $\det A_n$? Justify your answer.

Paper 4, Section I

1F Linear Algebra

Let V be a vector space and $\alpha : V \to V$ a linear map. What is the *dual space* V^*? If $\mathcal{B}$ is a finite basis of V, define what is meant by the *dual basis* $\mathcal{B}^*$ of V^* and prove that $\mathcal{B}^*$ is indeed a basis.

[*No result about dimensions of dual spaces may be assumed.*]

Let $V = P_2$ be the space of real polynomials of degree at most 2 and consider the linear maps from P_2 to $\mathbb{R}$

$$f_0(p) = p(0), \qquad f_1(p) = \int_0^1 p(t)dt, \qquad f_2(p) = \int_{-1}^0 p(t)dt.$$

Show that f_0, f_1, f_2 form a basis of P_2^* by exhibiting the basis of P_2 to which it is dual.

[*You may assume that* $\{1, t, t^2\}$ *is a basis of* P_2.]

Paper 1, Section II

8F Linear Algebra

State the rank-nullity theorem, explaining all the quantities that appear in it.

Let U, V, W be vector spaces where U and V are finite-dimensional. If $\alpha : V \to W$, $\beta : U \to V$ are linear maps, prove that

$$\mathrm{rk}(\alpha \circ \beta) \geqslant \mathrm{rk}(\alpha) + \mathrm{rk}(\beta) - \dim V.$$

If X and Y are matrices representing the same linear map between two finite-dimensional vector spaces with respect to different bases, write down the relation satisfied by X and Y. [You should explain the terms appearing in this relation.]

Let $M_k(\mathbb{C})$ denote the vector space of all $k \times k$ complex matrices. Let A be a block matrix of the form $A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$, where $P \in M_n(\mathbb{C})$, $S \in M_m(\mathbb{C})$ and P is invertible. Show that

$$\mathrm{rk}(A) = \mathrm{rk}(P) + \mathrm{rk}(S - RP^{-1}Q).$$

Deduce that

$$\mathrm{rk}(I_n - QR) = \mathrm{rk}(I_m - RQ) + n - m$$

where I_r is the identity matrix of size r.

Paper 2, Section II

8F Linear Algebra

Let m be a positive integer and $\alpha \in \mathbb{C}$. What is a *Jordan block* $J_m(\alpha)$?

Let $p(x)$ be a polynomial with complex coefficients. Show that

$$p(J_2(\alpha)) = \begin{pmatrix} p(\alpha) & p'(\alpha) \\ 0 & p(\alpha) \end{pmatrix}.$$

Let A be an $n \times n$ complex matrix. Define the *Jordan normal form* of A.

Show that the Jordan normal form of A is a diagonal matrix if and only if $\mathrm{Ker}(A - \lambda I_n)^2 = \mathrm{Ker}(A - \lambda I_n)$ for all $\lambda \in \mathbb{C}$, where I_n is the identity matrix of size n.

Let $A \in M_n(\mathbb{C})$ be an $n \times n$ complex matrix and let B be the Jordan normal form of A. Show that B is also the Jordan normal form of the transpose matrix A^T.

Show that A can be factorised as $A = CD$ where the matrices C and D are symmetric and C is non-singular.

Paper 3, Section II

9F Linear Algebra

Throughout this question V is an n-dimensional complex vector space and $\varphi : V \to V$ is a linear endomorphism of V.

(a) Define the *minimal polynomial* m_φ of φ and explain why m_φ is uniquely defined. State the Cayley–Hamilton theorem. What can we deduce about the relationship between m_φ and the characteristic polynomial of φ?

(b) Now suppose that V has a basis $(a_1, \ldots, a_n)$ such that $\varphi(a_k) = a_{k+1}$, for each $k = 1, 2, \ldots, n-1$. Let θ be a map assigning to each complex polynomial p the vector $p(\varphi)(a_1) \in V$. By considering θ, or otherwise, show that the minimal polynomial of φ is

$$m_\varphi(x) = x^n - \sum_{k=0}^{n-1} c_{k+1} x^k$$

where the coefficients c_j are determined by $\varphi(a_n) = \sum_{j=1}^{n} c_j a_j$. [*Hint: You do not need to determine the eigenvalues of* φ.]

(c) Show that if the minimal polynomial of a linear endomorphism $\varphi : V \to V$ is of the form $m_\varphi(x) = (x-\alpha)^n$ for some constant α (where as above $n = \dim V$), then V cannot be written as a direct sum $V_1 \oplus V_2$, where V_1, V_2 are non-zero, proper subspaces of V such that $\varphi(V_i) \subset V_i$, $i = 1, 2$. Show also that V has a basis $\mathcal{B} = (b_1, \ldots, b_n)$ such that $\varphi(b_k) = b_{k+1}$ for $k = 1, 2, \ldots, n-1$ and compute the matrix of φ with respect to the basis $\mathcal{B}$.

Paper 4, Section II

8F Linear Algebra

Let Q be a quadratic form on a real vector space. What is the *symmetric bilinear form* associated to Q and why does it exist? Define what it means for a symmetric bilinear form to be *positive semi-definite* and *positive definite.*

State and prove Sylvester's law of inertia, stating clearly any auxiliary result on the diagonalization of real quadratic forms that you require.

Let ϕ be a non-degenerate, symmetric bilinear form on a $2n$-dimensional real vector space V and suppose that $\phi(v,v) = 0$ for all v in a k-dimensional subspace E of V. Show that $k \leqslant n$.

Given two real quadratic forms $f(x) = \sum_{i,j=1}^{n} a_{ij}x_i x_j$ and $g(x) = \sum_{i,j=1}^{n} b_{ij}x_i x_j$, let (f,g) denote the quadratic form

$$(f,g)(x) = \sum_{i,j=1}^{n} a_{ij}b_{ij}x_i x_j.$$

Let a quadratic form $l^2(x) = (\sum_{i=1}^{n} l_i x_i)^2$ be the square of a real linear function. Determine the rank and signature of the quadratic forms l^2 and (l^2, s^2), where $s(x) = \sum_{i=1}^{n} s_i x_i$ is another real linear function.

Deduce that if f and g are positive semi-definite quadratic forms, then so is (f,g).

Paper 3, Section I

8H Markov Chains

(a) What does it mean to say a Markov chain is *recurrent*?

(b) Let $X_0 = 0$ and $X_n = Z_1 + \ldots + Z_n$ for $n \geqslant 1$, where $Z_1, Z_2, \ldots$ are independent and $\mathbb{P}(Z_n = +1) = p = 1 - \mathbb{P}(Z_n = -1)$ for each n. Prove that the Markov chain $(X_n)_{n \geqslant 0}$ is recurrent if and only if $p = 1/2$.

[*You may use the fact that there is a constant* $A > 0$ *such that* $k!\ (e/k)^k k^{-1/2} \to A$ *as* $k \to \infty$.]

Paper 4, Section I

7H Markov Chains

Consider a Markov chain $(X_n)_{n \geqslant 0}$ on the state space $\{1, 2, 3, 4\}$ with transition matrix

$$P = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1/4 & 0 & 3/4 \end{pmatrix}$$

(a) List the communicating classes of the chain. For each class say whether it is open or closed.

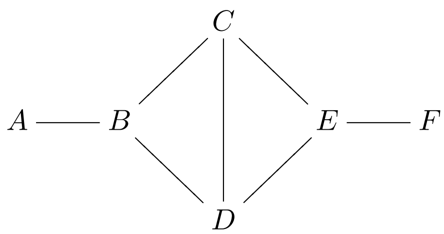
(b) Find $\lim_{n \to \infty} P^n$.

Paper 1, Section II

19H Markov Chains

(a) What does it mean to say a Markov chain is *reversible*? Show that a random walk on a finite connected graph is reversible.

Consider the random walk $(X_n)_{n \geqslant 0}$ on this graph, where $X_0 = A$.



(b) Find the expected number of steps until the walk first returns to A.

(c) Find the probability that the walk returns to A before hitting F.

(d) Given that the walk returns to A before hitting F, find the conditional expected number of steps until the walk first returns to A.

Paper 2, Section II

18H Markov Chains

Let $T_1, T_2, \ldots$ be independent and identically distributed random variables taking values in $\{1, \ldots, N\}$. Construct $(X_n)_{n \geqslant 0}$ as follows. First $X_0 = 0$. For $1 \leqslant n \leqslant T_1$, let $X_1 = T_1 - 1$ and $X_n = X_{n-1} - 1$. Note that $X_{T_1} = 0$. For $T_1 + 1 \leqslant n \leqslant T_1 + T_2$, let $X_{T_1+1} = T_2 - 1$ and $X_n = X_{n-1} - 1$ until $X_{T_1+T_2} = 0$. This pattern repeats forever with $X_{T_1+T_2+1} = T_3 - 1$ and so forth.

(a) Let $S_0 = 0$ and $S_k = T_1 + \ldots + T_k$ for $k \geqslant 1$. Show that

$$X_n = \max\{S_{k+1} : S_k < n\} - n$$

for $n \geqslant 1$.

(b) Find the transition probabilities of the Markov chain $(X_n)_{n \geqslant 0}$ in terms of the given constants $q_j = \mathbb{P}(T_1 = j)$ for $1 \leqslant j \leqslant N$.

(c) Show that there is a unique invariant distribution $(\pi_i)_{0 \leqslant i \leqslant N-1}$ for the Markov chain and compute it in terms of $(q_j)_{1 \leqslant j \leqslant N}$.

(d) Find an example of $(q_j)_{1 \leqslant j \leqslant N}$ such that $\mathbb{P}(X_n = 0)$ does not converge as $n \to \infty$.

Pick ε such that $0 < \varepsilon < 1$ and consider a Markov chain $(X_n^{(\varepsilon)})_{n \geqslant 0}$ on $\{0, \ldots, N-1\}$ with $X_0^{(\varepsilon)} = 0$ and transition matrix $P^{(\varepsilon)} = (p_{i,j}^{(\varepsilon)})_{i,j}$ given by

$$P^{(\varepsilon)} = (1 - \varepsilon)P + \varepsilon I$$

where $P = (p_{i,j})_{i,j}$ is the transition matrix for $(X_n)_{n \geqslant 0}$ found in part (b) and I is the $N \times N$ identity matrix.

(e) Show that $\mathbb{P}(X_n^{(\varepsilon)} = 0)$ converges as $n \to \infty$, and compute the limit in terms of $(q_j)_{1 \leqslant j \leqslant N}$ and ε.

Paper 2, Section I

3B Methods

Consider the initial value problem for a second-order differential operator with constant coefficients and a forcing term:

$$\mathcal{L}y(t) \equiv \alpha y'' + \beta y' + \gamma y = f(t), \quad t > a, \quad y(a) = y'(a) = 0\,,$$

with $\alpha \neq 0$. Write down the Green's function $G(t,\tau)$ constructed to satisfy $\mathcal{L}G = \delta(t-\tau)$.

Use the Green's function approach to determine an explicit solution for the forced oscillator problem

$$y'' + \omega^2 y = \sin(\lambda t), \quad t > 0\,, \quad y(0) = y'(0) = 0\,.$$

Paper 3, Section I

5D Methods

The Fourier transform $\tilde{f}(k)$ is defined in this question by $\tilde{f}(k) = \displaystyle\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$.

Prove the convolution theorem in the form

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\tilde{g}(k)e^{ikx}\,dk = \int_{-\infty}^{\infty} f(u)g(x-u)\,du$$

for suitably integrable functions $f(x)$, $g(x)$.

Determine the Fourier transform of the function

$$f(x) = \begin{cases} 1 \text{ for } -1 < x < 1, \\ 0 \text{ otherwise.} \end{cases}$$

Hence calculate $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 k}{k^2}\,dk$.

Paper 1, Section II

13B Methods

(a) Legendre's differential equation on the domain $-1 < x < 1$ is given by

$$\left(1 - x^2\right) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + \lambda \, y = 0 \, .$$

Put this equation in Sturm-Liouville form and show that the Sturm-Liouville operator is self-adjoint with respect to an inner product you should specify. Briefly state some key properties of the eigenvalues λ_k and eigenfunctions $y_k(x)$ of any Sturm-Liouville differential equation.

Consider a series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ of Legendre's equation and show that the coefficients a_n satisfy the recurrence relation

$$\frac{a_{n+2}}{a_n} = \frac{n(n+1) - \lambda}{(n+1)(n+2)} \, .$$

Hence, show that polynomial solutions $y(x) = P_\ell(x)$ of degree ℓ exist when $\lambda = \ell(\ell+1)$, where ℓ is a non-negative integer ($\ell \geqslant 0$). Find expressions for $P_1(x)$ and $P_3(x)$, adopting the convention that $P_n(1) = 1$.

(b) Laplace's equation in spherical polars for the axisymmetric case takes the form

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial x} \left(\left(1 - x^2\right) \frac{\partial \Phi}{\partial x} \right) = 0 \, ,$$

where $x = \cos\theta$. State the general form of the solution $\Phi(r, x)$ obtained using the method of separation of variables (derivation not required).

Suppose that on the sphere at $r = R$, the boundary condition is $\Phi(R, x) = x(1 - x^2)$. Find the regular solution in the interior of the sphere.

Paper 2, Section II

14B Methods

Consider a string of uniform mass density ρ that is stretched under tension τ along the x-axis. The string undergoes small transverse oscillations in the (x, y) plane, with displacement represented by $y(x, t)$. Derive the equation of motion governing $y(x, t)$, identifying the wave speed c in terms of ρ and τ (neglecting gravity).

The string is fixed at both ends, $x = 0$ and $x = L$. Determine the general solution for the oscillatory motion of the string using the method of separation of variables.

Assume the string is at rest for $t < 0$. At time $t = 0$, the string is struck by a hammer within the interval $[l - \epsilon/2, l + \epsilon/2]$, where $x = l$ represents the position along the string. The hammer's impact imparts a constant velocity $v/\sqrt{\epsilon}$ to the section of the string within this interval, while the rest of the string remains unaffected. Calculate the total energy imparted to the string by this blow. Determine the eigenmode coefficients for the resulting string solution and the energy excited in each mode relative to the total energy.

In musical terms, the $n = 7$ eigenmode is generally regarded as dissonant. Where can you strike the string in order to minimise the vibration of this mode? Briefly comment on the power law fall-off of the energy in each mode as the hammer head narrows, $\epsilon \to 0$?

Paper 3, Section II

14D Methods

Prove that, for scalar fields $\phi(\mathbf{x})$ and $\psi(\mathbf{x})$ in a three-dimensional domain $\mathcal{D}$ with boundary $\partial\mathcal{D}$,

$$\int_{\mathcal{D}} (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV \equiv \int_{\partial\mathcal{D}} \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) dS,$$

where $\mathbf{n}$ is the outward unit normal to $\partial\mathcal{D}$.

Let $\psi(\mathbf{x})$ satisfy

$$\nabla^2\psi = \delta(\mathbf{x} - \mathbf{x}_0) \quad \text{in} \quad z > 0$$

with

$$\psi = 0 \quad \text{on} \quad z = 0, \qquad \psi \to 0 \text{ as } |\mathbf{x}| \to \infty$$

in Cartesian coordinates $\mathbf{x} = (x, y, z)$. Use the method of images to determine $\psi(x, y, z)$.

Now use the identity above to solve the equation

$$\nabla^2\phi = 0 \quad \text{in} \quad z > 0, \qquad \phi \to 0 \text{ as } |\mathbf{x}| \to \infty$$

with $\phi = 1$ on $z = 0$, $x^2 + y^2 < 1$, while $\phi = 0$ on $z = 0$, $x^2 + y^2 > 1$ in terms of a surface integral. Find the closed-form solution for $\phi(0, 0, z)$.

Paper 4, Section II

14D Methods

The function $u(x,t)$ satisfies

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{on} \quad -\infty < x < \infty$$

with

$$u(x,0) = \exp(-x^2), \qquad \frac{\partial u}{\partial t}(x,0) = 0,$$

where x is a space coordinate and t is time.

Define the spatial Fourier transform

$$\tilde{u}(k,t) = \int_{-\infty}^{\infty} u(x,t) e^{-ikx}\, dx,$$

and determine the differential equation and initial conditions satisfied by $\tilde{u}(k,t)$. By solving this differential equation, determine $\tilde{u}(k,t)$ explicitly. Thence, by calculating an appropriate integral, calculate $u(x,t)$. Interpret your solution physically.

Paper 1, Section I

5A Numerical Analysis

Consider the ODE of the form

$$y'(t) = f(t, y(t)), \quad y(0) = y_0 \in \mathbb{R}, \tag{$*$}$$

where $y(t)$ exists and is unique for $t \in [0, T]$ and $T > 0$.

(a) State the Dahlquist equivalence theorem regarding convergence of a multistep method.

(b) Consider the following multistep method for $(*)$ with a parameter $\alpha \in \mathbb{R}$:

$$y_{n+3} + (2\alpha - 3)(y_{n+2} - y_{n+1}) - y_n = h\alpha\big(f(t_{n+2}, y_{n+2}) - f(t_{n+1}, y_{n+1})\big),$$

producing a sequence $\{y_n\}_{n \leqslant N}$, where $N = \lfloor \frac{T}{h} \rfloor$ and $h > 0$ is the step-size. It is given that the method is of at least order 2 for any α and also of order 3 for $\alpha = 6$. Determine all values of α for which the method is convergent, and find the order of convergence.

Paper 4, Section I

6A Numerical Analysis

Consider the quadrature formula

$$\int_0^1 f(x)x\,dx \approx \sum_{i=0}^{1} a_i f(x_i), \quad x_i \in [0,1], \qquad f \in C[0,1], \tag{$*$}$$

which is exact for polynomials of degree 1.

(a) For $i = 0, 1$, find expressions for the weights a_i in terms of the nodes x_0, x_1.

(b) Define what it means for $(*)$ to be a *Gaussian quadrature*, and determine the numerical values of the nodes x_0, x_1 in that case.

Paper 1, Section II
17A Numerical Analysis

(a) Define *Householder reflections* and show that a real Householder reflection is a symmetric and orthogonal matrix.

(b) Let $H \in \mathbb{R}^{n\times n}$ be a Householder reflection. Determine the eigenvalues of H and their multiplicities.

(c) Show that for any $A \in \mathbb{R}^{n\times n}$ there exist Householder reflections $H_1, \ldots, H_n$ such that $H_n H_{n-1} \cdots H_1 A = R$, where R is upper triangular.

(d) Show that if A is symmetric there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that $C = QAQ^T \in \mathbb{R}^{n\times n}$ is symmetric and tridiagonal (that is, only the diagonal, super and subdiagonal have non-zero entries), and C can be computed in finitely many operations $(+, -, \times, \div, \sqrt{\ })$.

Paper 2, Section II
17A Numerical Analysis

Consider the scalar autonomous ODE of the form

$$y' = f(y), \quad y(0) = y_0 \in \mathbb{R}, \tag{$*$}$$

where $y(t)$ exists and is unique for $t \in [0, T]$ and $T > 0$. Consider also the following two Runge-Kutta methods:

$$k_1 = f(y_n), \quad k_2 = f\big(y_n + \frac{h}{2}k_1 + \frac{h}{2}k_2\big), \quad y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2), \tag{$\dagger$}$$

$$\text{same as } (\dagger) \text{ except } k_2 = f\big(y_n + \frac{h}{4}k_1 + \frac{3h}{4}k_2\big), \tag{$\ddagger$}$$

both producing a sequence $\{y_n\}_{n\leqslant N}$, where $N = \lfloor \frac{T}{h} \rfloor$ and $h > 0$ is the step-size.

(a) Do the above Runge-Kutta methods have the same order? If so, determine the order. If not, determine which method has the highest order.
[*Hint: Think about how both methods can be written in terms of a single parameter.*]

(b) For a numerical method approximating the solution of $(*)$, define the linear stability domain. What does it mean for such a numerical method to be A-stable?

(c) Are any of the Runge-Kutta methods $(\dagger)$ and $(\ddagger)$ A-stable? If so, determine the linear stability domain for the method(s).

Paper 3, Section II

17B Numerical Analysis

Consider $C[-1,1]$ equipped with the inner product $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)w(x)\,dx$, where $w(x)>0$ for $x\in(-1,1)$. Moreover, for $n\in\mathbb{N}$, let

$$A_n = \begin{bmatrix} \alpha_1 & \sqrt{\beta_2} & 0 & \cdots & 0 \\ \sqrt{\beta_2} & \alpha_2 & \sqrt{\beta_3} & \ddots & \vdots \\ 0 & \sqrt{\beta_3} & \alpha_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \sqrt{\beta_n} \\ 0 & \cdots & 0 & \sqrt{\beta_n} & \alpha_n \end{bmatrix},$$

where $\alpha_n\in\mathbb{R}$ and $\beta_n>0$.

(a) Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of monic polynomials of degree n orthogonal with respect to the above inner product. Prove that for $n\geqslant 1$ each p_n has n distinct zeros in the interval $(-1,1)$.

(b) Let $P_0(x)=1$, $P_1(x)=x-\alpha_1$, and let P_n satisfy the following recurrence relation:

$$P_n(x) = (x-\alpha_n)P_{n-1}(x) - \beta_n P_{n-2}(x), \quad n\geqslant 2.$$

Prove that for $n>1$ we have $P_n(x)=\det(xI-A_n)$.

(c) Prove that if $p_0(x)=1$ and

$$\alpha_n = \frac{\langle p_n, xp_n\rangle}{\langle p_n,p_n\rangle}, \qquad \beta_n = \frac{\langle p_n,p_n\rangle}{\langle p_{n-1},p_{n-1}\rangle}$$

then all the eigenvalues of A_n are distinct and reside in $(-1,1)$.

[*Hint: You may quote the three-term recurrence relation theorem from the class notes.*]

Paper 1, Section I
7H Optimisation
(a) Derive the dual problem to

$$\text{maximise } c^\top x \text{ subject to } Ax \leqslant b,\ x \geqslant 0$$

where the vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and the $m \times n$ matrix A are given, and the inequalities are interpreted component-wise.

(b) Find the optimal solution to

$$\begin{aligned} \text{maximise } 2x_1 + 3x_2 + 4x_3 \quad \text{subject to} \quad & x_1 + 3x_2 + x_3 \leqslant 2, \\ & x_1 + x_2 + 4x_3 \leqslant 1, \\ & x_1, x_2, x_3 \geqslant 0. \end{aligned}$$

Paper 2, Section I
7H Optimisation
(a) What does it mean to say a set $X \subseteq \mathbb{R}^m$ is *convex*? Assuming X is convex, what does it mean to say a function $f : X \to \mathbb{R}$ is *convex*?

(b) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is convex. Let $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ be defined by $g(t, x) = tf\left(\frac{x}{t}\right)$, where $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$. Show that g is convex.

(c) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ has the property that there is a function $\lambda : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f(x) - f(y) \leqslant \lambda(x)^\top (x - y)$$

for all $x, y \in \mathbb{R}^n$. Prove that f is convex.

Paper 3, Section II
19H Optimisation
Let A be the $m \times n$ payoff matrix of a two-person, zero-sum game.

(a) Write down the necessary and sufficient conditions that a vector $p \in \mathbb{R}^m$ is an optimal mixed strategy for Player I in terms of the optimal mixed strategy $q \in \mathbb{R}^n$ for Player II and the value v of the game.

(b) In the anti-symmetric case where $m = n$ and $A = -A^\top$, show that the value of the game is zero.

(c) Suppose there are rows i_0 and i_1 such that $A_{i_0 j} \leqslant A_{i_1 j}$ for all $1 \leqslant j \leqslant n$. Show that there is an optimal strategy $p \in \mathbb{R}^m$ for Player I such that $p_{i_0} = 0$.

(d) Find the optimal strategies for both players for the game with payoff matrix

$$A = \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 2 \\ 1 & -1 & 0 & 1 \\ 0 & -2 & -1 & 0 \end{pmatrix}$$

Paper 4, Section II

18H Optimisation

Given supplies $(s_i)_{1\leqslant i\leqslant m}$, demands $(d_j)_{1\leqslant j\leqslant n}$ and transport costs $(c_{ij})_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n}$, consider the problem of minimising

$$\begin{aligned} \sum_{i,j} c_{ij}x_{ij} \text{ subject to } & \sum_j x_{ij} = s_i \text{ for all } i, \\ & \sum_i x_{ij} = d_j \text{ for all } j, \\ & x_{ij} \geqslant 0 \text{ for all } i, j. \end{aligned}$$

Assume that all supplies and demands are non-negative, that $\sum_i s_i = \sum_j d_j$ and that the problem is not degenerate.

(a) Derive the dual problem. State the necessary and sufficient conditions for optimality of the primal problem in terms of an optimal solution of the dual problem.

(b) Suppose $(x_{ij})_{ij}$ is a basic feasible solution of the problem. How many ordered pairs (i, j) are such that $x_{ij} > 0$?

(c) Explain the transportation algorithm. Your answer should include a method for choosing an initial basic feasible solution as well as the details of the pivot step. Why does the algorithm terminate at the optimal solution?

(d) Suppose that both the supplies $(s_i)_i$ and demands $(d_j)_j$ are integer-valued. Show that there is an integer-valued optimal solution $(x_{ij})_{ij}$.

Paper 3, Section I

6C Quantum Mechanics

Let $\psi(x,t)$ solve the time-dependent Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}kx^2\psi\,, \qquad -\infty < x < +\infty\,,$$

for a particle of mass m moving in a potential $V(x) = \frac{1}{2}kx^2$. If $\mathcal{O}$ is an operator representing an observable, its expectation value at time t in a normalized state ψ is

$$\langle\mathcal{O}\rangle(t) = \int \psi(x,t)^*\mathcal{O}\psi(x,t)dx\,.$$

Write down operators Q and P representing, respectively, the position and the momentum of the particle. Calculate the time derivative of $\langle P\rangle(t)$ as a function of $\langle Q\rangle(t)$ and interpret the answer.

[*Hint: You may assume ψ and its derivatives are smooth and decrease to zero at infinity as needed.*]

Paper 4, Section I

4C Quantum Mechanics

A quantum particle of mass m is confined to move inside the rectangular box

$$\{(x,y,z) : 0 \leqslant x \leqslant a\,, \quad 0 \leqslant y \leqslant b\,, \quad 0 \leqslant z \leqslant c\}.$$

Derive the energy eigenvalues and eigenfunctions under the assumption that $a < b < c$. (You need not normalize the eigenfunctions.)

What is the degeneracy of the ground state, i.e., the dimension of the eigenspace corresponding to the lowest energy eigenvalue, and similarly for the next to lowest energy eigenvalue (the first excited state)?

How do your conclusions change if $a < b = c$?

Paper 1, Section II

14C Quantum Mechanics

Consider the one-dimensional potential

$$V(x) = \begin{cases} 0 & \text{if } |x| > a\,, \\ -\frac{1}{2a} & \text{if } |x| \leqslant a\,, \end{cases}$$

where $a > 0$. Show that for positive a there exist *normalizable* and *even* solutions to the stationary Schrödinger equation

$$-\frac{\hbar^2}{2m}\psi'' + V(x)\psi = E\psi\,, \qquad -\infty < x < +\infty\,,$$

with energy $E = E(a) = -\frac{\hbar^2\kappa(a)^2}{2m}$, where $\kappa(a) > 0$ satisfies an equation which you should give. Show that for small positive a the energy is unique and the solution is unique up to multiplication by a constant.

Now consider the limit $a \to 0+$. Calculate the limiting value $E_0 = \lim_{a\to 0+} E(a)$, and show that this is the energy of a normalizable and even solution ψ_0 to the stationary Schrödinger equation with a singular potential V_0; give V_0 and ψ_0 explicitly.

Paper 2, Section II

15C Quantum Mechanics

This question concerns one dimensional quantum mechanics on the real line, with the momentum operator given by $p = -i\hbar \frac{d}{dx}$ as usual. A pair of distinct one dimensional Hamiltonians

$$H_+ = \frac{p^2}{2m} + V_+(x) \text{ and } H_- = \frac{p^2}{2m} + V_-(x)\,, \qquad -\infty < x < +\infty,$$

are said to be *partners* if there exists a function $f = f(x)$ such that

$$H_\pm = \frac{1}{2m}(p \pm if)(p \mp if)\,.$$

Show that $[f(x), p] = i\hbar f'(x)$. Taking the upper sign show that

$$V_+(x) = \frac{1}{2m}\left(f(x)^2 - \hbar f'(x)\right)$$

and find V_-.

Choosing f appropriately, find a partner Hamiltonian $H_+ = \frac{p^2}{2m} + V_+(x)$ to

$$H_- = \frac{p^2}{2m} + 2m$$

giving $V_+(x)$ explicitly in as simple a form as possible. [*Hint:* $\mathrm{sech}^2 z + \tanh^2 z = 1$.] Show that $\lim_{x\to\pm\infty} V_+(x) = 2m$.

By considering the solutions $e_k(x) = e^{ikx}$ to $H_- e_k = \left(\frac{\hbar^2 k^2}{2m} + 2m\right) e_k$ and applying the operator $p + if$, show that it is possible to generate a corresponding solution to the partner Hamiltonian H_+. Hence compute the reflection and transmission coefficients for the Hamiltonian

$$\frac{p^2}{2m} - 4m\,\mathrm{sech}^2\left(\frac{2mx}{\hbar}\right)\,.$$

Paper 4, Section II

15C Quantum Mechanics

(i) The angular momentum operators for a particle moving in three dimensional space are

$$L_a = -i\hbar\epsilon_{abc}x_b\frac{\partial}{\partial x_c}\,.$$

Show that if $f = f(r)$, where $r^2 = x_1^2 + x_2^2 + x_3^2$, is a smooth radial function, then if m, n are nonnegative integers $\chi_{m,n} = (x_1 + ix_2)^m x_3^n f(r)$ satisfies $L_3\chi_{m,n} = \lambda\chi_{m,n}$ for some λ depending on m, n which you should find. Find an analogous relation for $(x_1 - ix_2)^m x_3^n f(r)$.

(ii) The Hamiltonian for a particle moving in three spatial dimensions in a symmetric harmonic potential $V(x_1, x_2, x_3) = \frac{1}{2}m\omega^2(x_1^2 + x_2^2 + x_3^2) = \frac{1}{2}m\omega^2 r^2$ is

$$H\Psi = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi}{\partial x_1^2} + \frac{\partial^2\Psi}{\partial x_2^2} + \frac{\partial^2\Psi}{\partial x_3^2}\right) + V(x_1, x_2, x_3)\Psi\,, \qquad (x_1, x_2, x_3) \in \mathbb{R}^3\,.$$

Find the lowest eigenvalue of H and its corresponding eigenfunction Ψ_0. Next, find all the eigenfunctions and eigenvalues of H, and determine the degeneracy of each eigenvalue, i.e. the dimension of the corresponding eigenspace. (You are not required to normalize the eigenfunctions.)

Find χ such that $L_3\chi = 2\hbar\chi$ and $H\chi = \frac{7}{2}\hbar\omega\chi$.

[*Hint: in (ii) you may freely use the fact that the functions*

$$\psi_n(x) = h_n(x)e^{-\frac{1}{2}x^2},$$

where h_n is the Hermite polynomial of degree n, constitute a complete orthogonal set and satisfy

$$-\frac{1}{2}\frac{\partial^2\psi_n}{\partial x^2} + \tfrac{1}{2}x^2\psi_n = (n + \tfrac{1}{2})\psi_n\,, \qquad \int \psi_m(x)\psi_n(x)dx = 0 \textit{ if } n \neq m\,.$$

Explicitly, the first three Hermite polynomials are given by

$$h_0(x) = 1\,, \quad h_1(x) = x \quad \textit{and} \quad h_2(x) = (2x^2 - 1)\,.\,]$$

Paper 1, Section I

6H Statistics

The distribution of a random variable X depends on an unknown parameter θ. Consider testing the null hypothesis $H_0 : \theta = \theta_0$ versus the alternative hypothesis $H_1 : \theta = \theta_1$. State and prove the Neyman–Pearson lemma in the case where X has a probability density function $p(x;\theta)$.

Let X have probability density function $p(x;\theta) = \frac{1}{2}\theta e^{-\theta|x|}$ for $x \in \mathbb{R}$ and $\theta > 0$. Find the critical region of the most powerful test of size α when $\theta_0 < \theta_1$.

Paper 2, Section I

6H Statistics

Let X be a random variable with the $\mathrm{Exp}(\theta)$ distribution. Suppose the prior distribution of θ is $\Gamma(m,\lambda)$ for known parameters m and λ; that is, the prior density is $p(\theta) = C_{m,\lambda}\theta^{m-1}e^{-\lambda\theta}$ where $C_{m,\lambda} = \lambda^m/\Gamma(m)$.

(a) Find the posterior distribution of θ.

(b) Show that the Bayesian estimator of θ for the loss function $L(\theta,a) = (\theta-a)^2$ is given by $\hat{\theta}_{\text{Bayes}} = (m+1)/(\lambda+X)$.

(c) What is the Bayesian estimator of θ for the loss function $L(\theta,a) = \cosh\bigl(r(\theta-a)\bigr)$ for a given positive constant $r < \lambda$. [*Recall that* $\cosh u = \frac{1}{2}(e^u + e^{-u})$.]

Paper 1, Section II

18H Statistics

A data set contains the ordered pairs of observations $(X_1,Y_1),\ldots,(X_n,Y_n)$. A statistician models these data as $Y_i = X_i\beta + \varepsilon_i$, where $X_1,\ldots,X_n$ are known real parameters, the noise $\varepsilon_i \sim N(0,\sigma^2)$ are independent and identically distributed, and the real parameters β and σ^2 are unknown.

(a) Find the maximum likelihood estimator $\widehat{\beta}$ for β and $\widehat{\sigma^2}$ for σ^2. Using standard properties of normal random variables, show that $\widehat{\beta}$ and $\widehat{\sigma^2}$ are independent.

(b) Find a $(1-\alpha)$-confidence interval for β. Express your answer in terms of the cumulative distribution function of the t_k distribution for an appropriately chosen k.

(c) Let $\widetilde{\beta} = \sum_{i=1}^n c_i Y_i$, where $c_1,\ldots,c_n$ are known constants. If $\widetilde{\beta}$ is an unbiased estimator of β, show that

$$\mathrm{Var}(\widetilde{\beta}) \geqslant \frac{\sigma^2}{\sum_{i=1}^n X_i^2}.$$

For which choice of constants $c_1,\ldots,c_n$ is there equality for all (β,σ^2)? [If you use the Gauss–Markov theorem, you must prove it.]

(d) Another statistician models the same data as $X_i = Y_i b + e_i$, where now it is assumed that $Y_1,\ldots,Y_n$ are known parameters, the noise $e_i \sim N(0,s^2)$ are i.i.d., and the real parameters b and s^2 are unknown. Let $\widehat{b}$ and $\widehat{s^2}$ be the maximum likelihood estimators of b and s^2 respectively. Show that $\hat{b}\hat{\beta} \leqslant 1$, with equality only if $\widehat{\sigma^2} = 0 = \widehat{s^2}$.

Paper 3, Section II

18H Statistics

Let $X = (X_1, \ldots, X_n)$ be a discrete random vector with probability mass function $f(x;\theta)$, where θ is an unknown parameter.

(a) In this context, what is a *sufficient statistic* for θ? State and prove the factorisation criterion for sufficiency.

(b) State and prove the Rao–Blackwell theorem.

(c) Let $X_1, \ldots, X_n$ be independent and identically distributed Poisson random variables with mean $\sqrt{q}$ where q is unknown and $n \geqslant 2$ is given. Find a one-dimensional sufficient statistic T for q. Show that $\widetilde{q} = X_1^2 - X_1$ is an unbiased estimator of q. Find another unbiased estimator of q that is a function of T and that has strictly smaller variance than $\widetilde{q}$.

Paper 4, Section II

17H Statistics

Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent random variables. Assume $X_i \sim N(\lambda, 1)$ and $Y_j \sim N(\mu, 1)$ for each $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, where the constants λ and μ are unknown. Let $\bar{X} = \frac{1}{m}\sum_{i=1}^{m} X_i$ and $\bar{Y} = \frac{1}{n}\sum_{j=1}^{n} Y_j$.

(a) Find the generalised likelihood ratio test of size α for $H_0^{(a)} : \lambda = 0 = \mu$ versus $H_1^{(a)} : \lambda, \mu$ unrestricted. Express your answer in terms of the cumulative distribution function F_k of the χ_k^2 distribution, for a suitable k.

(b) Find the generalised likelihood ratio test of size α for $H_0^{(b)} : \lambda = \mu$ versus $H_1^{(b)} : \lambda, \mu$ unrestricted. Express your answer in terms of the cumulative distribution function Φ of the $N(0,1)$ distribution.

(c) Show, regardless of the true values of λ and μ, that there is a positive probability that the test from part (b) rejects $H_0^{(b)}$ but the test from part (a) does not reject $H_0^{(a)}$.

Paper 1, Section I

4C Variational Principles

Given a real symmetric $n \times n$ matrix A, consider the quadratic function

$$Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}\,,$$

on the unit sphere $S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T \mathbf{x} = 1\}$. Assume that $\mathbf{x_0}$ is a unit vector such that

$$Q(\mathbf{x_0}) \geqslant Q(\mathbf{x})\,, \quad \forall\, \mathbf{x} \in S^{n-1}\,.$$

Show that $\mathbf{x_0}$ is an eigenvector of the matrix A and determine the corresponding eigenvalue E. How does this eigenvalue compare to the other eigenvalues of A?

For the case that

$$A = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix} \qquad -\infty < t < +\infty$$

calculate E, as a function of $t \in \mathbb{R}$, and draw a sketch to show that it is convex.

Paper 3, Section I

4C Variational Principles

Consider the functional

$$S[x,y] = \frac{1}{2}\int_{t_0}^{t_1} \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + 2\sin\omega t\,\frac{dx}{dt} - y^2 \right) dt$$

defined on smooth curves $t \mapsto (x(t), y(t))$ in the plane. Assume $\omega \in \mathbb{R}$ is constant.

Write out the Euler-Lagrange equations for S and find the general solution.

What symmetries does the system have?

Find all the first integrals (conserved quantities) of the system.

[*You may either use the Noether theorem or work with the Euler-Lagrange equations. Consider all values of* $\omega \in \mathbb{R}$.]

Paper 2, Section II

13C Variational Principles

This question concerns the movement of a particle in space $\mathbb{R}^3$. Introduce cylindrical coordinates (ρ, ϕ, z) and assume that the trajectory of the particle can be parameterized as a curve

$$z \mapsto (\rho(z), \phi(z))$$

going from $A = (\rho(z_0), \phi(z_0), z_0)$ to $B = (\rho(z_1), \phi(z_1), z_1)$, and is such as to make the following functional stationary:

$$F[\rho, \phi] = \int_{z_0}^{z_1} n(\rho, \phi, z) \sqrt{1 + \left(\frac{\partial \rho}{\partial z}\right)^2 + \rho^2 \left(\frac{\partial \phi}{\partial z}\right)^2} dz\,, \quad \text{where } z_1 > z_0\,,$$

where the function $n = n(\rho, \phi, z)$ is positive and smooth. Write down the Euler-Lagrange equations for this functional.

In the case that $n = n(\rho)$ depends only on ρ, show that there are special solutions to the Euler-Lagrange equations of the form

$$\rho(z) = R, \qquad \phi(z) = \phi_0 + \omega(R) z,$$

where R and ϕ_0 are constants, and $\omega = \omega(R)$ solves an equation

$$n(R) R \omega^2 + a(1 + R^2 \omega^2) n'(R) = 0 \qquad (*)$$

for some constant a which you should find. [*You may assume (*) has two solutions $\pm\omega$, with $\omega > 0$.*]

Find a condition on a positive number L which implies that the points having cylindrical coordinates $(R, \phi_0, 0)$ and (R, ϕ_0, L) can be joined by means of these special solutions and sketch two of them.

Paper 4, Section II

13C Variational Principles

The equation of motion for a bead of mass m moving without friction on a cycloidal shaped wire is the Euler-Lagrange equation for the functional

$$S[\phi] = \int_0^T \Big(ma^2(1-\cos\phi)\dot{\phi}^2 - mga(1+\cos\phi) \Big)\, dt\,, \qquad T > 0\,.$$

Write down the Euler-Lagrange equation for this functional, and show it implies that $u = \cos(\frac{\phi}{2})$ satisfies

$$\ddot{u} + \omega^2 u = 0, \tag{$*$}$$

where ω^2 is a positive number which you should find. [*You should take m, g, a to be positive constants.*]

Using the change of dependent variable $\phi \to u = \cos(\frac{\phi}{2})$, define a new functional $\hat{S}[u] = \int_0^T f(u, \dot{u})\, dt$ such that $\hat{S}[u] = S[\phi]$; give a formula for f and give the Euler-Lagrange equation for $\hat{S}$. How is this equation related to $(*)$?

Give the second variation functional $\delta^2 \hat{S}(\eta)$, where the variation functions η vanish at the endpoints $t = 0$ and $t = T$. Consider the solution $u(t) = A\cos\omega t$ of $(*)$ with fixed endpoint conditions

$$u(0) = A, \qquad u(T) = A\cos\omega T,$$

on the interval $0 \leqslant t \leqslant T$. By considering the orthonormal collection of functions

$$e_n(t) = \sqrt{\frac{2}{T}}\, \sin\frac{n\pi t}{T}\,,$$

find a number t_0 such that $A\cos\omega t$ is a local minimizer of $\hat{S}$ if $T < t_0$ but not for $T > t_0$.

[*Hint: you may assume all variations to be of the form $\eta = \sum_{n=1}^{\infty} c_n e_n(t)$, and re-arrange and interchange sums with derivatives as needed. Observe that $\ddot{e}_n = -(n\pi/T)^2 e_n$.*]

END OF PAPER