UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA 2025

List of Courses

Analysis I
Differential Equations
Dynamics and Relativity
Groups
Numbers and Sets
Probability
Vector Calculus
Vectors and Matrices

Paper 1, Section I
3E Analysis I

(a) Write down without proof the power series that converge to the functions $\exp(z)$ and $\sin(z)$, respectively.

(b) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$. Give without proof a power series converging to $f'(z)$ and state its radius of convergence.

(c) Using the power series of $\exp(z)$, prove that $\exp(a+b) = \exp(a)\exp(b)$ for all $a, b \in \mathbb{C}$. [You may not use any other property of $\exp(z)$ without proving it. You may use without proof that a function $f : \mathbb{C} \to \mathbb{C}$ with $f'(z) = 0$ for all $z \in \mathbb{C}$ is constant.]

(d) Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \sin(z)/z$ for $z \neq 0$ and by $f(0) = 1$. Find $d^k f/dz^k$ at $z = 0$ for $k = 1, 2, 3, \ldots$. Justify your answer.

Paper 1, Section I
4E Analysis I

(a) State Riemann's integrability criterion and prove that it implies (Riemann) integrability.

(b) Let $a \leqslant b < c \leqslant d$ be real numbers and let $f : [a, d] \to \mathbb{R}$ be an integrable function. Using Riemann's integrability criterion, prove that f is also integrable on $[b, c]$. [You may not use any other criterion for integrability.]

Paper 1, Section II
9E Analysis I

(a) What is a *Cauchy sequence*?

(b) State and prove the general principle of convergence. [You may use without proof the Bolzano–Weierstrass theorem.]

(c) Let (a_n) be a sequence of real numbers and define the sequence (d_n) by $d_n = |a_{n+1} - a_n|$.

Determine whether each of the following two statements is true or false. Give a proof or a counterexample as appropriate.

(i) If the series $\sum_{n=1}^{\infty} d_n$ is convergent, then the sequence (a_n) is convergent.

(ii) If the sequence (a_n) is convergent, then the series $\sum_{n=1}^{\infty} d_n$ is convergent.

(d) Decide whether the sequence (b_n) defined as

$$b_n = \sum_{j=n+1}^{2n} \frac{j}{j^2 + j + 1}$$

is convergent. Justify your answer.

[*In parts (c) and (d) you may use without proof any convergence tests from lectures.*]

Paper 1, Section II
10E Analysis I

(a) Let $f : [0,1] \to \mathbb{R}$ be a function. Define what it means that the *limit* of f at a point $x \in [0,1]$ is ℓ for some $\ell \in \mathbb{R}$.

(b) Give a criterion in terms of limits for f to be *continuous* at a point $x \in [0,1]$. [You need not prove the criterion.]

(c) Let $A, B, C \subset \mathbb{R}$ be intervals, and let $f : A \to B$ and $g : B \to C$ be functions. Suppose f is continuous at some point $x \in A$, and g is continuous at $f(x)$. Prove that $g \circ f : A \to C$ is continuous at x. [You may use without proof the sequential characterisation of continuity.]

(d) Consider the functions $f_1, f_2 : [0,1] \to \mathbb{R}$ defined by $f_1(x) = \sin(1/x)$ and $f_2(x) = x \cdot \sin(1/x)$ for $x > 0$ and $f_1(0) = f_2(0) = 0$. Determine the set of points where these functions are continuous. [You may use without proof the continuity of functions constructed from continuous functions using arithmetic operations, provided you state the results you use clearly. You may also assume that $\sin(x)$ is continuous.]

(e) Give examples of functions $f : [0,1] \to \mathbb{R}$ that are continuous at the points given in each of the following sets and nowhere else:

(i) $\{0\} \cup \{1/n : n \in \mathbb{Z}, n > 0\}$,

(ii) $\{1/n : n \in \mathbb{Z}, n > 0\}$.

Justify your answers.

Paper 1, Section II
11E Analysis I

Let $a < b$ be real numbers and let $f : [a,b] \to \mathbb{R}$ be a function.

(a) What does it mean that f is *differentiable* at a point $x \in (a,b)$? Define the *derivative*.

(b) State the mean value theorem.

(c) Now suppose that f is continuous on $[a,b]$ and differentiable on (a,b). Prove that f is increasing on (a,b) if and only if $f'(x) \geqslant 0$ for all $x \in (a,b)$. [A function f is increasing on an interval (a,b) if $a < x \leqslant y < b$ implies $f(x) \leqslant f(y)$.]

(d) Suppose further that $a = 0$, $f(0) = 0$ and f' is increasing on (a,b). Prove that $f(x)/x$ is increasing on (a,b). [*Hint: You may find parts (b) and (c) useful.*]

Paper 1, Section II

12E Analysis I

(a) State the fundamental theorem of calculus.

(b) Let $f, g : [a, b] \to \mathbb{R}$ be differentiable functions with continuous derivatives. Suppose that $f(a) \leqslant g(a)$ and $f'(x) \leqslant g'(x)$ for all $x \in [a, b]$. Prove that $f(b) \leqslant g(b)$. [You may use without proof any version of the fundamental theorem of calculus provided you state it clearly.]

(c) Let $f : [0, \infty) \to \mathbb{R}$ be a differentiable function with continuous derivative. Suppose that $f(0) = 0$ and $0 \leqslant f'(x) \leqslant 1$ for all $x \in [0, \infty)$. Prove that

$$\int_0^x \big(f(t)\big)^3 dt \leqslant \Big(\int_0^x f(t)\, dt\Big)^2$$

for all $x \in [0, \infty)$.

(d) Does the conclusion in part (c) necessarily hold without the condition $f(0) = 0$? Justify your answer.

Paper 2, Section I

1C Differential Equations

Find the stationary points of the function

$$V(x,y) = \tfrac{1}{3}y^3 + x^2y - y$$

and classify them by using the Hessian. Sketch the form of the contours of V.

A particle with position $\big(x(t), y(t)\big)$ moves with velocity $(\dot{x}, \dot{y}) = -\boldsymbol{\nabla} V$. Show that its trajectory $y(x)$ satisfies the equation

$$2xy\frac{dy}{dx} - y^2 = x^2 - 1\,.$$

By use of a suitable integrating factor, or otherwise, find the equation of a general trajectory in the form $y^2 = f(x,c)$, where c is an arbitrary constant.

Paper 2, Section I

2C Differential Equations

(a) Find the fixed points of the differential equation

$$\frac{dx}{dt} = f(x), \text{ where } f(x) = ax + 2x^2 - x^3\,,$$

and determine, graphically or otherwise, the range of a in which each fixed point is stable.

(b) Find the fixed points of the difference equation

$$x_{n+1} = g(x_n), \text{ where } g(x) = b - \tfrac{1}{4}x^2\,,$$

and determine the finite range of b in which there exists a stable fixed point.

[*In both parts (a) and (b) you do not need to consider stability at the values of a or b where the number of fixed points changes.*]

Paper 2, Section II

5C Differential Equations

(a) Let $y_1(x)$ and $y_2(x)$ be any two complementary functions for the equation

$$y'' + p(x)y' + q(x)y = f(x)\,. \tag{1}$$

Define the *Wronskian* $W(x)$ of y_1 and y_2. State a differential equation that is satisfied by $W(x)$ for any choice of y_1 and y_2. Assuming that $W(x) \neq 0$ for any x, state briefly why any solution of (1) can be written in the vector form

$$\begin{pmatrix} y \\ y' \end{pmatrix} = u(x)\begin{pmatrix} y_1 \\ y_1' \end{pmatrix} + v(x)\begin{pmatrix} y_2 \\ y_2' \end{pmatrix} \tag{2}$$

for some (as yet unknown) coefficients u and v.

Obtain two coupled first-order differential equations for $u(x)$ and $v(x)$ by requiring consistency between the two components of (2) and between (1) and (2). Deduce that

$$u' = -\frac{y_2 f}{W}\,, \qquad v' = \frac{y_1 f}{W}\,.$$

(b) You are given that $y_1(x) = x^{-1/2}e^{-x}$ is a complementary function for the equation

$$x^2y'' + xy' - (x^2 + \tfrac{1}{4})y = x^{7/2}\,, \qquad x > 0\,. \tag{3}$$

Determine the Wronskian to within a multiplicative factor. Hence find a second complementary function $y_2(x)$, choosing a convenient amplitude for it and thus the Wronskian.

Use the results of part (a) to determine the solution to (3).

Paper 2, Section II

6C Differential Equations

(a) For $k, m > 0$, find the solution $x(t)$ of the equation

$$t^2\ddot{x} + t\dot{x} - m^2x = t^k$$

that satisfies the boundary conditions $x(0) = x(1) = 0$.

Show that no such solution exists if $k = m = 0$.

(b) Using matrix methods, or otherwise, find the general solution $x(t)$ and $y(t)$ to the linear system of equations

$$t\dot{x} - 4x + \frac{6y}{t} = t + 2\,,$$
$$t\dot{y} - 3tx + 4y = t^2 + t\,.$$

[*Hint: You may find the substitution $y(t) = t^n z(t)$ helpful for a suitably chosen integer n.*]

Paper 2, Section II

7C Differential Equations

(a) The position $x(t)$ of a mass m obeys the equation

$$m\ddot{x} + b\dot{x} + kx = P \sum_{n=1}^{\infty} \delta(t - nt^*) \,,$$

where b, k, P and t^* are positive constants with $b^2 < 4mk$ and δ denotes the Dirac delta function. By comparing the homogenous solutions, or otherwise, identify a frequency ω such that, with $\tau = \omega t$, the scaled position $y(\tau) \equiv m\omega x(t)/P$ satisfies

$$y'' + 2\kappa y' + (\kappa^2 + 1)y = \sum_{n=1}^{\infty} \delta(\tau - nT) \,, \qquad (*)$$

where the dimensionless parameters κ and T should be determined.

(b) Find the solution $\mathbf{x}_n$ to the vector recurrence relation

$$\mathbf{x}_{n+1} = A\mathbf{x}_n + \mathbf{b} \,, \qquad \mathbf{x}_0 = \mathbf{a} \,, \qquad (\dagger)$$

where A is a 2×2 matrix, $\det(A - I) \neq 0$ and $\mathbf{a}$, $\mathbf{b}$ are vectors in $\mathbb{R}^2$.

(c) For each time interval $nT < \tau < (n+1)T$, the solution to $(*)$ is written as a linear combination

$$y(\tau) = C_n f(\tau - nT) + D_n g(\tau - nT)$$

of time-translated homogeneous solutions, where $f(\tau) = e^{-\kappa\tau} \cos\tau$ and $g(\tau) = e^{-\kappa\tau} \sin\tau$.

By considering the variation of y and y' across an infinitesimal interval including $\tau = (n+1)T$, find A and $\mathbf{b}$ such that $\mathbf{x}_n \equiv (C_n, D_n)$ satisfies $(\dagger)$. [*Hint:* $f'(\tau) = -g(\tau) - \kappa f(\tau)$ *and* $g'(\tau) = f(\tau) - \kappa g(\tau)$.]

(d) Explain why y tends to a periodic function for $\kappa\tau \gg 1$, and show that it is given by an amplitude vector $\mathbf{x}_\infty$ with

$$|\mathbf{x}_\infty| = \left\{[1 - f(T)]^2 + g(T)^2\right\}^{-1/2}.$$

Consider the case $0 < \kappa \ll 1$. Find simple leading-order approximations to $|\mathbf{x}_\infty|$ when $T = k\pi$ for an integer k with $\kappa k \ll 1$. Comment physically on this result.

Paper 2, Section II

8C Differential Equations

(a) Consider a change of variables from x and t to $\eta = x/t^p$ and $\tau = t$, where p is a constant. Use the chain rule to express $\partial/\partial x$ and $\partial/\partial t$ in terms of η, τ, $\partial/\partial\eta$ and $\partial/\partial\tau$. Show that the substitution $\theta(x,t) = H(\tau)Y(\eta)$ transforms the nonlinear diffusion equation

$$\frac{\partial\theta}{\partial t} = \frac{1}{m+2}\frac{\partial}{\partial x}\left(\theta^m \frac{\partial\theta}{\partial x}\right),$$

where $m > 0$, into the ordinary differential equation

$$-\frac{d(\eta Y)}{d\eta} = \frac{d}{d\eta}\left(Y^m \frac{dY}{d\eta}\right) \qquad (*)$$

provided $H(\tau)$ and p are chosen appropriately. Find the solution to $(*)$ that satisfies $Y(\pm 1) = 0$. [You may assume $Y'(0) = 0$ by symmetry.]

(b) The solution in part (a) is perturbed by setting $\theta(x,t) = H(\tau)\{Y(\eta) + \epsilon\tau^{-\lambda}y(\eta)\}$, where $\epsilon \ll 1$. You are given that the function $y(\eta)$ must then obey the equation

$$-\frac{d(\eta y)}{d\eta} - (m+2)\lambda y = \frac{d^2}{d\eta^2}\left(Y^m y\right). \qquad (\dagger)$$

For the case $m = 1$ show that $(\dagger)$ reduces to

$$(1-\eta^2)y'' - 2\eta y' + 6\lambda y = 0\,.$$

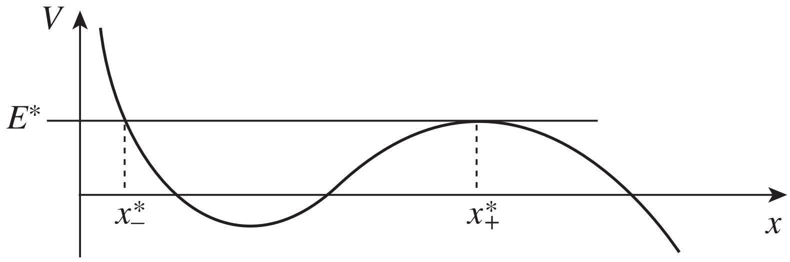
You are given that regularity at $\eta = \pm 1$ requires $y(\eta)$ to be a polynomial. By considering series solutions around $\eta = 0$, show that y is a polynomial if $\lambda = \frac{1}{6}k(k+1)$ for some positive integer k.

Paper 4, Section I

3B Dynamics and Relativity

A particle of mass m moves in one dimension with position $x(t)$, subject to a potential $V(x)$. Write down the equation of motion and also its first integral for a trajectory of total energy E.

Suppose the potential $V(x)$ takes the form shown in the figure below with a local maximum at x_+^* where $V = E^*$ and a second point $x_-^* < x_+^*$ where also $V = E^*$.



The particle is released from rest at a point x_- in (x_-^*, x_+^*) where $V = E < E^*$ and $V' < 0$. Express the period T of the motion as an integral over the interval $[x_-, x_+]$, where $V(x_+) = V(x_-) = E$.

Consider T as a function of $E < E^*$. Setting $\delta = 1 - E/E^*$, show that

$$T = \sqrt{\frac{m}{-V''(x_+^*)}} \Big(\log(1/\delta) + O(1) \Big) \quad \text{as} \quad \delta \to 0 \,.$$

[*Hint: It is helpful to approximate the integrand by Taylor expanding it around* $x = x_+^*$.]

Explain physically why T diverges as E approaches E^*.

Paper 4, Section I

4B Dynamics and Relativity

Two ice-hockey pucks A and B consist of rigid disks of radii R_A and R_B and masses M_A and M_B moving freely, without friction, in two dimensions. Initially Puck A is located at rest at the origin and Puck B is incident with speed u from $x = -\infty$ along the line $y = b$, with $0 < b < R_A + R_B$. The two pucks undergo an elastic collision during which the reaction force acts along the line between the two centres. Determine the velocities of the pucks after the collision.

Paper 4, Section II

9B Dynamics and Relativity

A particle of mass m moves in three dimensions with position $\mathbf{x}(t)$ and velocity $\dot{\mathbf{x}}(t)$ subject to a force

$$\mathbf{F}_1 = \dot{\mathbf{x}} \times \mathbf{B}\,,$$

where $\mathbf{B} = (0, 0, B)$ is a constant vector. Find two constants of motion and explain physically why they are constant.

If the particle starts from $\mathbf{x} = \mathbf{0}$ at $t = 0$ with velocity $\dot{\mathbf{x}}(0) = (u, 0, 0)$, show that the resulting trajectory is a circle whose radius and centre you should determine.

Suppose now that the particle is also subject to a frictional force

$$\mathbf{F}_2 = -\mu \dot{\mathbf{x}}\,,$$

where μ is a positive constant. Find the trajectory of the particle subject to the force $\mathbf{F}_1 + \mathbf{F}_2$ and starting from the same conditions as before. Sketch the resulting path in the (x, y)-plane.

Consider the final position of the particle as a function of the initial speed u. Show that this position lies on a certain straight line, to be specified, and find the final distance of the particle from the origin.

Paper 4, Section II

10B Dynamics and Relativity

A particle of mass m moves in the central potential $V(r) = -km/r$, where k is a positive constant. Show that the general form of its trajectory in polar coordinates is

$$r = \frac{r_0}{1 + e\cos\theta}\,.$$

Find the total energy E of the motion, and its angular momentum per unit mass l, in terms of the two integration constants r_0 and e.

(a) An asteroid approaches the solar system from outer space. At early times its trajectory is asymptotically a straight line with perpendicular distance b from the Sun and its speed is v. At late times its asymptotic trajectory is another straight line. Determine the angle between these lines as a function of b and v.

(b) Another asteroid approaches the solar system following a trajectory with total energy $E = 0$ which reaches a minimum distance d from the Sun. Find the resulting motion in polar coordinates, obtaining the polar angle as

$$\theta(t) = 2\tan^{-1}\big(T(t)\big),$$

where $T(t)$ is the root of a certain cubic equation, which you should identify.

Paper 4, Section II

11B Dynamics and Relativity

Consider a collection of N particles with positions $\mathbf{x}_i(t)$ and masses m_i, for $i = 1, 2, \ldots, N$, rotating about an axis passing through the origin with a common angular velocity $\boldsymbol{\omega}$. Show that the total kinetic energy of the system takes the form $T = I\omega^2/2$, where $\omega = |\boldsymbol{\omega}|$, and obtain a formula for the moment of inertia I.

State the corresponding formula for the moment of inertia of a solid body of non-uniform density $\rho(\mathbf{x})$ about the same axis.

A hollow bowling ball of mass M has a uniform density and occupies an annular region bounded by two concentric spheres of radii $R_+ > R_-$. Determine the moment of inertia of the bowling ball about its centre.

The bowling ball rolls from rest down an inclined plane of vertical height h without slipping. Find its resulting speed when it reaches the bottom of the incline in terms of the radius ratio $\mu = R_-/R_+$.

Two bowling balls, labelled A and B, roll down the incline starting from rest at different times but following the same path. The balls have the same mass M and outer radius R_+, but different values of μ, denoted μ_A and μ_B respectively. If $\mu_A > \mu_B$ which ball reaches the bottom of the incline in the least time? Justify your answer.

A defective bowling ball C has the same dimensions as A, but its inner cavity is slightly off-centre relative to its outer boundary. The ball rolls down the incline starting from rest. Without attempting detailed calculation, explain briefly whether it reaches the bottom faster or slower than ball A.

Paper 4, Section II

12B Dynamics and Relativity

In some inertial frame S with coordinates $x^\mu = (ct, \mathbf{x})$ a relativistic particle follows a world line $x^\mu(\tau)$ parametrized by its proper time τ.

Define the *four-velocity* U^μ of the particle and find the components of U^μ in frame S in terms of the three-velocity $\mathbf{v} = d\mathbf{x}/dt$ with magnitude $v = |\mathbf{v}|$. Show that $U_\mu U^\mu = c^2$.

The *four-acceleration* of the particle is defined as $A^\mu = dU^\mu/d\tau$. Find the components of A^μ in frame S in terms of $\mathbf{v}$ and the three-acceleration $\mathbf{a} = d\mathbf{v}/dt$.

Let S' be the instantaneous rest frame of the particle at time t. Show that in this frame, the four-acceleration takes the form

$$(A^\mu)' = \begin{pmatrix} 0 \\ \mathbf{a}' \end{pmatrix}.$$

Consider a particle moving along the x-axis in frame S with three-velocity $\mathbf{v} = v\hat{\mathbf{x}}$ and three-acceleration $\mathbf{a} = a\hat{\mathbf{x}}$. In frame S' its three-acceleration is $\mathbf{a}' = a'\hat{\mathbf{x}}$. By performing an appropriate Lorentz transformation, show that

$$a' = \frac{a}{(1 - v^2/c^2)^{3/2}}.$$

Suppose further that the particle starts from rest at $x = t = 0$, and that its acceleration a' in frame S' is constant and positive. Find the trajectory $x(t)$ of the particle. Draw a spacetime diagram of the trajectory. Use your diagram to show that a light signal emitted at $t = 0$ from the point $\mathbf{x} = (x_0, 0, 0)$ can never intercept the particle if $x_0 < -c^2/a'$.

Paper 3, Section I

1D Groups

What is the *centre* $Z(G)$ of a group G? Let H be a subgroup of $Z(G)$. Is H necessarily normal in G?

Calculate the centre of the dihedral group D_{2n} for all $n \geqslant 3$.

Paper 3, Section I

2D Groups

Let $\mathbb{Z}$ denote the group of integers under addition and let C_n denote the cyclic group of order n. Determine whether each of the following statements is true or false. Justify your answers.

(i) There exists an injective homomorphism from any finite cyclic group to the group $\mathbb{Z}$.

(ii) There exists a surjective homomorphism from C_n to C_m for all $n \geqslant m$.

(iii) Every proper subgroup of the Cartesian product group $\mathbb{Z} \times \mathbb{Z}$ is cyclic.

Paper 3, Section II

5D Groups

(a) Consider a group G acting on a set X.

(i) What is the *orbit* of an element of X? What is the *stabiliser* of an element of X? Suppose every element $x \in X$ has a nontrivial stabiliser. Can the group action be faithful?

(ii) Let x and y be elements of X that lie in the same orbit under the action of G. Let $\mathrm{Stab}_G(x)$ and $\mathrm{Stab}_G(y)$ be the stabilisers of these elements. Prove that $\mathrm{Stab}_G(x)$ is isomorphic to $\mathrm{Stab}_G(y)$.

(b) Let k and n be positive integers with $1 \leqslant k \leqslant n$. Consider the action of the symmetric group S_n on the set of k-element subsets of $\{1, \ldots, n\}$. For which values of k is this action faithful? Justify your answer.

(c) The alternating group A_n does not contain a proper normal subgroup for $n \geqslant 5$. Using this fact, find all the normal subgroups of S_n for $n \geqslant 5$.

Paper 3, Section II

6D Groups

(a) Let D_8 be the dihedral group of size 8. Determine all normal subgroups of D_8.

(b) Let G be a group and K a subgroup of G. Prove that K is normal if and only if there exists a surjective homomorphism of groups $\varphi\colon G \to H$ such that $\ker(\varphi)$ is K.

(c) Prove or give a counterexample to the following statement: if G is a group in which every subgroup is normal, then G is abelian.

(d) Let $\varphi\colon G \to H$ be a homomorphism of groups.

(i) Let N be a normal subgroup of H. Prove that

$$\varphi^{-1}(N) = \{g \in G \colon \varphi(g) \in N\}$$

is a normal subgroup of G.

(ii) If φ is surjective and K is a normal subgroup of G, prove that

$$\varphi(K) = \{\varphi(g) \colon g \in K\}$$

is a normal subgroup of H.

Paper 3, Section II

7D Groups

Prove directly from the definition that every Möbius transformation has at least one fixed point. Prove that every non-identity Möbius transformation has either one or two fixed points.

Let $m > 1$ be a positive integer. Prove that there exist infinitely many distinct Möbius transformations of order m.

Let $f(z) = \dfrac{az+b}{cz+d}$ and $g(z) = \dfrac{\alpha z+\beta}{\gamma z+\delta}$ be Möbius transformations and let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Prove that if A and B are conjugate in the group $\mathrm{SL}_2(\mathbb{C})$ then $f(z)$ and $g(z)$ are conjugate in the Möbius group.

Prove or give a counterexample to the following statement: if $f(z)$ and $g(z)$ are conjugate in the Möbius group, then either A and B are conjugate in $\mathrm{SL}_2(\mathbb{C})$ or A and $-B$ are conjugate in $\mathrm{SL}_2(\mathbb{C})$.

Paper 3, Section II

8D Groups

Let p be a prime number and let $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ denote the set of 2×2 matrices with entries in $\mathbb{Z}/p\mathbb{Z}$ and with non-zero determinant.

Prove that $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ is a group under matrix multiplication. Calculate the order of this group. [You may assume standard facts about matrix multiplication.]

Does $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$ contain an element of order 3? Does it contain an element of order 6? Justify your answers.

Does $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ contain a proper normal subgroup for all primes p? Justify your answer.

Prove that $\mathrm{GL}_2(\mathbb{Z}/11\mathbb{Z})$ contains a non-abelian subgroup of size 55. [It may be useful to consider upper-triangular matrices.]

Paper 4, Section I

1F Numbers and Sets

What does it mean to say that a series $\sum_{n=1}^{\infty} a_n$ *converges*? For $n \geqslant 1$, let

$$a_n = \frac{2(n+1)}{\sqrt{n^2+3n+2}+\sqrt{n^2+n}} - 1.$$

Does the series $\sum_{n=1}^{\infty} a_n$ converge? Justify your answer.

Paper 4, Section I

2E Numbers and Sets

(a) State the fundamental theorem of arithmetic and prove the uniqueness of factorization. [You may use without proof that if p is a prime and a, b are positive integers, then $p|ab$ implies $p|a$ or $p|b$.]

(b) Find a positive integer n such that $n/2$ is a square, $n/3$ is a cube and $n/5$ is a 5th power. [You do not need to compute the decimal representation of your example.]

Paper 4, Section II

5F Numbers and Sets

Prove that $\sqrt{2}+\sqrt{3}$ is an irrational number.

Prove that $\tan(\pi/8)$ is an irrational number. Is it an algebraic number?

Consider the grid $\mathbb{Z}^2$ as a subset of the plane $\mathbb{R}^2$. Show that the area of any triangle with vertices on the grid $\mathbb{Z}^2$ is a rational number. Deduce that the area of any convex polygon with vertices on the grid $\mathbb{Z}^2$ is a rational number.

Is it possible to draw a regular octagon with vertices on the grid $\mathbb{Z}^2$? If so, give an example; if not, justify your answer.

Paper 4, Section II

6E Numbers and Sets

(a) State the Chinese remainder theorem.

(b) Define *Euler's totient function* $\varphi(n)$.

(c) State the Fermat–Euler theorem.

(d) Let n, m be coprime positive integers. Prove $\varphi(nm) = \varphi(n)\varphi(m)$.

(e) Let n be a positive integer that is not divisible by the square of a prime. Let k be a positive integer such that $k \equiv 1 \mod \varphi(n)$. Show that $a^k \equiv a \mod n$ for all integers a. [You should *not* assume $(a, n) = 1$.]

(f) Explain briefly the *RSA public-key cryptography scheme*.

(g) Let n be a positive integer that is divisible by the square of a prime. Prove that there exist integers a, b such that $a \not\equiv b \mod n$, but $a^k \equiv b^k \mod n$ for all integers $k > 1$.

Paper 4, Section II

7D Numbers and Sets

(a) What does it mean for a function to be *injective*? Let $f\colon A \to A$ be an injective function on a set A. Is f necessarily surjective? Justify your answer.

(b) Let $f\colon A \to B$ be a function and suppose $S \subset A$ and $T \subset B$. Denote $f(S) = \{f(s)\colon s \in S\} \subset B$ and $f^{-1}(T) = \{a \in A\colon f(a) \in T\}$. Prove that

$$f\big(f^{-1}(T) \cap S\big) = T \cap f(S)\,.$$

If f is surjective, determine whether the following equality necessarily holds:

$$f(S \cap S') = f(S) \cap f(S')\,.$$

(c) Let $f\colon A \to B$ and $g\colon A' \to B$ be functions. Define

$$A \times_B A' = \big\{(a, a') \in A \times A'\colon f(a) = g(a')\big\}$$

and let $p\colon A \times_B A' \to A'$ be the function that maps (a, a') to a'. Determine whether the following statements are true or false. Justify your answers.

(i) If $f\colon A \to B$ is injective then p is injective.

(ii) If $f\colon A \to B$ is surjective then p is surjective.

Paper 4, Section II

8D Numbers and Sets

What does it mean for a set to be *countable*?

Let $A \subset \mathbb{R}$ be a countable subset of the real numbers and let $A[x]$ be the set of all polynomials in one variable with coefficients in A. Show that $A[x]$ is countable. Deduce that there exist uncountably many transcendental numbers.

Let N be a countable set and let $\mathcal{Q}$ denote the set of bijections $f\colon N \to N$ with the property that $f(x) \neq x$ for all $x \in N$. Is the set $\mathcal{Q}$ countable? Justify your answer.

Let $\{L_1, L_2, \ldots\}$ be a countable infinite set of straight lines in the plane $\mathbb{R}^2$. Prove that $\mathbb{R}^2$ is not equal to the union $\bigcup_{i=1}^{\infty} L_i$.

Paper 2, Section I
3F Probability

Let Y be a random variable taking values in $[0,\infty)$ with probability density function f_Y. Show that

$$\mathbb{E}[Y] = \int_0^\infty \mathbb{P}(Y \geqslant y)\, dy\,.$$

Let $X_1,\ldots,X_n$ be independent random variables uniformly distributed on $\{1,2,\ldots,n\}$, and let $M = \min\{X_1,\ldots,X_n\}$. Show that $\mathbb{E}[M] \leqslant e$ and, applying any inequality from the course, deduce that $\mathbb{P}(M \geqslant 6) \leqslant 0.5$. [*Hint:* $(1+x) \leqslant e^x$.]

Paper 2, Section I
4F Probability

(a) A train can stop at n stations on a railway line. At each station, the conductor makes a stop with probability $1/n$, independently of previous stations. Let X be the total number of stops made. What is the distribution of X?

For each $k = 0,1,2,\ldots$, find $\lim_{n\to\infty} \mathbb{P}(X = k)$.

(b) Consider a modification of this process where, initially, the conductor stops at each station with probability $1/n$, but after the first time that the train does stop, the probability of stopping at subsequent stations increases to $2/n$. Compute $\mathbb{E}[X]$ and find $\lim_{n\to\infty} \mathbb{E}[X]$.

Paper 2, Section II
9F Probability

In a group of $n > 3$ people, each pair is friends with probability $1/2$, independently of every other pair.

(a) A *triad* is a set of three people who are all friends with each other. Show that the probability that there are at least $n^3/24$ triads is at most $1/2$.

(b) The person, or persons, with the most friends has M friends, and the person, or persons, with the least friends has L friends. Show that the mean and median of $M+L$ are both equal to $n-1$. [The median is defined by $\inf\{x : \mathbb{P}(M+L \leqslant x) \geqslant 1/2\}$].

(c) Now suppose that each pair of people in the group are friends with probability $\dfrac{(1+\varepsilon)\log n}{n-1}$ for some $\varepsilon > 0$, independently of every other pair. Show that

$$\mathbb{P}(L = 0) \leqslant n^{-\varepsilon}.$$

Paper 2, Section II

10F Probability

Write $\mathbb{N} = \{1, 2, 3, \dots\}$. For any sequence $x = (x_1, \dots, x_n) \in \mathbb{N}^n$, let $N_i(x)$ denote the number of times that $i \in \mathbb{N}$ appears in the sequence x and let $M(x) = \max\{x_1, \dots, x_n\}$. Let (X_i) be a sequence of random variables taking values in $\mathbb{N}$, with $\mathbb{P}(X_1 = 1) = 1$ and

$$\mathbb{P}\big(X_{n+1} = i \mid X_1 = x_1, \dots, X_n = x_n\big) = \begin{cases} \dfrac{N_i(x) - \frac{1}{2}}{n+1} & \text{if } 1 \leqslant i \leqslant M(x)\,, \\ \dfrac{1 + \frac{1}{2}M(x)}{n+1} & \text{if } i = M(x) + 1\,, \\ 0 & \text{if } i > M(x) + 1 \end{cases}$$

for all $n \geqslant 1$.

(a) Let $A_n \subseteq \mathbb{N}^n$ be the set of sequences such that, if $x \in A_n$ then $x_1 = 1$ and $x_i \leqslant \max\{x_1, \dots, x_{i-1}\} + 1$ for all $i \leqslant n$. Prove that

$$\mathbb{P}\big((X_1, \dots, X_n) \in A_n\big) = 1\,.$$

(b) Letting $X = (X_1, \dots, X_n)$ and $M_n = M(X)$, show that

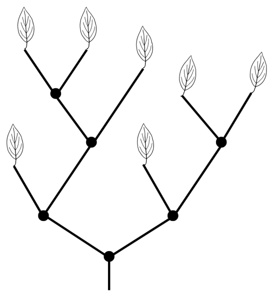
$$\mathbb{P}\big(N_1(X) = 1\big) = \frac{\prod_{i=1}^{n-1}(i + \frac{1}{2})}{n!} \quad \text{and} \quad \mathbb{P}\big(N_{X_n}(X) = 1\big) = \frac{1 + \frac{1}{2}\mathbb{E}[M_{n-1}]}{n}\,.$$

(c) Let x and y be two sequences in A_n, with $M(x) = M(y)$ and such that $\big(N_1(x), \dots, N_{M(x)}(x)\big)$ is a permutation of $\big(N_1(y), \dots, N_{M(y)}(y)\big)$. Show that

$$\mathbb{P}(X_1 = x_1, \dots, X_n = x_n) = \mathbb{P}(X_1 = y_1, \dots, X_n = y_n)\,.$$

(d) Using the result in part (c), prove that $\mathbb{P}\big(N_1(X) = 1\big) = \mathbb{P}\big(N_{X_n}(X) = 1\big)$. Obtain an expression for $\mathbb{E}[M_n]$. [*Hint: Consider a bijection* $\pi : A_n \to A_n$ *such that the frequency of the first element of* x *is the same as the frequency of the last element in* $\pi(x)$*.*]

Paper 2, Section II
11F Probability



A *tree*, like the example shown, has a root and splits into two branches at every branching point; branches never rejoin. Consider a random path from the root which, at each branching point, goes left or right with equal probability, and independently, until it reaches a leaf. Let X_1 be the number of branching points traversed by this path, and let (X_i) be a sequence of independent random variables with the same distribution as X_1.

Let ℓ be the number of leaves in the tree, and suppose there are at most b branching points on the path between the root and any leaf. Define a random variable $L = n^{-1} \sum_{i=1}^{n} 2^{X_i}$.

(a) State and prove Markov's inequality.

(b) Show that $\mathbb{E}[L] = \ell$.

(c) Prove that for all $\alpha > 0$,

$$\mathbb{P}\left(\left|\frac{L-\ell}{\ell}\right| \geqslant \alpha\sqrt{\frac{2^b}{n\ell}}\right) \leqslant \frac{1}{\alpha^2}\,.$$

(d) Let Φ be the distribution function of a standard normal random variable. Show that for all $\alpha > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{L-\ell}{\ell}\right| \geqslant \alpha\sqrt{\frac{2^b}{n\ell}}\right) \leqslant 2 - 2\Phi(\alpha).$$

[You may use without proof any theorem from the course in this part.]

Paper 2, Section II

12F Probability

Let $E_1, \ldots, E_n$ be independent random variables with an exponential distribution of mean 1, with $n \geqslant 2$. Let S be an independent random variable with probability density function

$$f(s) = \frac{s^{n-1} e^{-s}}{(n-1)!} \quad \text{for } s > 0\,.$$

Define

$$(X_1, \ldots, X_n) = \left(\left(1 + \frac{E_1}{S}\right)^{-n}, \ldots, \left(1 + \frac{E_n}{S}\right)^{-n} \right).$$

(a) Show that the moment generating function of S is $M_S(t) = (1-t)^{-n}$ for $t < 1$.

(b) Find the joint distribution function and joint probability density function of (X_1, X_2). What is the marginal distribution of X_1?

(c) Let K be the least integer $k \in \{1, \ldots, n-1\}$ such that $X_k < X_{k+1}$, with $K = n$ if there is no such integer. Show that

$$\mathbb{E}\left[\sum_{i=1}^{K} S^i \right] = \sum_{i=1}^{n} \binom{n+i-1}{i}.$$

Paper 3, Section I

3A Vector Calculus

Let D be a bounded region in $\mathbb{R}^2$ and $F : D \to \mathbb{R}$ a smooth function. Consider the surface in $\mathbb{R}^3$ defined by

$$S = \left\{ \big(x, y, F(x,y)\big) : (x,y) \in D \right\}.$$

Find an expression for the surface area of S.

Using this formula, compute the surface area in the following cases:

(i) D is the triangle with vertices $(0,0)$, $(2,0)$ and $(0,3)$ and $F(x,y) = 2x + 3y + 23$.

(ii) S is the spherical cap $\{x^2 + y^2 + z^2 = 1 : \ x^2 + y^2 \leqslant a, \ z \geqslant 0\}$, where $a \in (0,1)$.

Suppose that S is the unbounded surface $2z = x^2 - y^2$ in $\mathbb{R}^3$, defined for all x, y. Explain why the following integral exists and compute it:

$$\int_S \frac{dS}{(1 + x^2 + y^2)^2}.$$

Paper 3, Section I

4A Vector Calculus

State Green's theorem for integration in the plane.

Consider the region R expressed as $\left\{ (x,y) : x^4 - 2x^3 + 2x^2y^2 - 2xy^2 + y^4 - y^2 \leqslant 0 \right\}$. Express this region more simply in polar coordinates (r, θ).

Let C be the (positively oriented) boundary of R. Using Green's theorem, or otherwise, compute the line integral $\oint_C \mathbf{F} \cdot \mathbf{dr}$ for each of the following vector fields $\mathbf{F} = (F_x, F_y)$:

(i) $\mathbf{F} = \big(-y/\sqrt{x^2+y^2},\ x/\sqrt{x^2+y^2} \big)$ for $r > 0$, $\quad \mathbf{F} = (0,0)$ at $r = 0$,

(ii) $\mathbf{F} = (y^2, xy)$,

(iii) $\mathbf{F} = \big(0,\ \theta(x,y)\big)$ for $r > 0$ with $\theta \in (-\pi, \pi]$, $\quad \mathbf{F} = (0,0)$ at $r = 0$.

[*In parts (i) and (iii) you may assume that Green's theorem applies, despite the irregularity at* $r = 0$.]

Paper 3, Section II

9A Vector Calculus

Consider the vector field $\mathbf{F} = (3xz, y^2z, x^2 + y^2)$. Let $a, b > 0$ and let V be the region enclosed by the elliptical cylinder $ax^2 + by^2 = 1$ and the planes $z = 0$ and $z = 3$.

(a) State the divergence theorem and use it to compute the flux of $\mathbf{F}$ through the closed surface S, where S is the boundary of V.

(b) Without using the divergence theorem, directly calculate the flux of $\mathbf{F}$ through the top, bottom and side surfaces of S and confirm that the results are consistent with that in part (a).

(c) Is the vector field $\mathbf{F}$ conservative? Let C be the closed curve consisting of the intersection of the plane $z = 1$ and the cylinder $ax^2 + by^2 = 1$. Compute the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{dr} \, .$$

Paper 3, Section II

10A Vector Calculus

(a) A polynomial $p(x,y)$ of two variables x and y is called *homogeneous of degree* n if it satisfies $p(\lambda x, \lambda y) = \lambda^n p(x,y)$ for all λ.

By deriving a recurrence relation for the coefficients, prove that there are precisely two linearly independent homogeneous polynomials of each degree $n \geqslant 1$ that satisfy Laplace's equation in two dimensions.

[In parts (b) and (c), you may assume that there is exactly one solution to Poisson's equation with the specified forcing and boundary conditions.]

(b) Consider solving Poisson's equation in spherical polar coordinates,

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = f(r,\theta,\phi)\,,$$

in the region $r > 1$ subject to the boundary conditions $u = g(\theta,\phi)$ on $r = 1$ and $u \to 0$ as $r \to \infty$. In each of the following cases, find u:

(i) $f(r,\theta,\phi) = -2e^{-r}/r + e^{-r} + 12/r^6$ and $g(\theta,\phi) = 1\,.$

(ii) $f(r,\theta,\phi) = \cos(2\theta)/(r^3 \sin\theta)$ and $g(\theta,\phi) = \sin\theta\,.$

(iii) $f(r,\theta,\phi) = -\sin\phi/(r^3 \sin^2\theta)$ and $g(\theta,\phi) = \sin\phi\,.$

[*Hint: Try looking for solutions of the form* $g_1(r)g_2(\theta)g_3(\phi)$ *motivated by the form of the boundary conditions.*]

(c) Consider solving Poisson's equation in Cartesian coordinates,

$$\nabla^2 u = f(x,y,z)\,,$$

in the cube $[0,1]\times[0,1]\times[0,1]$ subject to the boundary condition $u = 0$ on its surface. In each of the following cases, find u:

(i) $f(x,y,z) = x^2y^2 + x^2z^2 + y^2z^2 - x^2y - xy^2 - x^2z - xz^2 - y^2z - yz^2 + xy + xz + yz.$
[*Hint: Try polynomials that satisfy all the boundary conditions.*]

(ii) $f(x,y,z) = \sin(2\pi x)\sin(2\pi y)\sin(4\pi z) - \sin(2\pi x)\sin(\pi y)\sin(5\pi z)$. [*Hint: Apply the Laplacian to* f.]

Paper 3, Section II

11A Vector Calculus

Define the *Jacobian matrix* J of a transformation between two sets of curvilinear coordinates in n dimensions.

Let $x_1, \ldots, x_n$ denote the usual Cartesian coordinates in $\mathbb{R}^n$. The *hyperspherical coordinate system* $(r, \theta_1, \theta_2, \ldots, \theta_{n-2}, \phi)$ extends plane-polar and spherical-polar coordinates to higher dimensions and is defined implicitly for $n \geqslant 4$ through

$$
\begin{aligned}
x_1 &= r\cos\theta_1\,, \\
x_j &= r\left(\prod_{k=1}^{j-1}\sin\theta_k\right)\cos\theta_j, \quad j = 2, \ldots, n-2, \\
x_{n-1} &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\phi\,, \\
x_n &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\phi\,,
\end{aligned}
$$

where $0 \leqslant r < \infty$, $0 \leqslant \theta_i \leqslant \pi$ and $0 \leqslant \phi < 2\pi$.

Find the Jacobian matrix J_n of the transformation from Cartesian coordinates to hyperspherical coordinates in $\mathbb{R}^n$. By expanding about a suitable row or column of J_n, prove that its determinant $|J_n|$ satisfies the recurrence relation

$$|J_n| = r f_n(\theta_1, \theta_2, \ldots, \theta_{n-2})\,|J_{n-1}|\,,$$

for a function f_n that you should specify.

Find a formula for the volume element dV in hyperspherical coordinates.

What is the surface area element dS on the surface $r = R$ in $\mathbb{R}^n$, where R is a positive constant?

(i) By integrating the volume element, find the volume of the region $0 \leqslant r \leqslant R$ and express it in terms of factorials.

(ii) Find the area of the surface $r = R$.

(iii) What is the value of the following surface integral

$$\int_{r=R} x_i x_j\, dS, \quad \text{where} \quad i, j \in \{1, \ldots, n\}\,?$$

Paper 3, Section II

12A Vector Calculus

Let $\mathbf{v}$ be a vector field and let T be a second-rank tensor field in $\mathbb{R}^3$, and define

$$[T\mathbf{n}]_i = T_{ij} n_j \quad \text{and} \quad [\mathrm{div}(T)]_i = \frac{\partial T_{ij}}{\partial x_j}\,.$$

Let $V \subseteq \mathbb{R}^3$ be a region with boundary ∂V and outward unit normal $\mathbf{n}$. Prove that

$$\int_V \mathrm{div}(T) \cdot \mathbf{v}\, dV = \int_{\partial V} (T\mathbf{n}) \cdot \mathbf{v}\, dS - \int_V T_{ij} \frac{\partial v_i}{\partial x_j}\, dV\,. \qquad (*)$$

Now let V be the unit cube $[0,1] \times [0,1] \times [0,1]$, with

$$T_{ij} = x_i x_j - \delta_{ij} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} x^2 \\ yx \\ zx \end{pmatrix}.$$

By computing the three integrals in $(*)$ separately, verify that the equation holds.

Paper 1, Section I

1B Vectors and Matrices

(a) Using the properties of complex numbers, show that a triangle inscribed in a circle is a right-angled triangle if and only if one of the sides is a diameter of the circle.

(b) A *chord* of a polygon is defined to be a line joining any pair of distinct vertices.

A regular N-gon is inscribed in the circle $|z+1| = 1$, for $z \in \mathbb{C}$, with one vertex chosen to lie at $z = 0$. Write down an Nth-order polynomial equation in z whose roots are the N vertices and thus determine the product of the lengths of all the chords emanating from the vertex at the origin. Hence show that the product of the lengths of all the chords of a regular N-gon inscribed in a circle of radius R is equal to $N^{N/2}R^{N(N-1)/2}$.

Paper 1, Section I

2C Vectors and Matrices

Consider the equation

$$A\mathbf{x} = \mathbf{b}\,, \text{ where } A = \begin{pmatrix} 2 & 0 & a \\ a & 1 & 1 \\ 1 & 1 & -2 \end{pmatrix} \text{ and } \mathbf{b} \in \mathbb{R}^3\,. \qquad (*)$$

(a) For which values of a does $(*)$ not have a unique solution for $\mathbf{x}$?

(b) For each of these values of a, find $\mathbf{n} \in \mathbb{R}^3$ such that $\mathbf{n} \cdot \mathbf{b} = 0$ is necessary for solutions to $(*)$ to exist.

(c) For $\mathbf{b} = (2, b, 0)^T$ and the same values of a, find the general solution of $(*)$ when solutions do exist.

Paper 1, Section II

5B Vectors and Matrices

Consider a non-degenerate tetrahedron T in $\mathbb{R}^3$ with four vertices specified by the position vectors $\mathbf{v}$ and $\mathbf{v}_i$ for $i = 1, 2, 3$. Let $\mathbf{e}_i = \mathbf{v}_i - \mathbf{v}$ denote the vectors along the three edges emanating from the vertex $\mathbf{v}$. Show that the volume of T is given by

$$V = \frac{1}{3!} \left| \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) \right| .$$

[You may assume that the volume of a tetrahedron is one third its base area times its height.]

Given the three vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ defined above, find explicit formulae for three vectors $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ such that $\mathbf{f}_i \cdot \mathbf{e}_j = \delta_{ij}$.

Now suppose that the three faces of T intersecting at the vertex $\mathbf{v}$ lie in planes defined for $\mathbf{x} \in \mathbb{R}^3$ by the vector equations $\mathbf{a}_i \cdot \mathbf{x} + b_i = 0$ and that the final face corresponds to the plane $\mathbf{c} \cdot \mathbf{x} + d = 0$. Here $\mathbf{a}_i$ and $\mathbf{c}$ are non-zero vectors and b_i and d are constants.

Show the following.

(i) $\mathbf{e}_i \cdot \mathbf{c} = -(\mathbf{c} \cdot \mathbf{v} + d)$.

(ii) There exists a real number λ_i such that $\mathbf{a}_i = \lambda_i \mathbf{f}_i$.

(iii) If γ_1, γ_2 and γ_3 are real numbers such that $\mathbf{c} = \gamma_1 \mathbf{a}_1 + \gamma_2 \mathbf{a}_2 + \gamma_3 \mathbf{a}_3$ then

$$\gamma_i = \frac{\mathbf{e}_i \cdot \mathbf{c}}{\lambda_i} .$$

Hence show that the volume of T can be expressed in terms of the planar faces (and the vertex $\mathbf{v}$) of the tetrahedron as

$$V = \frac{1}{3!} \left| \frac{(\mathbf{c} \cdot \mathbf{v} + d)^3}{(\gamma_1 \gamma_2 \gamma_3)\, \mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)} \right| .$$

Paper 1, Section II

6C Vectors and Matrices

(a) Write the real matrix

$$A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$$

as the sum of symmetric and antisymmetric parts. Hence, or otherwise, show that A can be written as $A_{ij} = \alpha\delta_{ij} + \beta n_i n_j + \gamma\epsilon_{ijk}n_k$, where $\mathbf{n}$ is a unit vector with $n_1 > 0$, and determine the constants α, β and γ in terms of a, b and c.

(b) Describe geometrically the action in $\mathbb{R}^3$ of A on the line through the origin parallel to $\mathbf{n}$ and on the plane through the origin perpendicular to $\mathbf{n}$. By what factor is any area in this plane multiplied?

(c) Deduce necessary and sufficient conditions on a, b and c for A to act on $\mathbb{R}^3$ either (i) as a reflection or (ii) as a rotation.

(d) Assume only $\det A \neq 0$. By considering the geometrical action of A^{-1}, or otherwise, find an expression for $(A^{-1})_{ij}$ in terms of α, β and γ.

(e) Show that $\det A = \frac{1}{2}(a+b+c)\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right]$.

Paper 1, Section II

7A Vectors and Matrices

(a) Define what is meant by the *eigenvalues* and *eigenspaces* of an $n \times n$ complex matrix A. Prove that any such matrix has at least one eigenvalue. What are the possible dimensions of the corresponding eigenspace?

(b) By computing a determinant, find the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 0 & -1 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 \\ -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and hence find the eigenvalues of A. What is the dimension of each eigenspace?

[*Hint: The sum of the coefficients of the characteristic polynomial should be zero.*]

(c) The *infinite* matrix B has entries $B_{ij} = 1$ if $j = i + 1$ and $B_{ij} = 0$ otherwise, for $i, j \in \{1, 2, 3, \ldots\}$. We say that a vector $\mathbf{u} = (u_1, u_2, u_3, \ldots)^T$, where $u_j \in \mathbb{C}$, is an *eigenvector* of B with *eigenvalue* λ if

$$\sum_{j=1}^{\infty} B_{ij} u_j = \lambda u_i \text{ for } i = 1, 2, \ldots \text{ and also } 0 < \sum_{i=1}^{\infty} |u_i|^2 < \infty\,.$$

Find all of the eigenvalues and eigenvectors of the matrix B.

(d) Let C be the transpose of the matrix B in part (c), as defined by $C_{ij} = B_{ji}$. Find all of the eigenvalues and eigenvectors of the matrix C.

[*Hint: In parts (c) and (d) do not assume that results for finite matrices necessarily carry over to infinite matrices.*]

Paper 1, Section II

8A Vectors and Matrices

(a) What does it mean for an $n \times n$ complex matrix to be *diagonalisable*?

(b) Let A be an $n \times n$ complex matrix (not necessarily diagonalisable) and $p(z) = \sum_{j=0}^{m} a_j z^j$ a polynomial. The $n \times n$ matrix $p(A)$ is defined by $p(A) = \sum_{j=0}^{m} a_j A^j$. Prove that the set of eigenvalues of $p(A)$ is $\{p(\lambda) : \lambda \text{ is an eigenvalue of } A\}$.

(c) The *exponential* of an $n \times n$ complex matrix A is defined by

$$\exp(A) = \sum_{j=0}^{\infty} \frac{A^j}{j!}\,.$$

[You may assume that the series converges.] Suppose that A is diagonalisable. Without considering a differential equation, prove that $\det\big(\exp(A)\big) = \exp\big(\mathrm{Tr}(A)\big)$.

(d) Let

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Using ideas from part (c), or otherwise, find $\exp(B)$.

END OF PAPER