UNIVERSITY OF CAMBRIDGE

MAT2

MATHEMATICAL TRIPOS Part II

Friday, 07 June, 2024 9:00am to 12:00pm

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most six questions** from Section I and from any number of questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Number Theory

Define the *Möbius function* $\mu : \mathbb{N} \to \mathbb{C}$, and state the Möbius inversion formula.

Let $k > 1$ be an integer. We say that $n \in \mathbb{N}$ is k^{th}*-power free* if for all $d \in \mathbb{N}$ with $d > 1$, d^k does not divide n. Show that

$$\sum_{\substack{1 \leqslant d \leqslant n \\ d^k \mid n}} \mu(d) = \begin{cases} 1 & n \text{ is } k^{\text{th}}\text{-power free,} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that for $n \in \mathbb{N}$ we have

$$\mu(n) = \sum_{\substack{1 \leqslant d \leqslant n \\ (d,n)=1}} e^{2\pi i d/n}.$$

2G Topics in Analysis

(a) Show that the collection of algebraic numbers is countable.

(b) Suppose that $a_j > 0$ and $\sum_{j=1}^{\infty} a_j$ converges. Show that we can find $\theta_j \in \{0, 1\}$ so that

$$\sum_{j=1}^{\infty} \theta_j a_j$$

is transcendental.

(c) Suppose $\theta_j \in \{0, 1\}$ and $\theta_j = 1$ for infinitely many values of j. Show that

$$\sum_{j=1}^{\infty} \frac{\theta_j}{j!}$$

is irrational.

3K Coding and Cryptography

(a) (i) Consider a cryptosystem $(\mathcal{K}, \mathcal{M}, \mathcal{C})$. Let e, d be the respective encryption and decryption functions. Model the key and messages as independent random variables K, M taking values in $\mathcal{K}, \mathcal{M}$, respectively and such that $M = d(C, K) \in \mathcal{M}$ and $C = e(K, M) \in \mathcal{C}$. Show that $H(M|C) \leqslant H(K|C)$.

(ii) Let $\mathcal{M} = \mathcal{C} = \mathcal{A}$, where $\mathcal{A}$ is a finite alphabet. Suppose we send n messages $(M_1, \ldots, M_n)$ encrypted as $(C_1, \ldots, C_n)$ using the same key. Define the *unicity distance*. By making some reasonable assumptions, give a closed formula for the unicity distance as a function of $|\mathcal{K}|$, $|\mathcal{A}|$ and a certain constant.

(b) Suppose we model English text by a sequence of random variables $(X_n)_{n \geqslant 1}$ taking values in $\mathcal{A} = \{A, B, \ldots, Z, \texttt{space}\}$. We define the entropy of English to be

$$H_E = \lim_{n \to \infty} H(X_1, \ldots, X_n)/n.$$

Assuming H_E exists, show that $0 \leqslant H_E \leqslant \log_2 27$. [You may assume Gibbs' inequality.]

4J Automata and Formal Languages

(a) Let $D = (\Sigma, Q, \delta, q_0, F)$ and $D' = (\Sigma, Q', \delta', q_0', F')$ be two deterministic automata. Define what it means that f is a *homomorphism* from D to D'.

(b) Prove that if f is a homomorphism from D to D', then $\mathcal{L}(D) = \mathcal{L}(D')$.

(c) Define the following partial order on deterministic automata: we write $D \leqslant D'$ if and only if there is an injective homomorphism from D to D'. Check whether the following statements are true or false. Justify your answers.

(i) There is a deterministic automaton D such that, up to isomorphism, there are only finitely many D' such that $D \leqslant D'$.

(ii) There is a deterministic automaton D such that, up to isomorphism, there are only finitely many D' such that $D' \leqslant D$.

5L Statistical Modelling

Let the concentrations of insecticides $Y_1, \ldots, Y_n$ relative to covariates $x_1, \ldots, x_n \in \mathbb{R}^p$ follow the model

$$Y_i = \mu + x_i^T \beta + \varepsilon_i,$$

where $\mu \in \mathbb{R}$ and $\beta \in \mathbb{R}^p$ are unknown and the ε_i are i.i.d. with distribution $N(0, \sigma^2)$ for known $\sigma^2 > 0$. Suppose that it is not possible to measure Y_i directly, but we only have access to a random variable Z_i, satisfying

$$Z_i = \begin{cases} 1 & \text{if } Y_i > \tau, \\ 0 & \text{otherwise,} \end{cases}$$

where $\tau \in \mathbb{R}$ is some fixed known tolerance.

(a) Find a binary regression model with a suitable link function and some linear predictor for the data $(Z_1, x_1), \ldots, (Z_n, x_n)$. How can the parameters μ and β be estimated from these data?

(b) Given a new data point $x^* \in \mathbb{R}^p$, find an asymptotic $1 - \alpha$ confidence interval for the expected mean response at x^* in the binary regression model.

6A Mathematical Biology

A discrete-time model of alien cell population dynamics considers the coupled dynamics of immature and mature cells. At each time step, a proportion λ of the immature cells mature and a proportion μ of mature cells divides. When a mature cell divides it is replaced by four new immature cells. Finally, a proportion k of mature cells die at each time step.

(a) Explain briefly how this model may be represented by the equations

$$\begin{aligned} a_{t+1} &= (1 - \lambda) a_t + 4\mu b_t, \\ b_{t+1} &= \lambda a_t + (1 - \mu - k) b_t \end{aligned}$$

where $0 \leqslant \mu, \lambda, k \leqslant 1$ and $\mu + k \leqslant 1$.

(b) What is the expected number of offspring per cell?

(c) By considering the total population of cells, show that population growth is only possible if $3\mu > k$. How does this relate to your answer to part (b)?

(d) At time T, the population is treated with a chemical that completely stops cells from maturing for all $t \geqslant T$, but otherwise has no direct effects. Explain what will happen to the population afterwards. In terms of a_T and b_T, what will be the total number of cells in the long term?

7D Further Complex Methods

The Papperitz equation is a second-order linear differential equation given by

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+\left(\frac{1-\alpha-\alpha'}{z-a}+\frac{1-\beta-\beta'}{z-b}+\frac{1-\gamma-\gamma'}{z-c}\right)\frac{\mathrm{d}w}{\mathrm{d}z} \qquad (\dagger)$$
$$-\frac{(b-c)(c-a)(a-b)}{(z-a)(z-b)(z-c)}\left(\frac{\alpha\alpha'}{(z-a)(b-c)}+\frac{\beta\beta'}{(z-b)(c-a)}+\frac{\gamma\gamma'}{(z-c)(a-b)}\right)w=0$$

with the constraint $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1$ where $\alpha,\alpha',\beta,\beta',\gamma,\gamma',a,b,c\in\mathbb{C}$ and a,b,c are pairwise distinct.

(a) State the general conditions for $z_0\in\mathbb{C}$ to be a *regular singular point* for a second-order linear differential equation. Determine the regular singular points of the Papperitz equation.

(b) Consider the Papperitz equation with non-integer $\alpha-\alpha'$, $\beta-\beta'$ and $\gamma-\gamma'$. Derive the leading-order exponents of two linearly independent solutions of this equation near each of its regular singular points.

(c) What is the *Papperitz symbol* for equation ($\dagger$)?

(d) Consider now the hypergeometric equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+\left(\frac{C}{z}+\frac{1+A+B-C}{z-1}\right)\frac{\mathrm{d}w}{\mathrm{d}z}+\frac{AB}{z(z-1)}w=0\,,$$

where $A,\,B,\,C\in\mathbb{C}$. This equation can be obtained as a special case of the Papperitz equation by setting $a=0$, $b=1$ and taking the limit $c\to\infty$. In the following you may set $c=\infty$ without further proof or derivation. By comparing the coefficients of the hypergeometric and the Papperitz equations, and setting $\alpha=0$, $\beta=0$, establish a Papperitz symbol for the hypergeometric equation.

8E Classical Dynamics

(a) What is meant by the *phase space* of a mechanical system with n degrees of freedom?

(b) Write down Hamilton's equations in terms of a Poisson bracket on phase space.

(c) A particle of charge e and mass m moves through a magnetic field $\mathbf{B}$ in three dimensions with electric field $\mathbf{E}=\mathbf{0}$. Show that its equations of motion can be obtained using the non-standard Poisson brackets

$$\{q_i,q_j\}=0\,,\qquad \{q_i,p_j\}=\delta_{ij}\,,\qquad \{p_i,p_j\}=e\,\epsilon_{ijk}B_k\,,$$

but with a free Hamiltonian, $H=\frac{|\mathbf{p}|^2}{2m}$, where q_i and p_i are generalised coordinates and momenta, respectively.

9D Cosmology

The energy density of photons ρ is given by the Stefan–Boltzmann law,

$$\rho = \frac{4\sigma}{c} T^4 \, ,$$

where T is the temperature and σ a constant. As the volume V of the Universe slowly expands, the first law of thermodynamics relates the photon entropy S to the energy $E = \rho V$ and the pressure $P = \rho/3$ (for vanishing chemical potential $\mu = 0$),

$$\mathrm{d}E = T\mathrm{d}S - P\mathrm{d}V \, .$$

(a) By substituting the Stefan–Boltzmann law, show that the entropy differential becomes

$$\mathrm{d}S = \frac{16\sigma}{3c}\Big(T^3\mathrm{d}V + 3T^2V\mathrm{d}T\Big) \, ,$$

which should be integrated to find an expression for the photon entropy density $s = S/V$.

(b) If the interaction rate Γ maintaining the photons in equilibrium is much greater than the Hubble expansion rate H (i.e. $\Gamma \gg H$), briefly give the key reason why the photon number and entropy are conserved as the Universe expands (provided the effective number of degrees of freedom g_* of particle species in equilibrium also does not change). Why does the photon temperature fall as $T \propto 1/a$, where a is the scale factor?

(c) Electrons and positrons annihilate and fall out of equilibrium after neutrino decoupling at around $k_B T \approx 1\,\mathrm{MeV}$. By counting the effective number of degrees of freedom g_* in equilibrium before and after this process, provide a brief explanation for why the photon temperature T_γ and neutrino temperature T_ν are related today by

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} .$$

10E Quantum Information and Computation

(a) Verify that for any 2×2 matrix, A, the following relation holds

$$(I \otimes A)\left|\Phi^+\right\rangle = (A^T \otimes I)\left|\Phi^+\right\rangle,$$

where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, I denotes the 2×2 identity matrix and T denotes transposition taken with respect to the standard basis $\{|0\rangle, |1\rangle\}$.

(b) Let $\theta \in [0, 2\pi)$,

$$U_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

$|\psi_1\rangle = U_\theta |0\rangle$ and $|\psi_2\rangle = U_\theta |1\rangle$.

(i) Show that $ZX|\psi_2\rangle = |\psi_1\rangle$, where X and Z are the usual one-qubit gates.

(ii) Show that the Bell state $|\Phi^+\rangle$ can be written as $\frac{1}{\sqrt{2}}(|\psi_1\rangle|\psi_1\rangle + |\psi_2\rangle|\psi_2\rangle)$. [*Hint: Use part (a).*]

(iii) Suppose Alice and Bob initially share the Bell state $|\Phi^+\rangle$, with the first qubit being with Alice and the second qubit being with Bob. Alice then applies U_θ^{-1} to her qubit and then measures it in the computational basis. What are Alice's possible outcomes and what are the corresponding states of Bob's qubit after Alice's measurement?

(iv) Suppose Alice knows the value of θ but Bob does not. Give a protocol by which Alice can transmit $|\psi_1\rangle$ to Bob by sending just one classical bit to him, given that they share the Bell state $|\Phi^+\rangle$. (This is in contrast to general teleportation of an unknown qubit, which uses one Bell state and two classical bits of communication.)

SECTION II

11F Number Theory

Let $\theta \in \mathbb{R}$ be irrational.

Define the *convergents* $(p_n, q_n)_{n \geqslant 0}$ of the continued fraction expansion $[a_0, a_1, a_2, \dots]$ of θ. Prove that $p_n/q_n \to \theta$ as $n \to \infty$, and that if $n \geqslant 1$ is odd then

$$\frac{p_{n-1}}{q_{n-1}} < \theta < \frac{p_n}{q_n}.$$

Compute the continued fraction expansion of $\sqrt{7}$.

For each of the following equations, either find a solution in strictly positive integers x, y, or prove that no such solution exists:

(i) $x^2 - 7y^2 = 1$.

(ii) $x^2 - 7y^2 = -1$.

(iii) $x^2 - 7y^2 = 5$.

(iv) $p_x/q_x = 34/13$, where $(p_n, q_n)_{n \geqslant 0}$ is the sequence of convergents of the continued fraction expansion of $\sqrt{7}$.

12G Topics in Analysis

(a) Suppose we have n points A_1, A_2, …, A_n, in order, along a straight line with each point coloured red or green. If A_1 is coloured red and A_n green, show that there must be an odd number of segments $A_j A_{j+1}$ with A_j and A_{j+1} of different colours.

(b) Consider an equilateral triangle DEF divided up into a grid of small equilateral triangles by 3 sets of n equidistant lines parallel to each side. We refer to the lines joining neighbouring points of the grid as segments.

(i) Suppose that each vertex of the grid is coloured red, green or blue, that every vertex on DE is coloured red or green, every vertex on EF is coloured green or blue and every vertex on FD is coloured blue or red. Show that there is an odd number of triangles in the grid with vertices of different colours.

(ii) Suppose instead that each vertex of the grid is coloured red, green or blue, and that the three vertices D, E and F have different colours. Must there be a triangle in the grid with vertices of different colours? Give reasons.

(c) Let us write

$$\triangle = \{\lambda_1 \mathbf{d} + \lambda_2 \mathbf{e} + \lambda_3 \mathbf{f} \, : \, \lambda_1 + \lambda_2 + \lambda_3 = 1, \lambda_u \geqslant 0\},$$

where $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ are the position vectors of D, E and F, and write I for the closed line segment DE, J for the closed line segment EF, K for the closed line segment FD.

Which of the following statements are true and which are false? Give a proof or a counterexample.

(i) There does not exist a continuous function $f : \triangle \to \partial\triangle$ with

$$f(\mathbf{x}) \in I \text{ for all } \mathbf{x} \in I, \; f(\mathbf{x}) \in J \cup K \text{ for all } \mathbf{x} \in J \cup K.$$

(ii) There does not exist a continuous function $g : \triangle \to \partial\triangle$ with

$$g(\mathbf{x}) \in I \text{ for all } \mathbf{x} \in I, \; g(\mathbf{x}) \in J \text{ for all } \mathbf{x} \in J, \; g(\mathbf{x}) \in K \text{ for all } \mathbf{x} \in K.$$

13L Statistical Modelling

A researcher has collected a dataset on 53 patients to study the occurrence of a certain disease. The dataset contains for each patient, a variable (`dis`) indicating if they are suffering from the disease (`dis` = 1) or not (`dis` = 0), a continuous variable (`pollution`) describing the level of air pollution near their home, as well as a variable (`risk`) recording which of two risk categories the patient falls under. Below is given the (shortened) `R` output of the analysis performed by the researcher.

```
> head(disease)
    dis  pollution risk
  1   0   58.44204    2
  2   0   45.29248    1

> binom.lm1 <- glm(dis ~ pollution + risk, family=binomial, data=disease)
> summary(binom.lm1)

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept) -11.83686    3.62821  -3.262  0.00110  **
pollution     0.18243    0.05679   3.212  0.00132  **
risk2         1.53893    0.74665  -2.061  0.03929  *

  (Dispersion parameter for binomial family taken to be 1)

      Null deviance: 73.5  on ?  degrees of freedom
  Residual deviance: 55.3  on ?  degrees of freedom
  AIC: 61.3
```

(a) Write down the generalised linear model being fitted. Why is there no coefficient for `risk1`?

(b) Give an interpretation for the coefficient of `pollution`.

(c) What are the missing degrees of freedom in the output?

(d) How should the `R` code above be changed to fit the model that corresponds to the 'Null deviance' in the output? What is the AIC value of that model?

(e) State the null and alternative hypotheses for the test performed in the code below and describe the form of the test.

What can the researcher conclude from the following ouput?

```
> pchisq(73.5-55.3, df = 2, lower.tail = FALSE)
[1] 0.0001142191
> binom.lm2 <- glm(dis ~ pollution * risk, family=binomial, data=disease)
> pchisq(binom.lm1$deviance-binom.lm2$deviance, 1, lower.tail=FALSE)
[1] 0.8705801
```

14A Mathematical Biology

An activator-inhibitor system is described by the equations

$$\frac{\partial u}{\partial t} = D_1 \frac{\partial^2 u}{\partial x^2} + u - uv + u^2,$$
$$\frac{\partial v}{\partial t} = D_2 \frac{\partial^2 v}{\partial x^2} + \alpha u^2 - \beta uv,$$

where $\alpha, \beta, D_1, D_2 > 0$.

(a) Find conditions on α and β for the spatially homogeneous system to have a stable stationary solution with $u > 0$ and $v > 0$. Sketch this region in the α–β plane (for the quadrant with $\alpha, \beta > 0$).

(b) Consider spatial perturbations of the form $\cos(kx)$ about the solution found in part (a), and set $\lambda = D_1/D_2$. Find conditions for the system to be unstable. For fixed β, sketch in the λ–α plane the region where spatial instability is possible for some $k \in \mathbb{R}$.

(c) Consider a general system of the form

$$\frac{\partial u}{\partial t} = D_1 \frac{\partial^2 u}{\partial x^2} + f(u, v),$$
$$\frac{\partial v}{\partial t} = D_2 \frac{\partial^2 v}{\partial x^2} + g(u, v).$$

A *Turing instability* of this system is when a stationary solution is stable to homogenous perturbations but is unstable to some spatial perturbation. Explain why a Turing instability is not possible when $D_1 = D_2$.

Verify that this is consistent with your answer for part (b) .

15E Classical Dynamics

Torque free rotation of a rigid body with principle moments of inertia (I_1, I_2, I_3) is described by Euler's equations

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3\,, \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1\,, \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2\,. \end{aligned}$$

(a) Write down expressions for the kinetic energy T and total angular momentum L, and show that they are each conserved.

(b) Suppose that $I_1 < I_2$ and $I_3 = I_1 + I_2$, and that initially $\omega_2(0) = 0$ while $\omega_1(0)\sqrt{I_2 - I_1} = \omega_3(0)\sqrt{I_2 + I_1}$. Show that subsequently

$$\dot{\omega}_1^2 = \left(\frac{2T}{I_2} - \omega_1^2\right)\left(\frac{I_2 - I_1}{I_2 + I_1}\right)\omega_1^2\,.$$

(c) Hence show that

$$\omega_1(t) = A\,\operatorname{sech}(\Omega t)$$

for constants A and Ω which you should find in terms of the kinetic energy and moments of inertia.

(d) Describe the motion as $t \to \infty$.

16I Logic and Set Theory

(a) In this part of the question, work within ZF.

Define the *rank* of a set. For a non-zero limit ordinal α, what is the rank of the set of all functions from α to α?

Define the *von Neumann hierarchy* and prove that it exhausts the set-theoretic universe, in the sense that every set is an element of some member of the von Neumann hierarchy.

(b) In this part of the question, work within ZFC.

Define the cardinal numbers $\aleph_\alpha$. Is every infinite cardinal equal to some $\aleph_\alpha$? Justify your answer.

Prove that $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$ for every ordinal α. Deduce that

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \aleph_\beta$$

for ordinals $\alpha \leqslant \beta$.

Prove that the following sentence is a theorem of ZFC, where $t \equiv x$ is taken to mean 't and x have the same cardinality.'

$$(\forall x)(\neg(x = \emptyset) \Rightarrow \neg(\exists y)((\forall t)((t \equiv x) \Rightarrow (t \in y))))$$

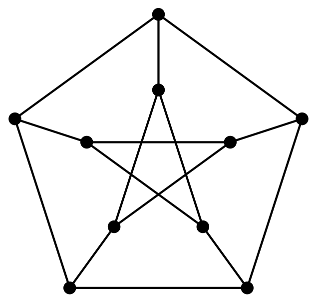
17I Graph Theory

What does it mean to say that a graph G is *strongly regular* with parameters (k, a, b)? Let G be a strongly regular graph with parameters (k, a, b) on n vertices such that $b \geqslant 1$ and $G \neq K_n$. Prove the *rationality condition,* namely that the numbers

$$\frac{1}{2}\left(n - 1 \pm \frac{(n-1)(b-a) - 2k}{\sqrt{(a-b)^2 + 4(k-b)}}\right)$$

are integers.

What are the eigenvalues (with their multiplicities) of the Petersen graph (shown below)?



Show that the set of edges of K_{10} cannot be partitioned into the edges of three copies of the Petersen graph. [*Hint: Suppose that it can be, and that the three Petersen graphs have adjacency matrices A, B and C. You may wish to consider an appropriate common eigenvector of A and B.*]

18H Galois Theory

(a) Give an example of a pair of monic quartic polynomials $f, g \in \mathbb{Z}[X]$ whose splitting fields over $\mathbb{Q}$ are isomorphic, but whose Galois groups over $\mathbb{Q}$ are not conjugate as subgroups of S_4.

(b) Let p be a prime and $q = p^d$ with $d \geqslant 1$. Show that there exists a field with q elements, and that it is unique up to isomorphism. [Standard results about splitting fields may be quoted without proof.] Show also that any irreducible factor of $X^q - X \in \mathbb{F}_p[X]$ has degree at most d.

(c) State a theorem which explains how reduction mod p may be used to help compute Galois groups. The polynomial $f(X) = X^4 - 21X^2 + 3X + 100$ has discriminant 547^2. Compute $\mathrm{Gal}(f/\mathbb{Q})$.

19H Representation Theory

What is the topological group S^1? Assuming any necessary facts about continuous homomorphisms with domain $(\mathbb{R}, +)$, show that every irreducible complex representation of S^1 is of the form

$$z \mapsto z^n : S^1 \to GL_1(\mathbb{C})$$

for some $n \in \mathbb{Z}$.

Let $\rho_V : SU(2) \to GL(V)$ be a complex representation of the topological group $SU(2)$ and let χ_V be its character.

(a) Show that χ_V is determined by its restriction to a subgroup T of $SU(2)$ isomorphic to S^1 and deduce that χ_V may be written in the form $\sum_{n\in\mathbb{Z}} a_n z^n$ for some non-negative integers a_n such that $\sum_{n\in\mathbb{Z}} a_n < \infty$. Show moreover that $a_n = a_{-n}$ for all $n \in \mathbb{Z}$.

(b) Let V_n be the $(n+1)$-dimensional irreducible representation of $SU(2)$. Write down χ_{V_n} in the form given in part (a). Decompose $V_4 \otimes V_4$, S^2V_4 and $\Lambda^2 V_4$ as a direct sum of irreducible representations up to isomorphism.

20F Number Fields

Let K be a number field.

Define the *norm* $N(I)$ of an ideal I in $\mathcal{O}_K$.

Let $d = [K : \mathbb{Q}]$. Define the *discriminant of a tuple* $(\alpha_1, \ldots, \alpha_d) \in K$. Define the *discriminant of* K.

State a formula for the norm of an ideal in terms of discriminants without proof.

Prove that $N(\alpha\mathcal{O}_K) = |N_{K/\mathbb{Q}}(\alpha)|$ for all $\alpha \in \mathcal{O}_K$.

Now let L/K be an extension of number fields, and let $P \subset \mathcal{O}_K$ and $Q \subset \mathcal{O}_L$ be non-zero prime ideals.

(a) Prove that $Q | P\mathcal{O}_L$ if and only if $P = Q \cap \mathcal{O}_K$.

(b) Suppose that $P = Q \cap \mathcal{O}_K$ and $[O_K : P] = [O_L : Q]$. Prove that if $Q = \alpha\mathcal{O}_L$ for some $\alpha \in \mathcal{O}_L$, then $P = N_{L/K}(\alpha)\mathcal{O}_K$.

21J Algebraic Topology

State the Mayer–Vietoris theorem for a simplicial complex K which is the union of two subcomplexes M and N.

For each $k = 2, 3, 4, \ldots$, construct an explicit simplicial complex K having

$$H_i(K) \cong \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/k & i = 1 \\ 0 & \text{else.} \end{cases}$$

Justify your answer.

22G Linear Analysis

In this question all spaces are complex.

(a) Let T be an operator on l_2. Prove that the spectrum of T is non-empty. [If you use a power series expression for the resolvent function then you must prove it.]

(b) Let $(e_n)_{n=1}^{\infty}$ be the usual basis of l_2. In each case below, an operator on l_2 is given; determine for each operator whether or not it is compact, and find its spectrum. [Any results about spectra from the course may be used without proof, as long as they are stated clearly.]

(i) $T(e_n) = e_{n+1}$ for all n.

(ii) $S(e_n) = e_{n-1}$ for all $n > 1$, with $S(e_1) = 0$.

(iii) $R(e_n) = \frac{1}{n} e_{n+1}$ for all n.

(iv) $Q(e_n) = \frac{1}{n} e_{n-1}$ for all $n > 1$, with $Q(e_1) = 0$.

23G Analysis of Functions

(a) For $f \in H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, show that there exists a unique weak solution $u \in \mathcal{S}'(\mathbb{R}^n)$ to the equation

$$\Delta^2 u + u = f. \qquad (\star)$$

(b) Show that if f has compact support in $\mathbb{R}^n$ and is infinitely differentiable, then the solution u is also infinitely differentiable.

(c) Now let $n = 3$ and let $f = \delta_0$ be the Dirac measure at zero. Show that there exists a unique continuous function u solving $(\star)$.

[*You may use results about Fourier transforms and Sobolev spaces from the course without proof, provided they are clearly stated.*]

24F Algebraic Geometry

In this question, all algebraic varieties are over the complex numbers $\mathbb{C}$.

Let $\varphi : \mathbb{P}^2 \to \mathbb{P}^1$ be a morphism. Show that φ is constant. [You may use without proof the fact that any two curves in $\mathbb{P}^2$ have non-empty intersection.]

Show that there is no closed subvariety of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ isomorphic to $\mathbb{P}^2$.

State the Riemann–Hurwitz theorem.

Consider a non-singular projective curve $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ defined by a bihomogeneous equation $f(x_0, x_1, y_0, y_1) = 0$ which is homogeneous of degree 2 in x_0, x_1 and homogeneous of degree 3 in y_0, y_1. Here x_0, x_1 are coordinates on the first $\mathbb{P}^1$ and y_0, y_1 are coordinates on the second $\mathbb{P}^1$. Compute the genus of X. Deduce that X is not isomorphic to a non-singular projective curve in $\mathbb{P}^2$. [You may use without proof the fact that a non-singular projective curve in $\mathbb{P}^2$ of degree d has genus $g = (d-1)(d-2)/2$.]

25J Differential Geometry

(a) Let X be a smooth, compact, connected manifold. State the homotopy lemma for smooth maps $f, g : X \to X$ and the homogeneity lemma for X. Prove that the number of pre-images modulo 2 of a smooth map $f : X \to X$ is the same at every regular value. Deduce a proof of the smooth Brouwer fixed-point theorem.

(b) Let $n \geqslant 1$ and $A := \{1 \leqslant \|x\| \leqslant 2\} \subset \mathbb{R}^{n+1}$. Consider a smooth map $\varphi : \mathbb{S}^n \to \mathbb{S}^n$ such that $\varphi(x) \perp x$ for all $x \in \mathbb{S}^n$, and extend it to $\tilde{\varphi} : A \to A$ by $\tilde{\varphi}(x) := \varphi(x/\|x\|)\|x\|$. For $\epsilon > 0$, define $\psi_\epsilon : A \to A$ by

$$\psi_\epsilon(x) := \frac{x + \epsilon\tilde{\varphi}(x)}{\sqrt{1+\epsilon^2}} \,.$$

Prove that, for ϵ small enough, ψ_ϵ is a diffeomorphism from A to A with $\det D\psi_\epsilon > 0$.

(c) Prove $\int_A \det(\mathrm{Id} + \epsilon D\tilde{\varphi}) = \mathrm{vol}(A)(1+\epsilon^2)^{(n+1)/2}$, where $\mathrm{vol}(A)$ is the volume of A.

(d) Deduce that, when n is even, there is no smooth map $\varphi : \mathbb{S}^n \to \mathbb{S}^n$ such that $\varphi(x) \perp x$ for all $x \in \mathbb{S}^n$.

26G Probability and Measure

(a) Define what it means for a sequence of real-valued random variables $(X_n)_{n\in\mathbb{N}}$ to be *uniformly integrable*. Show that if $(X_n)_{n\in\mathbb{N}}$ is bounded in L^p for some $p > 1$, then $(X_n)_{n\in\mathbb{N}}$ is uniformly integrable. Give, with justification, a counterexample to show that a sequence of random variables bounded in L^1 need not be uniformly integrable.

(b) Let $(X_n)_{n\in\mathbb{N}}$ be an identically distributed sequence of real-valued random variables in L^1. Let $S_n = X_1 + \cdots + X_n$. Show that the sequence $(S_n/n)_{n\in\mathbb{N}}$ is uniformly integrable. Assuming that $S_n/n \to \mathbb{E}(X_1)$ in probability, prove that $S_n/n \to \mathbb{E}(X_1)$ in L^1.

(c) Now assume that $(X_n)_{n\in\mathbb{N}}$ is an identically distributed sequence of random variables in L^2. Let $M_n = \max_{k\leqslant n} |X_k|$.

(i) Show that $n^{-1/2}M_n \to 0$ in probability as $n \to \infty$.

(ii) Show that $n^{-1/2}\mathbb{E}(M_n) \to 0$ as $n \to \infty$.

27K Applied Probability

(a) A train station has a platform where passengers arrive according to a renewal process with rate $\mu > 0$ (i.e. inter-arrival times have mean $1/\mu$). As soon as N passengers arrive, a train dispatches with all N passengers on board. The process continues. For $c > 0$, the company incurs a cost at the rate of nc per unit time when exactly n passengers are waiting, and a fixed cost K each time a train departs. What is the optimum value of N?

(b) A factory manufactures bulbs with independent lifetimes that are uniformly distributed in $[0, 3]$ (months). Whenever a bulb from this factory dies, it is immediately replaced by a new bulb from the same factory. You observe a bulb from this factory at the instant t. Let A_t be the time the bulb has been running (until the instant t) and let E_t be the remaining time the bulb will last (after instant t). For large t, find $\mathbb{P}(A_t \leqslant x)$, $\mathbb{P}(E_t \leqslant x)$ and $\mathbb{P}(A_t + E_t \leqslant x)$ for $x \in [0, 3]$.

(c) Let Π be the points of a non-homogeneous Poisson process on $\mathbb{R}^d$ with intensity function λ. Let $S = \sum_{x\in\Pi} g(x)$ where g is a smooth non-negative function. Show that $\mathbb{E}(S) = \int_{\mathbb{R}^d} g(u)\lambda(u)\,du$ and $\mathrm{var}(S) = \int_{\mathbb{R}^d} g(u)^2\lambda(u)\,du$, provided these integrals converge.

28L Principles of Statistics

Let $X_1, \ldots, X_n$ be i.i.d. draws from a distribution on $\mathbb{R}$, with $\mu = \mathbb{E}(X_1)$ and $\sigma^2 = \mathrm{Var}(X_1) > 0$.

(a) Show that $\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)^2 \xrightarrow{P} \sigma^2$, where $\bar{X}_n$ is the sample mean of the observations.

Let the function $g : \mathbb{R} \to \mathbb{R}$ be differentiable at μ with $g'(\mu) \neq 0$. Consider the plug-in estimator $T_n = T_n(X_1, \ldots, X_n) = g(\bar{X}_n)$ for the parameter $\theta = g(\mu)$.

(b) Using the Delta method, provide a formula for σ_n such that

$$\frac{T_n - \theta}{\sigma_n} \xrightarrow{d} N(0,1).$$

Consider now the jackknife variance estimator

$$v_{\mathrm{JACK}} = \frac{n-1}{n}\sum_{i=1}^n \left(T_{n-1,i} - \frac{1}{n}\sum_{j=1}^n T_{n-1,j} \right)^2,$$

where $T_{n-1,i} = T_{n-1}(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$, for $i = 1, \ldots, n$, is the leave-one-out estimator. In the following suppose further that g is twice-differentiable with $\sup_{x\in\mathbb{R}} |g''(x)| = 2M < \infty$.

(c) By considering a Taylor expansion of $T_{n-1,i} - T_n$, show that

$$v_{\mathrm{JACK}} = \frac{n-1}{n}(A_n + B_n + 2C_n),$$

where, for some $R_{n,i}$ satisfying $|R_{n,i}| \leqslant M(\bar{X}_{n-1,i} - \bar{X}_n)^2$ that you should specify,

$$A_n = (g'(\bar{X}_n))^2 \sum_{i=1}^n (\bar{X}_{n-1,i} - \bar{X}_n)^2, \qquad B_n = \sum_{i=1}^n \left(R_{n,i} - \frac{1}{n}\sum_{j=1}^n R_{n,j} \right)^2$$

and $|C_n| \leqslant \sqrt{A_n B_n}$. Here $\bar{X}_{n-1,i}$ is the sample mean of the observations with X_i excluded.

Further assuming that $\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)^4 \xrightarrow{P} c < \infty$, show that $v_{\mathrm{JACK}}/\sigma_n^2 \xrightarrow{P} 1$.

[*Hint: Use the fact (which you need not derive) that* $\bar{X}_{n-1,i} - \bar{X}_n = \frac{1}{n-1}(\bar{X}_n - X_i)$.]

(d) Give, with justification, an asymptotically valid $(1-\alpha)$-level confidence interval for θ based on the jackknife estimator.

29L Stochastic Financial Models

Let $(W_t)_{t \geqslant 0}$ be a Brownian motion.

(a) Using the definition of Brownian motion, show that $(W_t)_{t \geqslant 0}$ is a Gaussian process with $\mathbb{E}(W_t) = 0$ and $\mathbb{E}(W_s W_t) = s$ for all $0 \leqslant s \leqslant t$.

(b) Let

$$I_t = \int_0^t W_s \, ds \text{ for all } t \geqslant 0.$$

Show that $\mathbb{E}(I_t) = 0$ and $\mathbb{E}(I_s I_t) = \frac{1}{6} s^2 (3t - s)$ for all $0 \leqslant s \leqslant t$.

Consider a continuous time market with interest rate r and a stock whose time-t price is

$$S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t},$$

for a given volatility parameter $\sigma > 0$.

(c) Show that the discounted stock price $(e^{-rt} S_t)_{t \geqslant 0}$ is a martingale with respect to the filtration generated by the Brownian motion.

(d) Show that the time-0 Black–Scholes price of a European call with strike K and maturity date T is $S_0 \; F(\sigma^2 T, K e^{-rT}/S_0)$, where

$$F(v, m) = \mathbb{E}[(e^{-\frac{1}{2}v + \sqrt{v} Z} - m)^+] \text{ for all } v, m \geqslant 0$$

and Z is a standard normal random variable.

(e) Show that the time-0 Black–Scholes price of a European claim with time-T payout $\left(\exp\left(\frac{1}{T} \int_0^T \log S_t \, dt \right) - K \right)^+$ is

$$S_0 e^{-(\frac{1}{2}r + \alpha \sigma^2)T} F(\tfrac{1}{3}\sigma^2 T, K e^{-(\frac{1}{2}r - \alpha\sigma^2)T}/S_0),$$

for a constant α to be determined. [You may assume that the process $(I_t)_{t \geqslant 0}$ defined in part (b) is Gaussian.]

30K Mathematics of Machine Learning

(a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. What is meant by the *subdifferential* $\partial f(\alpha)$ of f at $\alpha \in \mathbb{R}^n$?

Prove that α minimises f if and only if $0 \in \partial f(\alpha)$.

(b) Consider a classification setting with input–output pairs $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^p \times \{-1, 1\}$. What is meant by the empirical ϕ-risk $\hat{R}_\phi(h)$ of hypothesis $h : \mathbb{R}^p \to \mathbb{R}$ given convex surrogate loss ϕ?

(c) In the following, we consider hypothesis class $\mathcal{H} := \{h_\beta : \beta \in \mathbb{R}^p\}$, where $h_\beta : x \mapsto x^\top \beta$. Fix $\lambda > 0$ and let $\hat{\beta}$ be the minimiser (assumed to exist) of $q : \mathbb{R}^p \to \mathbb{R}$ given by

$$q(\beta) = \hat{R}_\phi(h_\beta) + \lambda \|\beta\|_2^2.$$

Let $X \in \mathbb{R}^{n \times p}$ be the matrix with ith row x_i for $i = 1, \ldots, n$ and suppose that $XX^\top$ is invertible. Writing $P := X^\top (XX^\top)^{-1} X \in \mathbb{R}^{p \times p}$, show that $q(\beta) \geqslant q(P\beta)$. [*Hint: Consider the decomposition* $\beta = \{P + (I - P)\}\beta$.]

Hence conclude that $\hat{\beta} = X^\top \hat{\alpha}$ where $\hat{\alpha}$ is the minimiser of $r : \mathbb{R}^n \to \mathbb{R}$ given by

$$r(\alpha) = q(X^\top \alpha).$$

(d) Now take ϕ to be the hinge loss. Writing $k_i \in \mathbb{R}^n$ for the ith column of $XX^\top$, show that

$$\frac{1}{n} \sum_{i=1}^{n} k_i t_i = 2\lambda XX^\top \hat{\alpha},$$

where t_i is related to $y_i x_i^\top \hat{\beta}$ in a way you should specify. Hence conclude that whenever $y_i x_i^\top \hat{\beta} > 1$, then $\hat{\alpha}_i = 0$.

31C Asymptotic Methods

Consider the eigenvalue problem

$$\epsilon^2 \frac{d^2y}{dx^2} - \left(1 - \frac{\lambda^2}{x^2}\right) y = 0, \qquad (\star)$$

defined for $x \in [1, \infty)$, with eigenvalue $\lambda > 1$ and small parameter $0 < \epsilon \ll 1$, and with boundary conditions $y(1) = 0$ and $y \to 0$ as $x \to \infty$.

(a) Find the single turning point of this problem, $x = x_{\rm tp}$.

Write down the WKBJ form of the inner solution y_1, valid in the region $x_{\rm tp} - x \gg \epsilon$, and the outer solution y_2, valid in the region $x - x_{\rm tp} \gg \epsilon$. You may quote relevant WKBJ results from the lectures.

(b) Derive an approximate solution to $(\star)$ near the turning point. By matching it to y_1, obtain the quantisation condition

$$\int_1^{\lambda} \left(\frac{\lambda^2}{x^2} - 1\right)^{1/2} dx = \epsilon \left(n + \frac{3}{4}\right) \pi, \qquad (\dagger)$$

for integer n. [You may quote the following asymptotic behaviour of the Airy function:

$$\mathrm{Ai}(t) \sim \pi^{-1/2} (-t)^{-1/4} \cos\left[\tfrac{2}{3}(-t)^{3/2} - \tfrac{1}{4}\pi\right] \quad \text{as} \quad t \to -\infty. \,]$$

(c) We seek approximations to the smallest eigenvalues λ, which are close to 1. Using the Taylor-expansion method in $(\dagger)$, or otherwise, show that $\lambda \approx 1 + \alpha\epsilon^{2/3}$, where you should determine α. [*Hint: Consider the substitution* $x = \lambda(1-u)$.]

(d) Show that the largest eigenvalues, $\lambda \gg 1$, can be approximated by solutions to the transcendental equation

$$\lambda \ln \lambda = \epsilon \left(n + \frac{3}{4}\right) \pi.$$

32A Dynamical Systems

(a) Consider a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^2$.

(i) State the Poincaré–Bendixson Theorem.

(ii) Explain why every periodic orbit must enclose at least one fixed point.

(b) Consider the system in $\mathbb{R}^2$ given by

$$\begin{aligned} \dot{x} &= -x + ay + x^2 y, \\ \dot{y} &= \quad b - ay - x^2 y, \end{aligned}$$

where a and b are real, positive parameters.

(i) Find the fixed point of the system. On a sketch of the (a, b)-plane (for the quadrant where $a, b > 0$), show where the fixed point is stable.

(ii) A closed region $\mathcal{D}$ of the (x, y)-plane is given by the filled polygon with vertices at $(0, 0)$, $(0, b/a)$, $(b, b/a)$, (x^*, y^*) and $(x^*, 0)$, where (x^*, y^*) is such that $\dot{x} = 0$ at that point and the line segment between $(b, b/a)$ and (x^*, y^*) has gradient -1.

Show that trajectories do not leave $\mathcal{D}$.

[There is no need to determine (x^*, y^*) explicitly, but you may use that $b < x^*$.]

(iii) Use your results above to give conditions on the parameters a and b for the system to have a periodic orbit.

33B Principles of Quantum Mechanics

(a) For a harmonic oscillator A of unit mass and frequency ω, write down the Hamiltonian in terms of position and momentum operators (X_A, P_A) as well as in terms of creation and annihilation operators $(a_A, a_A^\dagger)$. Briefly describe the Hilbert space of the system.

(b) Now consider the AB system comprising oscillator A and a second harmonic oscillator B with the same frequency and unit mass. The two oscillators are coupled by an interaction Hamiltonian $H_{\text{int}} = \lambda X_A X_B$, where λ is a real parameter.

(i) Compute the corrections to both the ground state of the AB system and its energy to linear order in λ.

(ii) Using the corrected ground state obtained in part (i), compute the density operator of the full system and hence the reduced density operator ρ_A, by tracing over B. Your final answer should be a density operator ρ_A that is normalised up to and including $\mathcal{O}(\lambda^2)$. [You may assume that corrections to the ground state beyond linear order do not affect the normalised density operator up to and including $\mathcal{O}(\lambda^2)$.]

(iii) For any properly normalised density operator ρ, the purity is defined by $\gamma = \text{Tr}\big(\rho^2\big)$. Determine an upper and lower bound on γ for a general state and determine its value for a pure state.

(iv) Compute the purity of ρ_A to order λ^2. For what values of λ does it satisfy the upper and lower bounds? Interpret your findings.

34E Applications of Quantum Mechanics

Consider a particle of charge $-e$ and mass m moving in three dimensions with an electric field $\boldsymbol{E}$ and a magnetic field $\boldsymbol{B}$. The time-dependent Schrödinger equation for the particle's wave function $\psi(\boldsymbol{x}, t)$ is

$$i\hbar \left(\frac{\partial}{\partial t} - \frac{ie}{\hbar}\phi \right) \psi = -\frac{\hbar^2}{2m} \left(\boldsymbol{\nabla} + \frac{ie}{\hbar}\boldsymbol{A} \right)^2 \psi \ , \qquad (\star)$$

where $\boldsymbol{A}$ is the vector potential and ϕ is the electric potential, with $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ and $\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \boldsymbol{A}}{\partial t}$.

(a) The Schrödinger equation $(\star)$ should be invariant under the gauge transformations

$$\boldsymbol{A}(\boldsymbol{x}, t) \quad \rightarrow \quad \boldsymbol{A}(\boldsymbol{x}, t) + \boldsymbol{\nabla} f(\boldsymbol{x}, t) \ ,$$
$$\phi(\boldsymbol{x}, t) \quad \rightarrow \quad \phi(\boldsymbol{x}, t) - \frac{\partial}{\partial t} f(\boldsymbol{x}, t) \ ,$$

where f is an arbitrary smooth function, together with a suitable transformation of $\psi(\boldsymbol{x}, t)$. State such a suitable transformation of $\psi(\boldsymbol{x}, t)$ and check explicitly that $(\star)$ is invariant under these transformations.

(b) In the presence of electric and magnetic fields, the probability current is modified to

$$\boldsymbol{J}(\boldsymbol{x}, t) = -\frac{i\hbar}{2m} \left(\psi^* \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \psi^* \right) + \frac{e}{m} \psi^* \psi \boldsymbol{A} \ .$$

Show that $\boldsymbol{J}$ is gauge invariant and that it satisfies the conservation equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0 \ ,$$

where $\psi(\boldsymbol{x}, t)$ satisfies the Schrödinger equation and $\rho = \psi^* \psi$.

(c) Consider the situation with constant electromagnetic fields, where $\boldsymbol{B} = (0, 0, B)$, $\boldsymbol{E} = (0, \mathcal{E}, 0)$, and B and $\mathcal{E}$ are constants. Verify that the vector and electric potential can be chosen to have the form

$$\boldsymbol{A} = B(-y, 0, 0) \ , \qquad \phi = -\mathcal{E} y \ .$$

Restrict the motion to the (x, y) plane. Show that stationary states are given by

$$\psi(\boldsymbol{x}, t) = e^{ikx} e^{-iEt/\hbar} \varphi(y) \ ,$$

where $\varphi(y)$ is a harmonic oscillator wavefunction (i.e. it satisfies the time-independent Schrödinger equation with a harmonic oscillator potential), E is the energy and k is a real constant. At what position is the oscillator centered?

Find the allowed energies, and show that they can be written as

$$E = \hbar\omega_1 \left(n + \frac{1}{2} \right) + \mathcal{W} \ ,$$

where n is a non-negative integer, and you should determine both ω_1 and $\mathcal{W}$ explicitly. Briefly discuss what happens to the Landau levels in the presence of this electric field.

35B Statistical Physics

(a) What is meant by a *first-order phase transition* and what is meant by a *second-order phase transition*?

(b) Derive the Clausius-Clapeyron equation for dp/dT, with p the pressure, along the first-order phase transition curve between a liquid and a gas in terms of the temperature T, the latent heat L, and the volume per unit particle in the two phases v_{liquid} and v_{gas}. [You may assume that the chemical potentials of the two phases are equal on the phase transition curve.]

(c) Assuming that L is constant, $v_{\text{gas}} \gg v_{\text{liquid}}$, and that the gas obeys the ideal gas law, derive an equation for the phase transition curve $p(T)$. Determine the change in volume of the gas with T along the phase transition curve (i.e. dV_{gas}/dT along the curve).

(d) Consider the Dieterici equation of state

$$p = \frac{k_B T}{v - b} \exp\left(-\frac{a}{k_B T v}\right) .$$

Here v is the volume per unit particle.

(i) Provide a brief physical interpretation for the constants a and b.

(ii) Calculate the second virial coefficient.

(iii) Find the critical temperature T_c for the Dieterici equation of state in terms of a, b and k_B.

36D Electrodynamics

(a) Define a *dielectric* medium and qualitatively explain how the polarization $\boldsymbol{P}$ and magnetization $\boldsymbol{M}$ result in bound charge and bound current densities

$$\rho_{\rm bd} = -\boldsymbol{\nabla}\cdot\boldsymbol{P}\,, \qquad \boldsymbol{J}_{\rm bd} = \frac{\partial \boldsymbol{P}}{\partial t} + \boldsymbol{\nabla}\times\boldsymbol{M}\,.$$

(b) Starting from the *microscopic* Maxwell equations for the electric and magnetic fields $\boldsymbol{E}$, $\boldsymbol{B}$,

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{\rho}{\epsilon_0}\,, \qquad \boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}\,,$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0\,, \qquad \boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0\boldsymbol{J} + \mu_0\epsilon_0\frac{\partial \boldsymbol{E}}{\partial t}\,,$$

derive the *macroscopic* Maxwell equations for a dielectric medium in terms of the electric displacement $\boldsymbol{D}$ and the magnetising field $\boldsymbol{H}$, which you should define, as well as the free charge and current densities.

(c) Consider a spherical shell with radial extent $R_1 < r < R_2$, that consists of a linear magnetic medium with constant permeability μ such that $\boldsymbol{B} = \mu\boldsymbol{H}$. This shell is placed inside a magnetic field that approaches $\boldsymbol{B} = B_0\boldsymbol{e}_z$, $B_0 = \text{const}$, as $r \to \infty$. The electric field, the polarization, and the free charge and current densities are zero. Using the ansatz $\boldsymbol{B} = \boldsymbol{\nabla}\psi$, show that a solution for the magnetic field in all space is given by

$$\psi(r,\theta) = \begin{cases} (a_1 r + b_1/r^2)\cos\theta & \text{for } r < R_1 \\ (a_2 r + b_2/r^2)\cos\theta & \text{for } R_1 < r < R_2 \\ (a_3 r + b_3/r^2)\cos\theta & \text{for } r > R_2\,. \end{cases}$$

Using the boundary conditions at infinity, the origin, $r = R_1$ and $r = R_2$, derive six conditions for the free parameters a_1, b_1, a_2, b_2, a_3 and b_3. Express the parameters b_1, a_2, b_2 and a_3 in terms of B_0, μ, R_1, R_2 and a_1. Briefly describe how you would calculate the remaining parameters a_1 and b_3 (you do not need to compute a_1 and b_3).

[*Hint: In polar coordinates, the Laplace operator and the derivative in the Cartesian z direction are*

$$\nabla^2 f = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}f\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}f\,,$$

$$\frac{\partial}{\partial z}f = \cos\theta\frac{\partial}{\partial r}f - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}f\,.]$$

37B General Relativity

(a) For a spacetime with metric $g_{\alpha\beta}$, write down an explicit expression for the metric-preserving Levi-Civita connection $\Gamma^{\,\alpha}_{\beta\,\gamma}$.

For a spacetime that is nearly flat, the metric can be expressed in the form

$$g_{\alpha\beta} \;=\; \eta_{\alpha\beta} + h_{\alpha\beta}\,,$$

where $\eta_{\alpha\beta}$ is a flat metric with constant components (though not necessarily diagonal in the coordinates used) and the components $h_{\alpha\beta}$ and their derivatives are small. Show that to leading order in these small quantities,

$$2R_{\alpha\beta} \;= A\, h_{\alpha}{}^{\gamma}{}_{,\gamma\beta} \;+ B\, h_{\beta}{}^{\gamma}{}_{,\gamma\alpha} \;+ C\, h^{\gamma}{}_{\gamma,\alpha\beta} \;+ D\, h_{\alpha\beta,\gamma\rho}\,\eta^{\gamma\rho}\,,$$

for constants A, B, C, D which you should determine. Indices are raised and lowered using $\eta_{\alpha\beta}$.

(b) Consider the following metric

$$ds^2 \;=\; 2\,du\,dv \;+\; dx^2 \;+\; dy^2 \;+\; H(u,x,y)\,du^2\,,$$

where $H(u,x,y)$ is a smooth function that is not necessarily small. You may assume that the only nonzero connection coefficients are $\Gamma^{\,v}_{x\,u}, \Gamma^{\,v}_{y\,u}, \Gamma^{\,x}_{u\,u}, \Gamma^{\,y}_{u\,u}, \Gamma^{\,v}_{u\,u}$, and the coefficients related to these by symmetry. You may also assume that R_{uu} is the only coefficient of the Ricci tensor that is not trivially zero. Compute R_{uu} and hence obtain the Ricci scalar. Show that the full nonlinear vacuum field equations, with no cosmological constant, reduce to a partial differential equation for H, that you should determine.

[*You may assume in both parts of the question that*

$$R^{\alpha}{}_{\beta\,\mu\,\nu} \;=\; \Gamma^{\,\alpha}_{\nu\,\beta,\,\mu} \;-\; \Gamma^{\,\alpha}_{\mu\,\beta,\,\nu} \;+\; \Gamma^{\,\alpha}_{\mu\,\gamma}\Gamma^{\,\gamma}_{\nu\,\beta} \;-\; \Gamma^{\,\alpha}_{\nu\,\gamma}\Gamma^{\,\gamma}_{\mu\,\beta}\,. \qquad]$$

38C Fluid Dynamics II

Given a material interface $r = R(\theta, t)$ between two fluid regions in plane polar coordinates (r, θ), explain why

$$\frac{\partial R}{\partial t} + \frac{v}{r}\frac{\partial R}{\partial \theta} = u,$$

where $u\boldsymbol{e}_r + v\boldsymbol{e}_\theta$ is the fluid velocity at the interface.

An ocean gyre is modelled as a two-dimensional, circular patch of water of radius a in solid-body motion with angular velocity ω, surrounded by stationary water. Consider small, sinusoidal perturbations to the edge of the gyre $r = a + \eta(\theta, t)$, where $\eta = \epsilon \exp(ik\theta + \sigma t)$ and $\epsilon \ll ka$. Assuming potential flow, determine the relationship between the growth rate σ and the wave number k.

Briefly describe the subsequent motion of the gyre. Do disturbances propagate upstream or downstream relative to the original motion of the gyre?

[*Hint: In plane-polar coordinates,* $\nabla^2\phi = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial \phi}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2}$.]

39D Waves

Consider one-dimensional homentropic flow in an ideal gas with ratio of specific heats γ and equilibrium sound speed c_0.

(a) Starting from the conservation equations for mass and momentum, show that the Riemann invariants $R_{\pm}$, defined as

$$R_{\pm} = u \pm 2\frac{(c - c_0)}{(\gamma - 1)},$$

are constant on characteristics $C_{\pm}$ given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u \pm c,$$

where u is the fluid velocity, and $c = \sqrt{\mathrm{d}p/\mathrm{d}\rho}$ is the associated sound speed.

(b) Consider a semi-infinite tube filled with such an ideal gas in the region $x > X(t)$ to the right of a piston at $x = X(t)$. At time $t = 0$, the piston and the gas are at rest, $X = 0$, and the gas is uniform with $c = c_0$. For $t > 0$, the piston accelerates smoothly in the positive $x-$direction.

(i) Show that, prior to the formation of a shock, the motion of the gas can be written in terms of a parameter τ by

$$u(x, t) = \dot{X}(\tau) \quad \text{on} \quad x = X(\tau) + \left[c_0 + \frac{1}{2}(\gamma + 1)\dot{X}(\tau)\right](t - \tau),$$

in a region which you should specify carefully.

(ii) Suppose $X(t)$ is given by

$$X(t) = \frac{c_0 t^3}{T^2},$$

where T is a positive constant. Show that a shock first forms in the gas when $t/T = f(\gamma)$, for some function $f(\gamma)$ which you should determine.

40A Numerical Analysis

(a) Describe the *power method*, and the *inverse iteration method* with shift $s \in \mathbb{R}$. Briefly explain the purpose of these algorithms.

(b) Let A be an $n \times n$ real symmetric matrix. The QR algorithm is as follows. Set $A_0 = A$. For $k = 0, 1, \ldots$, set $A_{k+1} = R_k Q_k$ where R_k and Q_k are determined by the QR factorisation $A_k = Q_k R_k$. Here, Q_k is an $n \times n$ orthogonal matrix and R_k is an $n \times n$ upper triangular matrix with all of its diagonal elements strictly positive.

(i) Briefly explain the purpose of the QR algorithm.

(ii) Show that for all k, A_k is symmetric and has the same eigenvalues as A.

A matrix M is r*-banded* if $M_{ij} = 0$ whenever $|i - j| > r$. Show that if A_k is r-banded then so is A_{k+1}.

(iii) For any $k \geqslant 1$, let $\widetilde{Q}_k = Q_0 \ldots Q_{k-1}$ and $\widetilde{R}_k = R_{k-1} \ldots R_0$. Show that $A^k = \widetilde{Q}_k \widetilde{R}_k$.

(iv) Consider the first and last columns of $\widetilde{Q}_k$. How do these relate to the inverse iteration method and the power method?

END OF PAPER