UNIVERSITY OF CAMBRIDGE

MAT2

MATHEMATICAL TRIPOS Part II

Thursday, 06 June, 2024 9:00am to 12:00pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most six questions** from Section I and from any number of questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Number Theory

Let N be an odd positive integer, and let a be an integer.

Define the *Jacobi symbol* $\left(\frac{a}{N}\right)$, and write down a formula for $\left(\frac{2}{N}\right)$.

State the law of quadratic reciprocity for the Jacobi symbol, and use it to compute $\left(\frac{3}{N}\right)$ in terms of the value of N modulo 12.

Show that if d is a positive integer such that $d \equiv 0$ or $3 \bmod 4$, and a, b are positive odd integers such that $a \equiv b \bmod d$, then $\left(\frac{-d}{a}\right) = \left(\frac{-d}{b}\right)$.

2G Topics in Analysis

In this question, a contour Γ will be given by a piecewise smooth map $\gamma : [0,1] \to \mathbb{C}$ that is injective on $[0,1)$ and satisfies $\gamma(0) = \gamma(1)$. You may assume the Cauchy integral formula for a square contour.

If Γ is a contour and $f : \Gamma \to \mathbb{C}$ is continuous, show that

$$F(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(w)}{w - z}\, dw$$

defines a continuous function on $\mathbb{C} \setminus \Gamma$. Give an example to show that, even if f is analytic, F cannot always be extended to a continuous function on $\mathbb{C}$.

Suppose Ω is an open subset of $\mathbb{C}$ and K is a compact non-empty subset of Ω. If $f : \Omega \to \mathbb{C}$ is analytic, show that we can find a finite collection of contours $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ lying in $\Omega \setminus K$ such that

$$f(z) = \sum_{j=1}^{n} \frac{1}{2\pi i} \oint_{\Gamma_j} \frac{f(w)}{w - z}\, dw.$$

3K Coding and Cryptography

In the following, equivalent definitions of the Reed–Muller code can be used without justification.

(a) Let $n = 2^d$ $(d \geqslant 1)$. For $0 \leqslant r \leqslant d$, state and prove a formula for the rank of the Reed–Muller code $\mathrm{RM}(d, r)$ of length n. State its minimum distance.

(b) Consider the Mariner 9 code RM(5,1). What is its information rate? What proportion of errors can it correct in a single codeword? How do these two properties compare to the Hamming code of length 31?

(c) Show that all but two codewords in $\mathrm{RM}(d, 1)$ have the same weight.

4J Automata and Formal Languages

(a) Provide an example of a grammar $G = (\Sigma, V, P, S)$ such that for

$$\begin{aligned} Q &:= P \cup \{S \to \varepsilon\} \text{ and} \\ H &:= (\Sigma, V, Q, S), \end{aligned}$$

we have $\mathcal{L}(G) \cup \{\varepsilon\} \neq \mathcal{L}(H)$. Justify your claim.

(b) Let $G = (\Sigma, V, P, S)$ be an arbitrary grammar. Let $S^* \notin V$ and $V^* := V \cup \{S^*\}$. If $\alpha \to \beta \in P$, let $\alpha^* \to \beta^*$ be the rule where all instances of S are replaced by S^* (both on the left and on the right of the arrow). Let

$$\begin{aligned} P^* &:= \{\alpha^* \to \beta^* \,:\, \alpha \to \beta \in P\} \text{ and} \\ P^+ &:= P^* \cup \{S \to S^*\}. \end{aligned}$$

Show that for $G^+ := (\Sigma, V^*, P^+, S)$, we have that $\mathcal{L}(G) = \mathcal{L}(G^+)$. [You may use without proof that isomorphic grammars produce the same language.]

(c) Again, let $G = (\Sigma, V, P, S)$ be an arbitrary grammar and use the notation from (b). Let

$$\begin{aligned} Q^+ &:= P^+ \cup \{S \to \varepsilon\} \text{ and} \\ H^+ &:= (\Sigma, V^*, Q^+, S) \end{aligned}$$

and prove that $\mathcal{L}(G) \cup \{\varepsilon\} = \mathcal{L}(H^+)$.

5L Statistical Modelling

Explain mathematically why the two results returned in the `R` output below are as they are.

```
> n <- 50
> Y <- rnorm(n)
> p <- 2
> Z1 <- matrix(rnorm(n*p), nrow=n)
> lm1 <- lm(Y ~ Z1)
> sum(lm1$residuals)
[1] 0
>
> p <- 49
> Z2 <- matrix(rnorm(n*p), nrow=n)
> lm2 <- lm(Y ~ Z2)
> summary(lm2)$r.squared
[1] 1
```

6A Mathematical Biology

Consider a birth-death process in a population of size n. The birth rate per individual is λ and the death rate per individual is $\gamma + \beta n$, where λ, γ and β are positive constants.

Let $p_n(t)$ be the probability that the population has size n at time t. Write down the master equation for the system. Show that

$$\frac{d}{dt}\langle n\rangle = (\lambda - \gamma)\,\langle n\rangle - \beta\langle n^2\rangle\,,$$

where $\langle . \rangle$ denotes the mean.

From this result, deduce a bound on the mean in a steady state in the case that $\lambda > \gamma$.

What can be said about the mean at steady state when $\lambda \leqslant \gamma$?

7D Further Complex Methods

Let $F(z)$ be defined on the half plane $H = \left\{z \in \mathbb{C} \;:\; \mathsf{Re}(z) > 0\right\}$ such that

1. $F(z)$ is analytic on H.
2. $F(z+1) = zF(z)$ for all $z \in H$,
3. $F(z)$ is bounded in the strip $\left\{z \in \mathbb{C} \;:\; 1 \leqslant \mathsf{Re}(z) \leqslant 2\right\}$,
4. $F(1) = 1$.

(a) By using $F(z+1) = zF(z)$ for all $z \in \mathbb{C}$, extend F meromorphically to $z \in \mathbb{C}$ with $\mathsf{Re}(z) \leqslant 0$. Identify and characterize the singular points of the extended function.

(b) Now consider a function $\Gamma : H \to \mathbb{C}$ that also satisfies the four conditions listed above. By meromorphically extending Γ in the same way as done for F in part (a), show that the function $f(z) = F(z) - \Gamma(z)$ is entire.

(c) Assuming additionally that $f(z)$ is bounded on the strip $0 \leqslant \mathsf{Re}(z) \leqslant 1$, show that the function $S(z) = f(z)f(1-z)$ is entire and bounded on $\mathbb{C}$. You may conclude without further proof that $S(z)$ is therefore constant. Use this result to prove Wielandt's theorem, i.e. that $F(z) = \Gamma(z)$ for $z \in \mathbb{C}$.

8E Classical Dynamics

(a) Using spherical polar coordinates, (r, θ, ϕ), write down the Hamiltonian for a particle of mass m moving in a spherically symmetric potential.

(b) Show that p_ϕ and $p_\theta^2 + (p_\phi^2 / \sin^2\theta)$ are each conserved. Interpret them physically. [You may state Hamilton's equations without proof.]

(c) The particle executes circular motion in a plane through the origin inclined at angle ψ to the plane $\theta = \pi/2$. Evaluate $p_\theta(\theta)$ and show that it vanishes when $\sin\theta = \pm\cos\psi$. Interpret this result physically.

9D Cosmology

The Friedmann and acceleration equations for a universe without a cosmological constant ($\Lambda = 0$) are given by

$$\mathcal{H}^2 + kc^2 = \frac{8\pi G}{3c^2}\rho a^2 \,, \qquad \mathcal{H}' = -\frac{4\pi G}{3c^2}(\rho + 3P)a^2 \,,$$

where ρ is the energy density, P is the pressure, k is the curvature and primes denote differentiation with respect to conformal time τ (defined by $\mathrm{d}\tau = \mathrm{d}t/a(t)$). Here, a is the scale factor and $\mathcal{H}$ is the conformal Hubble parameter defined by $\mathcal{H} \equiv a'/a$.

(a) Assume that the universe is filled with a single-component fluid with equation of state $P = w\rho$, where w is a constant. By introducing the density parameter $\Omega = 8\pi G a^2 \rho/(3c^2\mathcal{H}^2)$, show that the two evolution equations can be recast as

$$\frac{kc^2}{\mathcal{H}^2} = \Omega - 1 \,, \qquad 2\mathcal{H}' + (1+3w)\left(\mathcal{H}^2 + kc^2\right) = 0 \,.$$

Hence, find the following evolution equation for the density parameter,

$$\Omega' = (1+3w)\mathcal{H}\Omega(\Omega - 1) \,. \qquad (\dagger)$$

(b) Using the time evolution of Ω from equation $(\dagger)$, qualitatively describe the flatness problem of an expanding universe ($\mathcal{H} > 0$) for models with an equation of state parameter in the range $0 \leqslant w \leqslant 1$. In particular, roughly sketch the time evolution of Ω in a radiation-filled universe with $w = 1/3$ taking initial values for both $\Omega < 1$ and $\Omega > 1$.

(c) Briefly discuss how the flatness problem can be alleviated by an early epoch of inflation during which $w \approx -1$.

10E Quantum Information and Computation

Recall the Schmidt decomposition theorem for bipartite states: For any bipartite state $|\psi_{AB}\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} X_{ij}|i\rangle|j\rangle$ of an $n-$dimensional system $A \simeq \mathbb{C}^n$ and an $m-$dimensional system $B \simeq \mathbb{C}^m$, there exist (non-unique) orthonormal bases $\{|\alpha_i\rangle\}_{i=1}^{n} \subset \mathbb{C}^n$, $\{|\beta_j\rangle\}_{j=1}^{m} \subset \mathbb{C}^m$, and a (unique) set of non-negative numbers $\lambda_1, \lambda_2, \ldots, \lambda_d$ (for $d = \min\{n, m\}$) such that

$$|\psi_{AB}\rangle = \sum_{i=1}^{d} \lambda_i |\alpha_i\rangle|\beta_i\rangle.$$

The number of non-zero λ_i is called the *Schmidt rank* of $|\psi_{AB}\rangle$ and is equal to rank(X).

(a) Using the Schmidt decomposition theorem, show that a bipartite state $|\psi_{AB}\rangle$ is entangled if and only if its Schmidt rank is at least 2.

(b) Consider the following tripartite states of 3 qubits:

$$\begin{aligned}
|\chi_{ABC}^{(1)}\rangle &= |0_A\rangle|0_B\rangle|0_C\rangle\,, \\
|\chi_{ABC}^{(2)}\rangle &= |0_A\rangle|\phi_{BC}^{+}\rangle\,, \\
|\chi_{ABC}^{(3)}\rangle &= \frac{1}{\sqrt{2}}(|0_A\rangle|0_B\rangle|0_C\rangle + |1_A\rangle|1_B\rangle|1_C\rangle)\,.
\end{aligned}$$

Here, $|\phi_{BC}^{+}\rangle = \frac{1}{\sqrt{2}}(|0_B\rangle|0_C\rangle + |1_B\rangle|1_C\rangle)$ is a Bell state of two qubits. Show that

(i) $|\chi_{ABC}^{(1)}\rangle$ is product across all three bipartitions,

(ii) $|\chi_{ABC}^{(2)}\rangle$ is product across the $A - BC$ bipartition, and is entangled across the $B - AC$ and the $C - AB$ bipartitions,

(iii) $|\chi_{ABC}^{(3)}\rangle$ is entangled across all three bipartitions.

(c) Suppose that Alice (A), Bob (B), and Charlie (C) share either the state $|\chi_{ABC}^{(3)}\rangle$ or $|\chi_{ABC}^{(4)}\rangle = \frac{1}{\sqrt{3}}(|0_A\rangle|0_B\rangle|1_C\rangle + |0_A\rangle|1_B\rangle|0_C\rangle + |1_A\rangle|0_B\rangle|0_C\rangle)$. Charlie performs a measurement in the computational basis and obtains the result 0. Show that the resulting state of Alice and Bob will be

(i) product in the case they share $|\chi_{ABC}^{(3)}\rangle$,

(ii) entangled in the case they share $|\chi_{ABC}^{(4)}\rangle$.

SECTION II

11F Number Theory

Let $k \in \mathbb{N}$.

Let p be an odd prime and $H_k = \{a + p^k\mathbb{Z} \mid a \equiv 1 \bmod p\}$. Show that H_k is a cyclic group under multiplication, and determine its order.

In the following, I denotes the $k \times k$ identity matrix.

Let p be an odd prime and let A be a $k \times k$ matrix with integer entries. Suppose that $A = I + p^m B$ for some $m \in \mathbb{N}$ and some $k \times k$ matrix B with integer entries, and that $A^n = I$ for some $n \in \mathbb{N}$. Show that $A = I$.

Now find the smallest integer $r \geqslant 1$ such that if A is a $k \times k$ matrix with integer entries satisfying $A = I + 2^r B$ for some $k \times k$ matrix B with integer entries, and $A^n = I$ for some $n \in \mathbb{N}$, then in fact $A = I$.

12J Automata and Formal Languages

(a) Define what it means for a language $L \subseteq \mathbb{W}$ to *satisfy the regular pumping lemma.*

(b) Prove that every regular language satisfies the regular pumping lemma. [You may assume that "regular" is equivalent to "accepted by a deterministic automaton" without proof.]

(c) Let L be a finite language such that there is a $w \in L$ with $|w| = 100$. Show that there can be no deterministic automaton D with at most 100 states such that $L = \mathcal{L}(D)$.

(d) Let $\Sigma = \{0, 1, 2\}$ and let $L \subseteq \{0, 1\}^*$ be an arbitrary language. Show that

$$\widehat{L} := \{u2v \,:\, u \in \mathbb{W}, v \in L\} \cup \{0, 1\}^*$$

satisfies the regular pumping lemma.

(e) Using part (d) or otherwise, provide an example of a language L such that L satisfies the regular pumping lemma, but for any grammar G, we have that $L \neq \mathcal{L}(G)$.

13A Mathematical Biology

An experiment is run with bacteria in a thin channel. The concentration of bacteria at time t is given by $C(x,t)$, where x is position in the channel. The ends of the channel are at $x = 0$ and $x = L$. The bacteria cannot survive outside the channel. Within the channel, the concentration of bacteria is modelled by

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} + \mu C\,,$$

where D and μ are positive constants.

Initially, the bacteria are at a uniform concentration C_0 for $0 < x < L$.

(a) Suppose that the ends of the channel are open so that bacteria may diffuse out of the ends.

(i) Find the concentration $C(x,t)$ for $t > 0$.

(ii) What is the flux of bacteria out of the channel? Comment on this flux at $t = 0$.

(iii) Show that the population will grow if $\mu > \mu_c$, where $\mu_c(D,L)$ should be given. Give a brief explanation for the dependence of μ_c on D and L.

(iv) Sketch the bacterial concentration as a function of x for a range of times (on the same sketch), paying particular attention to early and late times. Consider separately the cases when $\mu < \mu_c$ and $\mu > \mu_c$ (i.e. two sketches are required).

(b) The experiment is run again with the $x = L$ end of the channel closed ($x = 0$ remains open). Again initially $C(x,t) = C_0$. What is the condition now for population growth in the long term? Comment briefly on how this compares to the condition for growth in part (a).

(c) The experiment is run once more with the channel ends both open, but now the per capita growth rate is a function of time (as the experimental conditions fluctuate each day), given by $\mu(t) = \mu_0 + \mu_1 \cos(t)$. Find the condition for population growth in the long term. Give a brief interpretation of this result.

14D Cosmology

In a flat expanding universe dominated by non-relativistic matter with energy density $\rho(\mathbf{x},t)$, we represent small density inhomogeneities $\delta(\mathbf{x},t)$, $|\delta(\mathbf{x},t)| \ll 1$, relative to the homogeneous background $\bar{\rho}(t) = \bar{\rho}_0/a(t)^3$ by the relation $\rho = \bar{\rho}(1+\delta)$, with scale factor $a(t)$ (take $a(t_0)$=1 today). The continuity, Euler and Poisson equations are respectively

$$\begin{aligned}
&\dot{\delta} + \boldsymbol{\nabla}\cdot[\mathbf{v}(1+\delta)] = 0\,,\\
&\dot{\mathbf{v}} + 2\frac{\dot{a}}{a}\mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla}\phi - \frac{c^2}{\rho}\boldsymbol{\nabla}P\,,\\
&\nabla^2\phi = \frac{4\pi G}{c^2}\bar{\rho}\,\delta\,,
\end{aligned}$$

where $P(\mathbf{x},t)$ is the pressure and ϕ the perturbation of the Newtonian potential. Here, we will neglect rotational modes and assume the velocity $\mathbf{v}$ is solely compressional, that is, the divergence is the only non-vanishing part and $\theta \equiv \boldsymbol{\nabla}\cdot\mathbf{v}$. We also assume that pressure effects can be described using the sound speed c_s^2 defined by $c_\mathrm{s}^2 \equiv c^2(\mathrm{d}P/\mathrm{d}\rho)$.

(a) Show that for an equation of state given by $P = P(\rho)$, the pressure term in the linearised Euler equation can be written as $(c^2/\rho)\boldsymbol{\nabla}P \approx c_\mathrm{s}^2\boldsymbol{\nabla}\delta$. Linearise both the continuity equation and the divergence of the Euler equation (using $\theta = \boldsymbol{\nabla}\cdot\mathbf{v}$) and substitute the Poisson equation in the latter. Transforming to Fourier space with comoving wavemodes $\mathbf{k}$ (i.e. $\boldsymbol{\nabla} \to i\mathbf{k}/a$), combine these to obtain the evolution equation for density perturbations in an expanding universe,

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - \left(\frac{4\pi G}{c^2}\bar{\rho} - \frac{c_\mathrm{s}^2k^2}{a^2}\right)\delta = 0\,, \tag{$\dagger$}$$

where $k = |\mathbf{k}|$. Briefly explain the qualitative implications of each term in this equation. Define the Jeans length λ_J and discuss its significance.

(b) Suppose that the non-relativistic pressure is given by an equation of state $P = \alpha\rho^{4/3}$ where $0 < \alpha \ll 1$; assume that this does not influence the background evolution of a matter-dominated FLRW universe with $\bar{\rho}/c^2 = 1/(6\pi G t^2)$. Show that the varying sound speed has the following time-dependence,

$$\frac{c_\mathrm{s}^2(t)}{a^2} \approx \frac{L_0^2}{t^2}\,,$$

where L_0^2 is a constant. Using this sound speed, seek power law solutions $\delta \propto t^\gamma$ of the density perturbation equation ($\dagger$) to find the general solution

$$\delta(\mathbf{k},t) = A_\mathbf{k}t^{n_+} + B_\mathbf{k}t^{n_-}\,, \qquad \text{where} \qquad n_\pm = -\frac{1}{6} \pm \left[\left(\frac{5}{6}\right)^2 - L_0^2k^2\right]^{1/2}.$$

Given $\tilde{k}_\mathrm{J} = 5/(6L_0)$, describe these solutions at late times in the asymptotic limits where $k \ll \tilde{k}_\mathrm{J}$ and where $k \gg \tilde{k}_\mathrm{J}$. What would be the consequence of choosing α such that $L_0 \approx 50\,\mathrm{Mpc}$ today?

15E Quantum Information and Computation

Suppose $\mathcal{H}_n \simeq (\mathbb{C}^2)^{\otimes n}$ can be partitioned into 2 mutually orthonormal subspaces $\mathcal{S}_g$ and $\mathcal{S}_b$. Let P_g and P_b denote the orthogonal projections onto $\mathcal{S}_g$ and $\mathcal{S}_b$, respectively. We call $\mathcal{S}_g$ the *good* subspace and $\mathcal{S}_b$ the *bad* subspace. All kets in this question are normalised.

(a) Show that any $|\varphi\rangle \in \mathcal{H}_n$ can be uniquely decomposed as follows:

$$|\varphi\rangle = a\,|\psi_1\rangle + b\,|\psi_2\rangle\,,$$

where $|\psi_1\rangle \in \mathcal{S}_g$ and $|\psi_2\rangle \in \mathcal{S}_b$. Express a, b, $|\psi_1\rangle$ and $|\psi_2\rangle$ in terms of $|\varphi\rangle$, P_g and P_b.

For any $|\alpha\rangle \in \mathcal{H}_n$, consider a unitary operator $R_{|\alpha\rangle}$ which acts as follows:

$$\begin{aligned} &R_{|\alpha\rangle}\,|\alpha\rangle = -\,|\alpha\rangle\,, \\ &R_{|\alpha\rangle}\,|\beta\rangle = |\beta\rangle \quad \forall\,|\beta\rangle \text{ such that } \langle\beta|\alpha\rangle = 0\,. \end{aligned}$$

In particular define $R_0 := R_{|0_n\rangle}$, where $|0_n\rangle := |0\rangle^{\otimes n}$. Further, let A denote a unitary operator which acts on $|0_n\rangle$ as follows:

$$|\Omega\rangle := A\,|0_n\rangle := \sqrt{p}\,|\psi_g\rangle + \sqrt{1-p}\,|\psi_b\rangle\,,$$

where $|\psi_g\rangle \in \mathcal{S}_g$, $|\psi_b\rangle \in \mathcal{S}_b$, and $p \in (0,1)$. Let $R_g := R_{|\psi_g\rangle}$.

Suppose you are given quantum circuits which implement R_0, R_g and A. Below, we consider a protocol that increases the amplitude $\sqrt{p}$ of the *good* state $|\psi_g\rangle$.

(b) Show that $AR_0A^{-1}\,|\Omega\rangle = R_{|\Omega\rangle}\,|\Omega\rangle$.

(c) Consider the 2-dimensional real Euclidean plane $\mathcal{P}$ spanned by $|\psi_g\rangle$ and $|\psi_b\rangle$, and let θ denote the angle between $|\Omega\rangle$ and the horizontal axis $|\psi_b\rangle$. Express θ in terms of p.

(d) How can you geometrically interpret the actions of R_g and $(-R_{|\Omega\rangle})$ on any vector $|\phi\rangle$ in $\mathcal{P}$?

(e) Justify that R_g and $(-R_{|\Omega\rangle})$ leave the plane $\mathcal{P}$ invariant.

(f) Start with the state $|\Omega\rangle$ and apply the operator $Q := -R_{|\Omega\rangle}R_g$ to it. What is the action of a single use of Q on the state $|\Omega\rangle$ and k successive uses of Q on the state $|\Omega\rangle$? [*Hint: Draw clear diagrams.*] What is the final state $|\Omega_k\rangle$ obtained after k successive applications of Q on $|\Omega\rangle$?

(g) What is the probability that a measurement of $|\Omega_k\rangle$ will yield the *good* state $|\psi_g\rangle$? Show that $k = \mathcal{O}(1/\sqrt{p})$ iterations suffice to make the amplitude of the *good* state close to 1.

16I Logic and Set Theory

(a) State Zorn's Lemma, the Axiom of Choice and the Well-ordering Principle, and prove that they are equivalent.

(b) State Gödel's Completeness Theorem and the Compactness Theorem for First-order Logic.

Let $T_0, T_1, \ldots, T_n$ be first-order theories in some language L that partition the collection of all L-structures in the sense that every L-structure is a model of exactly one T_i. Show that each T_i is finitely axiomatisable.

17I Graph Theory

(a) Show that every graph of average degree at least d contains a subgraph of minimum degree at least $d/2$.

(b) Let $\delta, g \geqslant 3$ be fixed. Using random graph methods, or otherwise, show that there exists a graph G on at most $n = (100\delta)^g$ vertices such that the minimum degree of G is at least δ and the girth of G is at least g.

18H Galois Theory

(a) Let $M/L_1/K$ and $M/L_2/K$ be finite extensions of fields. Define the composite L_1L_2. Show that if L_1/K is Galois then L_1L_2/L_2 is Galois, and that there is an injective group homomorphism $\mathrm{Gal}(L_1L_2/L_2) \to \mathrm{Gal}(L_1/K)$.

(b) Let K be a field of characteristic not dividing n, and with algebraic closure $\overline{K}$. Let $\zeta_n \in \overline{K}$ be a primitive nth root of unity. Prove that $[K(\zeta_n) : K]$ divides $\phi(n)$, and that equality holds if $K = \mathbb{Q}$.

(c) Let $L_1 = \mathbb{Q}(\zeta_9)$ and $L_2 = \mathbb{Q}(\zeta_{15})$. Write each of the fields $L_1 \cap L_2$ and L_1L_2 in the form $\mathbb{Q}(\zeta_n)$ for suitable n. Justify your answers.

19H Representation Theory

Let p be a prime number and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the field with p elements. Consider the group of unitriangular 3×3-matrices with coefficients in $\mathbb{F}_p$,

$$G := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{F}_p \right\}.$$

(a) Describe the conjugacy classes of G.

(b) Show that G has precisely p^2 complex representations of degree 1 and describe them explicitly.

(c) Find an abelian subgroup A of G of order p^2 and construct $p-1$ irreducible complex representations of G of degree p by induction from A.

(d) Determine the character table of G.

20J Algebraic Topology

Let L be a simplicial complex and $K \leqslant L$ be a subcomplex, with associated chain complexes $C_\bullet(L)$ and $C_\bullet(K)$. Setting

$$C_k(L, K) := \frac{C_k(L)}{C_k(K)},$$

show that the boundary map of L descends to give $C_\bullet(L, K)$ the structure of a chain complex. Describe a long exact sequence relating $H_*(K)$, $H_*(L)$, and the homology $H_*(L, K)$ of $C_\bullet(L, K)$.

In the following, Δ^{n+1} denotes the standard $(n+1)$-simplex and $\partial\Delta^{n+1}$ denotes its boundary, i.e. the union of all its simplices of dimension $< n+1$. Suppose that $n \geqslant 2$.

Using that Δ^{n+1} has the same homology as a point, calculate the homology of $\partial\Delta^{n+1}$.

Recall that the rank $\operatorname{rk} A$ of an abelian group A is the maximal r such that $\mathbb{Z}^r \leqslant A$. If $K \leqslant \partial\Delta^{n+1}$ is a sub-simplicial complex, by considering $H_*(\partial\Delta^{n+1}, K)$ show that for each $0 < k \leqslant n$ we have

$$\operatorname{rk} H_k(K) \leqslant \binom{n+2}{k+2} - \#(k+1)\text{-simplices of } K.$$

[You may use that the rank of a subgroup or quotient of A is at most that of A.]

If $K < \partial\Delta^{n+1}$ is a proper sub-simplicial complex, show that $H_n(K) = 0$.

21G Linear Analysis

State and prove the Stone–Weierstrass theorem. [You may assume that the function $x^{1/2}$ is uniformly approximable by polynomials on $[0,1]$.]

Let $C(\mathbb{R})$ denote the space of all bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$, equipped with the uniform norm. Explain briefly why $C(\mathbb{R})$ is a Banach space.

If A is a subalgebra of $C(\mathbb{R})$ that contains the constants and separates the points, must A be dense in $C(\mathbb{R})$? Justify your answer.

22G Analysis of Functions

State and prove the Sobolev embedding theorem $H^s(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$.

Show that $H^1(\mathbb{R}^3)$ contains an unbounded function.

[*You may use results from Fourier analysis without proof, provided they are carefully stated.*]

23H Riemann Surfaces

State and prove the open mapping theorem for Riemann surfaces R and S. [You may assume the identity theorem.] Suppose that $f : R \to S$ is analytic and non-constant. Deduce that if R is compact then f is surjective.

Suppose that $f : \mathbb{D} \to \mathbb{D}$ is non-constant and analytic with $f(0) = 0$. Show that $g(z) = f(z)/z$ is analytic on $\mathbb{D}$, and that for the open disc D_r of radius $r < 1$ and centre 0 we have $|g(z)| \leqslant 1/r$ for $z \in D_r$.

Describe all of the analytic isomorphisms $h : \mathbb{D} \to \mathbb{D}$ such that $h(0) = 0$.

Suppose that $F : \mathbb{C} \to \mathbb{C}$ is an analytic isomorphism with $F(0) = 0$. Show that $|F(z)| \to \infty$ as $|z| \to \infty$ and that F extends to an analytic isomorphism $\mathbb{C}_\infty \to \mathbb{C}_\infty$.

Given an analytic isomorphism $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$, does f extend to an analytic isomorphism $\mathbb{C} \to \mathbb{C}$? Does every analytic isomorphism $\mathbb{D} \to \mathbb{D}$ extend to an analytic isomorphism $\mathbb{C} \to \mathbb{C}$? Justify your answers.

24F Algebraic Geometry

In this question, all algebraic varieties are over an algebraically closed field k.

State the Riemann–Roch theorem.

Let X be a non-singular projective curve of genus 1, and let $P_0 \in X$ be a point. Show that if D is a divisor of degree 0 on X, then there exists a unique $P \in X$ such that D is linearly equivalent to $P - P_0$.

Use this to describe a group law on X with P_0 as the identity element. [You do not need to prove this law satisfies the group axioms.]

Consider the group law on the non-singular cubic curve

$$X = Z(y^2 z - x^3 + 4xz^2 - z^3) \subseteq \mathbb{P}^2$$

with identity element $P_0 = (0 : 1 : 0) \in X$. Let $A = (2 : 1 : 1)$ and $B = (-2 : -1 : 1)$ be two points on X. Find $A + B$.

Let $X \subseteq \mathbb{P}^2$ be a non-singular cubic curve. A point $P_0 \in X$ is an *inflection point* of X if there exists a line $L \subseteq \mathbb{P}^2$ such that $X \cap L = \{P_0\}$. Let P_0 be an inflection point of X, and give X the group structure with identity element P_0. A point $P \in X$ is said to be 3-*torsion* if $P + P + P = P_0$. Prove that P is a 3-torsion point if and only if P is an inflection point.

25J Differential Geometry

Let $n \geqslant 1$ and $1 \leqslant k \leqslant N$ be integers. Let $\mathbf{Id}$ denote the identity matrix.

(a) State the definition of a *k-dimensional smooth manifold* $X \subset \mathbb{R}^N$. Define the *tangent space* $T_x X$ at a point $x \in X$ in terms of a parametrisation about x, and prove that the tangent space is independent of the parametrisation.

(b) Let $GL(n) \subset \mathbb{R}^{n^2}$ be the set of real $n \times n$ invertible matrices. Prove that it is a manifold, give its dimension, and compute $T_{\mathbf{Id}} GL(n)$.

(c) Let $SL(n) \subset \mathbb{R}^{n^2}$ be the set of real $n \times n$ matrices with determinant 1. Prove that it is a manifold, give its dimension, and compute $T_{\mathbf{Id}} SL(n)$.

(d) Let $SO(n) \subset \mathbb{R}^{n^2}$ be the set of real $n \times n$ matrices with determinant 1 and orthonormal column vectors. Prove that it is a manifold, give its dimension, and compute $T_{\mathbf{Id}} SO(n)$.

26G Probability and Measure

(a) (i) State Lévy's convergence theorem for characteristic functions.

(ii) Let $(X_n)_{n\in\mathbb{N}}$, X be random variables on $\mathbb{R}^d$. Show that $X_n \to X$ weakly if and only if for all $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$, $\langle u, X_n\rangle \to \langle u, X\rangle$ weakly. [For $a = (a_1, \ldots, a_d), b = (b_1, \ldots, b_d) \in \mathbb{R}^d$, $\langle a, b\rangle := \sum_{i=1}^d a_i b_i$.]

(b) Let X be a real-valued random variable with characteristic function $\phi(t) = \mathbb{E}(e^{itX})$.

(i) Show that for any $u > 0$

$$\frac{1}{u}\int_{-u}^{u} (1 - \phi(t))\, dt = 2\mathbb{E}\left(1 - \frac{\sin(uX)}{uX}\right).$$

(ii) Show that

$$\mathbb{P}(|X| > 2/u) \leqslant \frac{1}{u}\int_{-u}^{u} (1 - \phi(t))dt\,.$$

(iii) Now let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real-valued random variables with characteristic functions $(\phi_n)_{n\in\mathbb{N}}$. If for all $t \in \mathbb{R}$, $\phi_n(t) \to \phi(t)$ for some function ϕ that is continuous at 0, show that given any $\varepsilon > 0$, there exists $M > 0$ such that $\mathbb{P}(|X_n| > M) \leqslant \varepsilon$ for all n.

[*You may use convergence results for integrals given in the course and Fubini's theorem, without proof.*]

27K Applied Probability

(a) State and prove Little's formula for a regenerative process.

(b) What is an $M/G/1$ *queue*? Find the expected length of the busy period for an $M/G/1$ queue.

(c) Suppose customers arrive at a single server at rate $\lambda > 0$ and require an exponential amount of service time with rate $\mu > 0$. Customers not being served are impatient and will leave at rate $\delta > 0$, independently of their position in the queue. Let X_t denote the length of the queue at time t.

(i) Show that the system $X = (X_t)_{t\geqslant 0}$ has an invariant distribution.

(ii) Is X positive recurrent? Justify your answer.

28L Principles of Statistics

Consider a binomial model $X \sim \text{Binomial}(n, \theta)$, where $\theta \in \Theta = [0,1]$, and let $\delta_{\text{MLE}}(X) = \frac{X}{n}$ denote the maximum likelihood estimator. In all of the following, we use the weighted quadratic loss

$$L(a, \theta) = \frac{(a-\theta)^2}{\theta(1-\theta)}.$$

(a) Compute the risk $R(\delta_{\text{MLE}}, \theta)$, for each $\theta \in (0,1)$.

(b) Suppose π is a prior for θ. What is meant by the π-*Bayes risk* of an estimator δ of θ? Prove that any estimator of θ which minimises the posterior risk also minimises the π-Bayes risk.

(c) Now suppose π is the uniform prior on $[0,1]$. By differentiating the expression for the posterior risk with respect to $\delta(x)$, prove that δ_{MLE} is the unique π-Bayes rule. [In your calculations, you may interchange differentiation and integration without justification. You may also use the formulas $\int_0^1 \theta^{a-1}(1-\theta)^{b-1} d\theta = \text{Beta}(a,b)$ and $\text{Beta}(a+1,b) = \text{Beta}(a,b) \cdot \frac{a}{a+b}$ without proof.]

(d) Prove that the estimator δ_{MLE} is minimax. What does it mean for an estimator δ of θ to be *admissible*? Is the estimator δ_{MLE} admissible? Justify your answer. [Results from the course should *not* be used without proof.]

29L Stochastic Financial Models

Consider a discrete-time market model with interest rate $r > -1$ and one stock with time-n price S_n. Suppose $S_0 > 0$ is given and that $S_n = S_{n-1}\xi_n$ for all $n \geqslant 1$, where the sequence $(\xi_n)_{n\geqslant 1}$ is IID, and

$$\mathbb{P}(\xi_1 = 1 + b) = p = 1 - \mathbb{P}(\xi_1 = 1 + a)$$

for constants $-1 < a < b$ and $0 < p < 1$. Assume that there exists a risk-neutral measure $\mathbb{Q}$ for the model.

(a) Show that $a < r < b$, and that the sequence $(\xi_n)_{n\geqslant 1}$ is IID under $\mathbb{Q}$. Find $\mathbb{Q}(\xi_1 = 1 + b)$.

Fix a maturity date $N > 0$ and a payout function g. For all $s > 0$ define

$$\begin{aligned} V(N, s) &= g(s), \\ V(n-1, s) &= \frac{q}{1+r}V(n, s(1+b)) + \frac{1-q}{1+r}V(n, s(1+a)) \text{ for all } 1 \leqslant n \leqslant N, \end{aligned}$$

where $q = \mathbb{Q}(\xi_1 = 1 + b)$.

(b) Consider the case where $g(s) = \log s$. Find, with justification, a formula for $V(n, s)$ of the form $V(n, s) = A_n \log s + B_n$, for families of constants $(A_n)_{0\leqslant n\leqslant N}$ and $(B_n)_{0\leqslant n\leqslant N}$ which should be specified.

(c) Prove that an investor with initial capital $X_0 = V(0, S_0)$ can replicate the payout of the vanilla European claim with time-N payout $g(S_N)$. How many shares of the stock should the investor hold during the time interval $(n - 1, n]$?

Now assume $a \leqslant 0$, and fix a barrier $B > S_0$. For all $s > 0$ define

$$\begin{aligned} U(N, s) &= g(s), \\ U(n-1, s) &= \frac{q}{1+r}U(n, s(1+b))\mathbf{1}_{\{s(1+b)<B\}} + \frac{1-q}{1+r}U(n, s(1+a)) \text{ for all } 1 \leqslant n \leqslant N. \end{aligned}$$

(d) In terms of the function U, derive an explicit formula for the number of shares of the stock an investor should hold during the time interval $(n-1, n]$ in order to replicate the payout of the up-and-out European claim with time-N payout $g(S_N)\mathbf{1}_{\{\max_{0\leqslant n\leqslant N} S_n<B\}}$.

30C Asymptotic Methods

(a) State *Watson's lemma.*

(b) Consider the integral

$$I(x) = \int_0^1 \sqrt{t}\, \mathrm{e}^{ixt}\, dt\,,$$

where x is real and positive. Use the method of steepest descent to show that

$$I(x) \sim \frac{\sqrt{\pi}}{2}\,(-ix)^{-3/2} - \mathrm{e}^{ix} \sum_{n=0}^{\infty} a_n\,(-ix)^{-n-1}\,,$$

as $x \to \infty$, where the a_n are coefficients that you should determine.

[Hint: $\int_0^\infty \sqrt{u} e^{-u} du = \sqrt{\pi}/2$.]

31A Dynamical Systems

(a) A dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^2$ has a periodic orbit $\mathbf{x} = \mathbf{X}(t)$ with period T. The linearised evolution of a small perturbation $\mathbf{x} = \mathbf{X}(t) + \boldsymbol{\eta}(t)$ is given by $\eta_i(t) = \Phi_{ij}(t)\eta_j(0)$. Obtain the differential equation and initial condition satisfied by the matrix $\boldsymbol{\Phi}(t)$.

Explain how the stability of the orbit $\mathbf{X}(t)$ is connected to the quantity

$$\exp\left[\int_0^T \boldsymbol{\nabla}\cdot\mathbf{f}\,(\mathbf{X}(t))\;dt\right].$$

(b) Using the energy-balance method for nearly Hamiltonian systems, find a condition on the parameter a for a limit cycle to exist in the system given by

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x + \epsilon(1 - x^2 + ay^2)y,\end{aligned}$$

where $0 < \epsilon \ll 1$. Determine the stability of the limit cycle.

[*Hint: $\int_0^{2\pi} \sin^2\theta \cos^2\theta\, d\theta = \pi/4$ and $\int_0^{2\pi} \sin^4\theta\, d\theta = 3\pi/4$.*]

32C Integrable Systems

Explain how to find Lie symmetries for an ordinary differential equation $\Delta(x, u, u', \ldots, u^{(N)}) = 0$ by considering the action of the Nth prolongation

$$\mathrm{pr}^{(N)} V = V_1 \partial_x + \phi\, \partial_u + \phi_1 \partial_{u'} + \cdots + \phi_N \partial_{u^{(N)}}$$

of the vector field $V = V_1(x,u)\partial_x + \phi(x,u)\partial_u$. Give the inductive formula for the computation of the $\phi_j = \phi_j(x, u, u', \ldots, u^{(j)})$ which determines the prolongation.

For the special case in which $V_1 = f(x)$ depends only on x, compute the third prolongation, and show that

$$\phi_3 = \phi_{xxx} + u'(\alpha\phi_{xxu} - f''') + 3(u')^2\phi_{xuu} + (u')^3\phi_{uuu} + \beta(\phi_{xu} - f'' + u'\phi_{uu})u'' + (\phi_u - 3f')u''',$$

for integers α, β which you should determine.

Find the Lie symmetries of the equation

$$\frac{d^3u}{dx^3} = \frac{1}{u^3}$$

that are generated by a vector field of the form $V = f(x)\partial_x + \phi(x,u)\partial_u$.

33B Principles of Quantum Mechanics

(a) Let the components of the angular momentum operator be J_i, for $i = 1, 2, 3$. Define the states $|j, m\rangle$ in terms of the action of certain angular momentum operators and state the possible values of j and m. Let the operators $J_\pm \equiv J_1 \pm iJ_2$ and let $C_\pm(j, m)$ be defined by $J_\pm |j, m\rangle = C_\pm(j, m) |j, m \pm 1\rangle$. Considering particular cases, or otherwise, explicitly derive the four constants $\lambda_1^\pm$ and $\lambda_2^\pm$ appearing in

$$C_\pm(j, m) = \sqrt{j(j + 1) + \lambda_1^\pm m(m + \lambda_2^\pm)}\,.$$

Hence, or otherwise, prove that if O is a linear operator that commutes with all components of the angular momentum operator, then

(i) $\langle j, m| O |j', m'\rangle \propto \delta_{jj'}$

(ii) $\langle j, m| O |j', m'\rangle \propto \delta_{mm'}$

(iii) $\langle j, m| O |j, m\rangle$ is independent of m.

(b) Now consider a two qubit system with states $|j_1, m_1; j_2, m_2\rangle$, with $j_1 = j_2 = 1/2$. Let $|j, m\rangle$ be the eigenstates of the total angular momentum of this system. State which values of j and m occur and write down a resolution of the identity in terms of the $|j, m\rangle$ eigenstates. Write down the values of the Clebsch–Gordan coefficients

$$C(j, m; j_1, m_1, j_2, m_2) \equiv \langle j, m|j_1, m_1, j_2, m_2\rangle$$

for all appropriate values of j, m, m_1, m_2.

Suppose that an operator O commutes with all the components of the total angular momentum of this two qubit system. Using the results derived above and the results from part (a), or otherwise, write down the most general expression for

$$\big\langle j_1, m_1; j_2, m_2\big| O \big|j_1, m_1'; j_2, m_2'\big\rangle$$

for all distinct cases where it does not vanish.

34E Applications of Quantum Mechanics

(a) Consider a one-dimensional system with a periodic potential $V(x) = V(x+a)$, where a is a positive constant. The Schrödinger equation for the system is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\,\psi(x) \ , \tag{$\star$}$$

where E is the energy of the system. The Floquet matrix $F(E)$ is given by

$$\begin{pmatrix} \psi_1(x+a) \\ \psi_2(x+a) \end{pmatrix} = F(E) \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} ,$$

where ψ_1 and ψ_2 are two linearly independent solutions to ($\star$).

(i) Show that $\det(F) = 1$.

(ii) Explain why the trace $\mathrm{Tr}F$ is real.

(iii) Explain what the implications on the band structure are when $(\mathrm{Tr}F)^2 < 4$ and when $(\mathrm{Tr}F)^2 > 4$.

(b) Consider a tight-binding Hamiltonian that acts upon a single band of localised states in one dimension,

$$H_t = t \sum_{j\in\mathbb{Z}} \Big(|j\rangle\langle j+1| + |j\rangle\langle j-1| + (-1)^j |j\rangle\langle j| \Big) \ ,$$

where t is a positive constant. The integer j should be thought of as indexing sites along a chain of atoms separated from each other by a distance a; the state $|j\rangle$ locates an electron on atom j.

(i) Consider the translation operator T_ℓ which acts on the states $|j\rangle$ by shifting them to another site, $T_\ell|j\rangle = |j+\ell\rangle$, with $\ell \in \mathbb{Z}$. Determine the values of ℓ for which T_ℓ commutes with the Hamiltonian H_t. Hence determine the lattice spacing of the system.

(ii) Using Bloch's theorem, show that the Hamiltonian of the system can be reduced to a 2×2 matrix. Find the eigenvalues of this matrix.

(iii) What is the range of the first Brillouin zone? Plot the energy bands for the first Brillouin zone. In your plot you should clearly label the axes and indicate the values of the energies at the boundaries of the Brillouin zone. What is the bandwidth of each of the bands?

35B Statistical Physics

(a) What systems are described by a *grand canonical ensemble*? Use the grand partition function to derive the formula for the Fermi-Dirac distribution for the mean occupation numbers n_r of discrete single-particle states r with energies $E_r \geqslant 0$ in a gas of non-interacting identical Fermions in terms of $\beta = 1/(k_B T)$ and the chemical potential μ.

(b) Show that the density of states $g(E)$ for a gas of non-interacting non-relativistic identical Fermions with spin degeneracy g_s in a large d-dimensional volume V, where $d \geqslant 2$, is

$$g(E) = g_s B V E^{(d-2)/2}\,,$$

where B is a constant that you should determine. Compute the Fermi energy E_F, *i.e.* the chemical potential at zero temperature and with N particles. Find an expression at zero temperature for pV in terms of N and E_F, where p is the pressure. [You may denote the surface area of a unit $(d-1)$-dimensional sphere by S_{d-1}.]

(c) Consider a semi-infinite block of metal in 3-dimensional space occupying $z \leqslant 0$ with infinite extent in the x and y directions, and empty space for $z > 0$. An electron in the metal with momentum (p_x, p_y, p_z) will escape from the metal if p_z is large enough, $\frac{p_z^2}{2m} \geqslant E_F + V_0$, where m is the mass of the electron and V_0 is the potential barrier.

The flux density, J_z, of electrons escaping from the metal is the number of electrons escaping per unit area per unit time. Model the electrons as an ideal non-relativistic gas of Fermions, and assume that $k_B T \ll V_0$ and $k_B T \ll E_F$. You may assume that $\mu = E_F$ in this limit. Thereby derive an expression for the flux density of electrons escaping from the metal.

[*Hint:* $\displaystyle\int_{-\infty}^{\infty} e^{-a\,x^2} dx = \sqrt{\pi/a} \quad \textit{for} \quad a > 0.$]

36D Electrodynamics

The Maxwell equations for charge and current densities $\rho(\boldsymbol{x},t)$ and $\boldsymbol{J}(\boldsymbol{x},t)$ are

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=\frac{\rho}{\epsilon_0}\,,\qquad \boldsymbol{\nabla}\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t}\,,$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}=0\,,\qquad \boldsymbol{\nabla}\times\boldsymbol{B}=\mu_0\boldsymbol{J}+\mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\,.$$

The energy and momentum density of the electromagnetic field are defined by

$$\mathcal{E}=\frac{1}{2}\left(\epsilon_0\boldsymbol{E}^2+\frac{\boldsymbol{B}^2}{\mu_0}\right)\qquad\text{and}\qquad \boldsymbol{g}=\varepsilon_0\boldsymbol{E}\times\boldsymbol{B}\,.$$

(a) Use the Maxwell equations to show that $\boldsymbol{g}$ obeys the local conservation law

$$\partial_t g_j+\nabla_i\sigma_{ij}=-(\rho\boldsymbol{E}+\boldsymbol{J}\times\boldsymbol{B})_j\,,$$

with

$$\sigma_{ij}=\tilde{a}\,E_iE_j+\tilde{b}\,\boldsymbol{E}^2\delta_{ij}+\tilde{c}\,B_iB_j+\tilde{d}\,\boldsymbol{B}^2\delta_{ij}\,,$$

where δ_{ij} is the Kronecker delta and $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$ are constants you should determine. [*Hint: A vector field* $\boldsymbol{a}$ *satisfies* $(\boldsymbol{\nabla}\times\boldsymbol{a})\times\boldsymbol{a}=(\boldsymbol{a}\cdot\boldsymbol{\nabla})\boldsymbol{a}-\frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{a}^2)$]

(b) Provide a brief physical interpretation of the tensor σ_{ij} and, in particular, its diagonal and off-diagonal components.

(c) The electric field of a homogeneously charged sphere with total charge q and radius R centered on the origin is given by

$$\boldsymbol{E}=E(r)\boldsymbol{e}_r\quad\text{with}\quad E(r)=\begin{cases}\frac{q}{4\pi\epsilon_0}\frac{r}{R^3} & \text{for } r\leqslant R\\ \frac{q}{4\pi\epsilon_0}\frac{1}{r^2} & \text{for } r>R\,,\end{cases}$$

where $r=|\boldsymbol{x}|$ and $\boldsymbol{e}_r$ is the unit vector in the radial direction. The magnetic field $\boldsymbol{B}$ is zero. Calculate the energy density $\mathcal{E}$ and, thus, the total energy W_{em} contained in the electromagnetic field.

Under a Lorentz transformation with velocity $\boldsymbol{v}=(0,0,v)$ in the z direction and Lorentz factor $\gamma=1/\sqrt{1-(v^2/c^2)}$, the electromagnetic field changes to $\boldsymbol{E}'$ and $\boldsymbol{B}'$ given by

$$\boldsymbol{E}'_{||}=\boldsymbol{E}_{||}\,,\qquad \boldsymbol{E}'_{\perp}=\gamma(\boldsymbol{E}_{\perp}+\boldsymbol{v}\times\boldsymbol{B})\,,$$

$$\boldsymbol{B}'_{||}=\boldsymbol{B}_{||}\,,\qquad \boldsymbol{B}'_{\perp}=\gamma\left(\boldsymbol{B}_{\perp}-\frac{\boldsymbol{v}}{c}\times\frac{\boldsymbol{E}}{c}\right)\,,$$

where the subscripts $||$ and $\perp$, respectively, denote field components parallel and perpendicular to $\boldsymbol{v}$. Compute the linear momentum $\boldsymbol{P}=(0,0,P_z)$ contained in the electromagnetic field of a homogeneously charged sphere moving with $\boldsymbol{v}$ in the slow-velocity limit, i.e. ignoring terms of order $(v/c)^2$ or higher. [*Hint: Vector fields* $\boldsymbol{a}$, $\boldsymbol{b}$ *and* $\boldsymbol{c}$ *satisfy the relation* $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$ *and* $\int_0^\pi\sin^3\theta\,\mathrm{d}\theta=\frac{4}{3}$.]

37B General Relativity

(a) Consider a curve $x^\alpha(\lambda)$ with tangent vector $T^\alpha = dx^\alpha/d\lambda$, in a spacetime with metric $g_{\mu\nu}$. For any vector fields U^α and W^α, show that, on the curve,

$$\frac{d}{d\lambda}(\, g_{\mu\nu} U^\mu W^\nu \,) \;=\; (\nabla_T U)^\alpha W_\alpha \,+\, U_\alpha (\nabla_T W)^\alpha \,,$$

where $\nabla_V = V^\alpha \nabla_\alpha$ denotes the covariant directional derivative along a vector V^α.

(b) Now consider a one-parameter family of geodesics defined by the functions with two arguments, $x^\alpha(\tau,\sigma)$. Fixing a value of σ in these functions gives a timelike geodesic parametrised by proper time τ, with tangent vector $T^\alpha = \partial x^\alpha/\partial\tau$ satisfying $T^\alpha T_\alpha = -1$, while fixing a value of τ gives a curve parametrised by σ, with tangent vector $S^\alpha = \partial x^\alpha/\partial\sigma$.

Show that the commutator $[T,S]^\alpha = (\nabla_T S)^\alpha - (\nabla_S T)^\alpha = 0$ and hence derive the equation of geodesic deviation in the form

$$(\nabla_T \nabla_T S)^\alpha \,+\, E^\alpha{}_\beta S^\beta \;=\; 0\,,$$

where $E^\alpha{}_\beta$ is a tensor to be defined in terms of T^α and the Riemann tensor $R^\alpha{}_{\beta\mu\nu}$.

(c) Suppose that in part (b),

$$R_{\alpha\beta\mu\nu} \;=\; K(\, g_{\alpha\mu}\, g_{\beta\nu} \,-\, g_{\alpha\nu}\, g_{\beta\mu}\,)\,,$$

where K is a scalar. Show that, if $T^\alpha S_\alpha = 0$ at some point on a particular geodesic, then

$$(\nabla_T \nabla_T S)^\alpha \,-\, K S^\alpha \;=\; 0$$

at all points on that geodesic.

[*Hint: You may use, without proof, standard properties of the metric connection, symmetries of the Riemann tensor, and the Ricci identity in the form*

$$(\nabla_T \nabla_S V)^\alpha \,-\, (\nabla_S \nabla_T V)^\alpha \;=\; (\nabla_{[T,S]} V)^\alpha \,+\, R^\alpha{}_{\beta\mu\nu} V^\beta T^\mu S^\nu$$

for vector fields T^α, S^α and V^α.]

38C Fluid Dynamics II

A long, thin organism can be modelled as a body that occupies the region $y = 0$, $x > 0$ in two dimensions, surrounded by an incompressible fluid. The organism is at rest relative to the fluid that is far away from it, and it exerts a tangential stress $Sx^{-1/2}$, with $S > 0$ a constant, on the surrounding fluid.

(a) Write down the boundary-layer equations describing steady high-Reynolds-number flow around the organism. What boundary conditions apply to this flow?

(b) Use scaling to show that the width δ of the boundary layer in the y direction satisfies $\delta \sim (\rho\nu^2/S)^{1/3}x^{1/2}$, where ρ and ν are the density and kinematic viscosity of the fluid, respectively. What is the associated velocity scale in the direction parallel to the organism?

(c) Determine the form of a similarity solution to the boundary-layer equations in which the stream function is proportional to a dimensionless function $f(\eta)$ of a suitable similarity variable η. Determine the differential equation and boundary conditions satisfied by $f(\eta)$. Explain briefly how one could solve the differential equation numerically to determine the velocity field and, specifically, the velocity of the fluid adjacent to the organism.

39D Waves

Let $\boldsymbol{u} = (u, 0, w)$ denote the components of a solenoidal two-dimensional velocity field with $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$. Consider small-amplitude vertical-velocity perturbations $w(x, z, t)$ about a state of rest in a two-dimensional stratified fluid of sufficiently slowly varying background density $\rho_0(z)$ such that

$$\left| \frac{1}{\rho_0} \frac{\mathrm{d}\rho_0}{\mathrm{d}z} \frac{\partial w}{\partial z} \right| \ll \left| \frac{\partial^2 w}{\partial z^2} \right| .$$

(a) Show that $w(x, z, t)$ satisfies the equation

$$\nabla^2 \left(\frac{\partial^2 w}{\partial t^2} \right) + N^2(z) \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{where} \quad N^2(z) = -\frac{g}{\rho_0} \frac{\mathrm{d}\rho_0}{\mathrm{d}z} ,$$

and N is the buyoancy frequency.

(b) Consider a semi-infinite region of fluid with constant buoyancy frequency N, flowing with uniform velocity $U > 0$ in the $x-$direction and bounded below by a sinusoidal range of hills at $h = h_0 \sin kx$, where $k > 0$.

(i) Determine the frequency of the motion induced in a frame of reference in which the background fluid is stationary.

(ii) You may assume that the vertical component of the fluid velocity at $z = 0$ is given by the vertical component of the velocity of fluid particles that follow the undulations of the lower boundary, i.e. $w = U \partial h / \partial x$ at $z = 0$. Hence establish that for $z > 0$, the vertical velocity satisfies

$$w = w_0 \exp[i(kx + mz - \omega t)],$$

where $w_0 = U k h_0$, and m satisfies a dispersion relation which you should express in terms of N, U and k.

(iii) The time- and horizontally-averaged vertical wave energy flux at $z = 0$ is given by $\langle \overline{I_z} \rangle_0 = \langle \overline{\tilde{p} w} \rangle_0$, where $\tilde{p}$ is the pressure perturbation, the angle brackets denote an average over the wave period and the overline denotes an average over the hill wavelength. Calculate $\langle \overline{I_z} \rangle_0$ for sufficiently large wavelengths $k \leqslant k_c = N/U$ making clear the reasoning for selecting the sign of all terms.

(iv) Show that $\langle \overline{I_z} \rangle_0 = 0$ for sufficiently short wavelengths $k > k_c$.

(v) Give a brief physical interpretation of the cut-off wavenumber $k_c = N/U$.

40A Numerical Analysis

Let A be an $n \times n$ real symmetric positive definite matrix and $\mathbf{b} \in \mathbb{R}^n$. Consider the linear system

$$A\mathbf{x} = \mathbf{b}.$$

(a) An iterative method is defined by $\mathbf{x}_{k+1} = H\mathbf{x}_k + \mathbf{v}$ with $n \times n$ matrix H and $\mathbf{v} \in \mathbb{R}^n$.

(i) State, giving a brief justification, necessary and sufficient conditions for $\mathbf{x}_k$ to converge to $\mathbf{x}^* = A^{-1}\mathbf{b}$ as $k \to \infty$.

(ii) Give H and $\mathbf{v}$ in terms of A and $\mathbf{b}$ for the Jacobi method. Prove that the Jacobi method is convergent if A is strictly diagonally dominant. [You may use Gershgorin's circle theorem without proof.]

(b) Define the function $f : \mathbb{R}^n \to \mathbb{R}$ by $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x}$.

(i) Show that the unique global minimizer of f is at $\mathbf{x}^* = A^{-1}\mathbf{b}$. Show also that for any $\mathbf{x}$ we have the identity

$$f(\mathbf{x}) - f(\mathbf{x}^*) = \tfrac{1}{2}\nabla f(\mathbf{x})^T A^{-1} \nabla f(\mathbf{x})\,,$$

where $\nabla f(\mathbf{x})$ is the gradient of f at $\mathbf{x}$.

(ii) Consider the gradient method with exact line search to find the $\mathbf{x}$ that minimises $f(\mathbf{x})$:

$$\mathbf{x}_{k+1} = \mathbf{x}_k - t_k \nabla f(\mathbf{x}_k) \qquad \text{where} \qquad t_k = \arg\min_{t \in \mathbb{R}} f(\mathbf{x}_k - t\nabla f(\mathbf{x}_k))\,.$$

Give an explicit expression for t_k in terms of the residual $\mathbf{r}_k = \mathbf{b} - A\mathbf{x}_k$.

(iii) Deduce that

$$f(\mathbf{x}_{k+1}) - f(\mathbf{x}^*) = \left[1 - \frac{(\mathbf{r}_k^T\mathbf{r}_k)^2}{(\mathbf{r}_k^T A^{-1}\mathbf{r}_k)(\mathbf{r}_k^T A\mathbf{r}_k)}\right](f(\mathbf{x}_k) - f(\mathbf{x}^*))\,.$$

Conclude that

$$f(\mathbf{x}_k) - f(\mathbf{x}^*) \leqslant \left[1 - \frac{l}{L}\right]^k (f(\mathbf{x}_0) - f(\mathbf{x}^*))\;,$$

where l and L are respectively the smallest and largest eigenvalues of A.

What does this inequality tell us in the case when $n = 1$?

END OF PAPER