UNIVERSITY OF CAMBRIDGE

MAT2

MATHEMATICAL TRIPOS Part II

Tuesday, 04 June, 2024 1:30pm to 4:30pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most six questions** from Section I and from any number of questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Number Theory

Let d be a positive integer.

Define what it means for a positive definite binary quadratic form $f(x, y) = ax^2 + bxy + cy^2$ to be *reduced*. If d is congruent to 0 or 3 mod 4, define the *class number* $h(-d)$.

Show that if d is odd and has k distinct prime factors, then $h(-4d) \geqslant 2^{k-1}$.

Give an example with $d > 1$ to show that the inequality $h(-4d) \geqslant 2^{k-1}$ can be strict.

2G Topics in Analysis

State Baire's category theorem. Define an *isolated point* for a metric space.

Which of the following statements are true and which are false? Give a proof or a counterexample. (By a metric space we mean a non-empty metric space.)

(i) A countable metric space must have isolated points.

(ii) A complete metric space cannot have isolated points.

(iii) An uncountable metric space without isolated points must be complete.

(iv) A complete metric space must be uncountable.

(v) A countable complete metric space must have isolated points.

(vi) All the points in a countable complete metric space are isolated.

(vii) A countable complete metric space containing at least two points must contain at least two isolated points.

(viii) A countable complete metric space containing infinitely many points must contain infinitely many isolated points.

3K Coding and Cryptography

(a) Show that Hamming's original code is perfect.

(b) Consider the code obtained by using Hamming's original code for the first 7 bits and the final bit as a check digit, so that

$$x_1 + x_2 + \cdots + x_8 \equiv 0 \pmod 2.$$

Find the minimum distance for this code. How many errors can it detect? How many errors can it correct?

(c) Given a code of length n which corrects e errors, can you always construct a code of length $n+1$ which detects $2e+1$ errors? Give a brief justification of your answer.

4J Automata and Formal Languages

(a) Define what an *index set* is and when it is called *non-trivial*.

(b) Define the index set **Inf**.

(c) State Rice's theorem.

(d) Let $X \subseteq \mathbb{W}$ and let $\mathbf{Inf}_X := \{w \in \mathbf{Inf} : \mathrm{ran}(f_{w,1}) \subseteq X\}$. Show, by modifying the proof of Rice's theorem or otherwise, that for each nonempty X, the set $\mathbf{Inf}_X$ is not computable.

5L Statistical Modelling

Suppose we observe the proportion $Y \sim n^{-1}\mathrm{Binomial}(n,p)$ where $n \in \mathbb{N}$ is known and $p \in (0,1)$ is unknown. Compute the score function and the Fisher information for this statistical model.

State the Newton–Raphson and Fisher scoring algorithms for computing the maximum likelihood estimator.

How many steps do these algorithms take to converge to the maximum likelihood estimator when initialised at $p^{(0)} = Y$? How many steps does Fisher scoring take when initialised at some $p^{(0)} \neq Y$?

6A Mathematical Biology

The model of a viral infection in a population is given by the system

$$\frac{dX}{dt} = \mu N - \beta XY - \mu X,$$
$$\frac{dY}{dt} = \beta XY - (\mu + \nu)Y,$$
$$\frac{dZ}{dt} = \nu Y - \mu Z,$$

where μ, β and ν are positive constants and X, Y, and Z are respectively the number of susceptible, infected and immune individuals in a population of size N, independent of t, where $N = X + Y + Z$.

(a) Interpret the biological meaning of each of the parameters μ, β and ν.

(b) Show that there is a critical population size $N_c(\mu, \beta, \nu)$ such that if $N < N_c$ there is no steady state with the infection maintained in the population. Show that in this case the numbers of infected and immune individuals decrease to zero for all possible initial conditions.

(c) Show that for $N > N_c$ there is a steady state $(X, Y, Z) = (X^*, Y^*, Z^*)$ with $0 < X^*, Y^*, Z^* < N$. Show that this steady state is stable.

7D Further Complex Methods

Consider the differential equation

$$x \frac{\mathrm{d}^3 y(x)}{\mathrm{d}x^3} + 2y(x) = 0 \qquad (\dagger)$$

on the domain $x \in (0, \infty)$.
(a) Write

$$y(z) = \int_\gamma e^{zt} f(t)\, \mathrm{d}t,$$

where γ is a contour in the complex plane, and substitute this expression for y into the differential equation $(\dagger)$. Explain how the resulting integral equation can be solved by finding an appropriate function $f(t)$ and contour γ. Determine this function $f(t)$ and clearly state any required conditions on γ.
(b) Express the solution $y(x)$ in integral form. [You do *not* have to evaluate this integral, but you should simplify it as far as possible.] [*Hint: Consider a subset of the real axis for your choice of the contour γ.*]

8E Classical Dynamics

(a) In Lagrangian mechanics, explain what is meant by the *generalised momentum* associated to a generalised coordinate q.

(b) What does it mean for a generalised coordinate to be *ignorable*? Show that the generalised momentum associated to an ignorable coordinate is conserved. [You may state the Euler–Lagrange equations without proof.]

(c) A certain system has generalised coordinates (q_1, q_2, q_3) and Lagrangian

$$L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2\right) - \frac{1}{2}\left(q_1^2 + q_2^2 + q_3^2\right) - \alpha\left(q_1 q_2 + q_2 q_3 + q_3 q_1\right) ,$$

where α is constant. Show that L is invariant under rotations around the $(1,1,1)$ axis in q-space. Hence find two conserved quantities.

9D Cosmology

A non-relativistic particle species in equilibrium with mass m, temperature T and chemical potential μ satisfying $k_{\rm B}T \ll mc^2$ and $\mu \ll mc^2$, is described by the Maxwell-Boltzmann distribution. This can be integrated over momenta to give the total number density

$$n = g_s \left(\frac{2\pi m k_{\rm B} T}{h^2}\right)^{3/2} \exp\left[(\mu - mc^2)/(k_{\rm B}T)\right] ,$$

where g_s is the degeneracy.

(a) Deuterium is in equilibrium with non-relativistic protons and neutrons at around $t \approx 100$ seconds ($k_{\rm B}T \approx 0.1\,\mathrm{MeV}$) through the interaction $D \longleftrightarrow n + p$. Show that the ratio of the number densities can be expressed as

$$\frac{n_D}{n_p n_n} \approx \left(\frac{\pi m_p k_{\rm B} T}{h^2}\right)^{-3/2} e^{B_D/(k_{\rm B}T)} ,$$

where the deuterium binding energy is $B_D = (m_p + m_n - m_D)c^2 = 2.2\,\mathrm{MeV}$, and m_p, m_n and m_D are the proton, neutron and deuterium masses, respectively. [*Hint: The degeneracy factor for Deuterium is* $g_D = 4$.]

(b) Now use fractional densities relative to the baryon number density n_B (e.g. $X_p = n_p/n_B$) to find an expression for $X_D/(X_p X_n)$. In this case, replace $n_B = \eta\, n_\gamma$ where η is the baryon-to-photon ratio and the photon number is

$$n_\gamma = \frac{16\pi\zeta(3)}{(hc)^3}(k_{\rm B}T)^3 ,$$

where ζ is the Riemann zeta function. Briefly explain how the fractional density ratio $X_D/(X_p X_n)$ offers insight into the "deuterium bottleneck", that is, the delay in forming deuterium nuclei to temperatures well below the binding energy, $k_{\rm B}T \ll B_D$?

(c) In an alternative cosmology, the baryon-to-photon ratio η is larger. Assuming that the decoupling of neutrons and protons is unaffected by this change, would the helium abundance $Y_{\rm He}$ be larger or smaller in this scenario than the standard result $Y_{\rm He} \approx 0.25$? Explain your reasoning.

10E Quantum Information and Computation

Consider a function $f : \mathbb{Z}_{12} \to \mathbb{Z}_9$ defined by

$$f(x) = 4^x \bmod 9.$$

(a) The function f is periodic. Find its period r.

(b) Suppose we are given the following quantum state of 2 registers:

$$|f\rangle := \frac{1}{\sqrt{12}} \sum_{x \in \mathbb{Z}_{12}} |x\rangle \, |f(x)\rangle \, ,$$

and a measurement of the second register yields a value y.

What are the possible values of y and what are the corresponding probabilities?

(c) If $y = 4$, find the resulting state $|\alpha\rangle$ of the first register after the above measurement.

(d) Let QFT_{12} denote the quantum Fourier transform modulo 12. How does it act on a state $|x\rangle$ for $x \in \mathbb{Z}_{12}$?

(e) Suppose a measurement of the state $\mathrm{QFT}_{12} \, |\alpha\rangle$ yields a value c. What are the possible values of c and what are the corresponding probabilities?

SECTION II

11G Topics in Analysis

Suppose that a_j, b_j are real and strictly positive for all $j \geqslant 0$, and we have $p_0 = a_0$, $p_{-1} = 1$, $q_0 = 1$, $q_{-1} = 0$ with

$$\begin{aligned} p_j &= a_j p_{j-1} + b_{j-1} p_{j-2} \\ q_j &= a_j q_{j-1} + b_{j-1} q_{j-2} \end{aligned}$$

for $j \geqslant 1$. Show that

$$\frac{p_n}{q_n} = a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{\ddots \, \cfrac{}{a_{n-1} + \cfrac{b_{n-1}}{a_n}}}}}}$$

and that

$$\begin{pmatrix} p_n & b_n p_{n-1} \\ q_n & b_n q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & b_0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & b_n \\ 1 & 0 \end{pmatrix}.$$

We now specialise to the case when $b_j = 1$ and the a_j are strictly positive integers for all j. Show that $p_n q_{n-1} - q_n p_{n-1} = (-1)^{n+1}$ for all $n \geqslant 1$.

Show that p_n/q_n tends to a limit x and

$$\left| \frac{p_n}{q_n} - x \right| + \left| \frac{p_{n+1}}{q_{n+1}} - x \right| = \frac{1}{q_n q_{n+1}}$$

for each $n \geqslant 0$.

Now specialise still further to the case when $a_j = 1$ for all j. Show that $p_n = F_{n+2}$, $q_n = F_{n+1}$ for all $n \geqslant 1$ where $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Solve this difference equation to obtain an expression for F_n in terms of powers of $\phi = (1+\sqrt{5})/2$. Hence show that, in this case, the limit x discussed in the previous paragraph is ϕ. Show also that

$$F_n F_{n+1} \left| \frac{F_{n+1}}{F_n} - \phi \right| \to \frac{\phi}{\sqrt{5}} \quad \text{and} \quad F_n F_{n+1} \left| \frac{F_{n+2}}{F_{n+1}} - \phi \right| \to \frac{1}{\phi\sqrt{5}}$$

as $n \to \infty$.

12K Coding and Cryptography

(a) (i) Describe briefly the *Rabin cryptosystem*, including how to encrypt and decrypt messages. Show that breaking the Rabin cryptosystem is essentially as difficult as factoring the public modulus, N.

(ii) Criticise the following authentication procedure:

Alice chooses N as the public modulus for the Rabin cryptosystem. To be sure you are in communication with Alice, you send her a "random item" $r = m^2 \pmod{N}$. On receiving r, Alice proceeds to decode using her knowledge of the factorisation of N, and finds a square root m_1 of r. She returns m_1 to you and you check that $r = m_1^2 \pmod{N}$.

(b) (i) Describe briefly the *RSA cryptosystem* with public modulus N.

A budget internet company decides to provide each of its customers with their own RSA ciphers using a common modulus N. Customer j is given the public key (N, e_j) and sent secretly their decrypting exponent d_j. The company then sends out the same message, suitably encrypted, to each of its customers. You intercept two of these messages to customers i and j where e_i and e_j are coprime. Explain how you would decipher the message.

You are one of the customers, and so also know your own decrypting exponent. Can you decipher any message sent to another customer?

(ii) Explain why it might be a bad idea to use RSA with public modulus $N = pq$ with $|p - q|$ small.

A user of RSA accidentally chooses the public modulus N to be a large prime number. Explain why this system is not secure.

13D Further Complex Methods

(a) When is a function $f : \mathbb{C} \to \mathbb{C}$ called *elliptic*? Define a *fundamental cell* explicitly and state the property of an elliptic function regarding the number of its zeros and poles in a fundamental cell.

(b) Show that an elliptic function without poles is constant. [You may use without proof that an entire and bounded function is constant.]

(c) Let z_j denote the poles of an elliptic function f in a fundamental cell. Show that

$$\sum_j \operatorname{Res}(f; z_j) = 0.$$

Can there exist an elliptic function with a single pole of order one in a fundamental cell?

(d) If h is a meromorphic function on and inside a simple closed clockwise contour γ and h has no zeros or poles on γ, then

$$\frac{1}{2\pi i} \oint_\gamma \frac{h'(z)}{h(z)} \mathrm{d}z = P - Z\,,$$

where P and Z denote respectively the number of poles and zeros, counting multiplicities, of $h(z)$ inside the contour γ. Using this relation, show that a non-constant elliptic function f takes each value the same number of times in a cell, counting multiplicities.

(e) An example of an elliptic function is the Weierstrass function

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{(m,n)} \left[\frac{1}{(z - w_{m,n})^2} - \frac{1}{w_{m,n}^2} \right],$$

where $w_1,\ w_2 \in \mathbb{C} \setminus \{0\}$ with $\frac{w_1}{w_2} \notin \mathbb{R}$, $w_{m,n} = m w_1 + n w_2$ and the sum extends over $(m,n) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$. Identify and characterize the singularities of the Weierstrass function.

(f) The Laurent series of the Weierstrass function about $z_0 = 0$ can be written as

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{k=0}^{\infty} a_{2k} z^{2k}.$$

Calculate the coefficients a_{2k}.

14E Classical Dynamics

A mass m_1 is suspended from a fixed point with coordinates $(x, y, z) = (0, 0, 0)$ by a spring with spring constant k_1. A second mass m_2 is suspended from the first mass by a spring with spring constant k_2. Each spring has natural length ℓ. The motion of the masses is restricted to the (x, y)-plane, with gravity acting in the $-y$ direction. The position of the first mass is $(x_1, y_1, 0)$ and the position of the second mass is $(x_2, y_2, 0)$.

(a) Write down the Lagrangian of the system and hence determine the equations of motion.

(b) Find the equilibrium position of each mass that has $y_2 < y_1 < 0$.

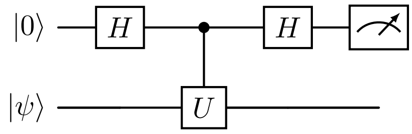
(c) For the remainder of this question suppose that the x-coordinate of mass m_2 is held fixed at its equilibrium value, and consider the case $m_1 = m_2 = m$ and $k_1 = k_2 = k$. One of the system's normal modes has the first mass moving in the x direction with no motion in the y direction and the other mass stationary. Show that this mode's frequency ω_1 satisfies

$$\omega_1^2 = \frac{k}{m}\left(2 - \frac{1}{\frac{2mg}{k\ell} + 1} - \frac{1}{\frac{mg}{k\ell} + 1}\right).$$

Find the other normal modes and corresponding frequencies, showing that they are independent of the strength of gravity.

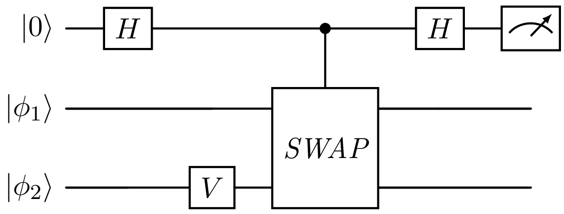
15E Quantum Information and Computation

(a) Consider a circuit which uses a controlled unitary gate with unitary operator U:



Here H represents a Hadamard gate, $|\psi\rangle$ denotes a single qubit state which is an eigenstate of U, $U|\psi\rangle = e^{2\pi i\theta}\,|\psi\rangle$, and the measurement is done in the computational basis. Express the phase θ in terms of the probability of the measurement giving zero.

(b) Consider the following circuit which uses a controlled $SWAP$ gate. A $SWAP$ gate acts as: $SWAP|i\rangle\,|j\rangle \mapsto |j\rangle\,|i\rangle\ \forall\ |i\rangle\,,|j\rangle \in \mathbb{C}^2$.



Here V is a unitary operator, $|\phi_1\rangle\,,|\phi_2\rangle$ are single qubit states, and the measurement is done in the computational basis. Find the probability of the measurement giving zero.

(c) Consider the operator U and the state $|\psi\rangle$ introduced in part (a). It is given that $\theta = j/2^m$ for some $j \in \{0, 1, 2, \ldots, 2^m - 1\}$ and some $m \in \mathbb{N}$. Define the controlled unitary operator $\Theta_m(U)$ which acts on a state $|k\rangle\,|\psi\rangle$ of $m+1$ qubits as:

$$\Theta_m(U)\,|k\rangle\,|\psi\rangle = |k\rangle\,U^k\,|\psi\rangle\,,$$

where $U^k\,|\psi\rangle$ is the state obtained by k successive applications of the operator U on the state $|\psi\rangle$ and $k \in \{0, 1, 2, \ldots, 2^m - 1\}$.

(i) Find an expression for the following state of $(m+1)$ qubits:

$$|\Phi\rangle := \Theta_m(U)\left(H^{\otimes m} \otimes I\right)|0\rangle^{\otimes m}\,|\psi\rangle\ .$$

[You should write out the result of applying the operators to $|0\rangle^{\otimes m}\,|\psi\rangle$.]

(ii) Write an expression for the corresponding state $|\phi_j\rangle$ of the first m qubits. Show that $\{|\phi_j\rangle\}_{j=0}^{2^m-1}$ is an orthonormal basis.

(iii) Let F be an operator acting on m qubits as $F\,|j\rangle = |\phi_j\rangle$. Justify why F is a unitary operator.

(iv) State the 2 sequential operations that you can do on $|\phi_j\rangle$ to find the value of j. What is the probability p_j of finding the value of j (and hence θ)?

[QUESTION CONTINUES ON THE NEXT PAGE]

(d) Recall the Breidbart basis used in the intercept and resend attack by Eve in the BB84 protocol. Suppose you do a measurement in this basis to discriminate between two equiprobable states $|\psi_1\rangle = |0\rangle$ and $|\psi_2\rangle = |-\rangle$. What is your probability of success? Justify why this is the best measurement that you can do to distinguish between the states (clearly stating any relevant result from the course).

16I Logic and Set Theory

State and prove Hartogs' Lemma.

Define *ordinal exponentiation.* Show that $\alpha^{\beta+\gamma} = \alpha^{\beta} \cdot \alpha^{\gamma}$ for all ordinals α, β, γ. Given ordinals γ and α, show that there exist unique ordinals β and δ such that $\gamma = \omega^{\alpha} \cdot \beta + \delta$ with $\delta < \omega^{\alpha}$. [You may assume standard properties of ordinal addition and multiplication. Other results used must be proved.]

Let X be a well-ordered set. Say $x \in X$ is a *limit point of* X if $I_x \neq \emptyset$ and I_x has no greatest element, where $I_x = \{y \in X : y < x\}$. Let X' denote the set of limit points of X and define $X^{(\alpha)}$ for all ordinals α by recursion as follows:

$$\begin{aligned} X^{(0)} &= X \\ X^{(\alpha+1)} &= \left(X^{(\alpha)}\right)' \\ X^{(\lambda)} &= \bigcap_{\alpha<\lambda} X^{(\alpha)} \qquad \text{(for non-zero limit ordinal } \lambda) \end{aligned}$$

Show that if $X \neq \emptyset$, then $X' \neq X$. Deduce that $X^{(\alpha)} = \emptyset$ for some α. The least such α is the *index of* X.

Show that if ξ is an ordinal, then

$$\xi' = \{\gamma < \xi : \exists \beta > 0 \ \text{ such that } \gamma = \omega \cdot \beta\}.$$

Describe ξ''. Find the index of ω and the index of ω^2.

17I Graph Theory

State and prove Turán's theorem.

A *rhombus* is the graph formed by two triangles sharing an edge. Prove that if G is a graph on $n \geqslant 4$ vertices that has more edges than $T_2(n)$, then G contains a rhombus.

Find a graph G on 6 vertices such that $e(G) = e(T_2(6))$ and G does not contain a rhombus, but G is not isomorphic to $T_2(6)$.

18H Galois Theory

(a) Let L/K be a field extension. Explain what it means to say that

(i) L/K is *finite*;

(ii) L/K is *separable*;

(iii) L/K is *simple*.

Which pairs of these properties together imply the third? In each case give a proof or counterexample.

(b) Let L be the splitting field of $f(X) = X^3 - X - 1$ over $\mathbb{Q}$. Compute $\mathrm{Gal}(L/\mathbb{Q})$. Show that L has a unique quadratic subfield, and write it in the form $\mathbb{Q}(\sqrt{d})$ for d a squarefree integer. Show also that if α is a root of f then $L = \mathbb{Q}(\alpha + \sqrt{d})$.

19H Representation Theory

Let G be a finite group. What is the *character* χ_V of a complex representation (ρ, V) of G?

Suppose that (ρ, V) and (σ, W) are complex representations of G. Show that the vector space $\mathrm{Hom}_{\mathbb{C}}(V, W)$ of $\mathbb{C}$-linear maps $\alpha\colon V \to W$ can be made into a representation of $G \times G$ via

$$((g,h)\cdot\alpha)(v) = \sigma(h)\left(\alpha(\rho(g^{-1})v)\right) \text{ for } (g,h) \in G \times G, \alpha \in \mathrm{Hom}_{\mathbb{C}}(V,W) \text{ and } v \in V.$$

Show that the character $\chi_{\mathrm{Hom}_{\mathbb{C}}(V,W)}$ satisfies

$$\chi_{\mathrm{Hom}_{\mathbb{C}}(V,W)}(g,h) = \overline{\chi_V(g)}\chi_W(h).$$

Consider the permutation representation $\mathbb{C}G$ of $G \times G$ arising from the action of $G \times G$ on G via

$$(g,h)\cdot x = gxh^{-1} \text{ for } (g,h) \in G \times G, x \in G$$

What is $\chi_{\mathbb{C}G}$?

Suppose that $V_1, \ldots, V_r$ are all the simple representations of G (up to isomorphism). Show there is an isomorphism

$$\mathbb{C}G \cong \bigoplus_{i=1}^{r} \mathrm{Hom}_{\mathbb{C}}(V_i, V_i)$$

of representations of $G \times G$.

20F Number Fields

State Dirichlet's unit theorem.

Let $K = \mathbb{Q}(\sqrt{5})$, and determine the units in $\mathcal{O}_K$. [You may use without proof the description of $\mathcal{O}_K$, as long as you state it clearly.]

For $K = \mathbb{Q}(\sqrt{5})$, what are the possible degrees of extensions L/K of number fields such that $1 < |\mathcal{O}_L^\times / \mathcal{O}_K^\times| < \infty$? Give an example for each possible degree.

21J Algebraic Topology

State the Seifert–van Kampen theorem.

If (X, x_0) is a based topological space, and $f : (S^{n-1}, *) \to (X, x_0)$ is a map of based spaces, define the space $X \cup_f D^n$ obtained by attaching an n-cell to X along f. For $n = 2$, carefully prove a formula describing $\pi_1(X \cup_f D^2, x_0)$ in terms of the group $\pi_1(X, x_0)$ and the element $[f] \in \pi_1(X, x_0)$.

Writing $S^1 \vee S^1$ for the wedge of two circles, calculate $\pi_1(S^1 \vee S^1, *)$.

Explain how to attach 2-cells to $S^1 \vee S^1$ to obtain a space whose fundamental group is the symmetric group on 3 letters, proving carefully that this is indeed the fundamental group obtained.

[*You may use any description of the group $\pi_1(S^1, *)$, provided it is clearly stated. You should justify any presentation of the symmetric group on 3 letters that you use.*]

22G Linear Analysis

State and prove the Baire Category Theorem.

Let X be a Banach space, and let S be a non-empty subset of X that is closed, convex and symmetric (S symmetric means $x \in S$ implies $-x \in S$). Show that if $\cup_{n=1}^{\infty} nS = X$ then S is a neighbourhood of the origin.

Give an example to show that the condition that S is convex cannot be omitted.

23G Analysis of Functions

State (without proof) the Hahn–Banach theorem for linear functionals on a normed real vector space X.

Now consider the topological dual space X'. For each $x \in X$, define $\hat{x} : X' \to \mathbb{R}$ by the action

$$\hat{x}(f) = f(x), \;\; f \in X'.$$

Show carefully that $x \mapsto \hat{x}$ defines a linear isometry from X into the bidual space X'' (the topological dual space of X'), and that

$$\|x\|_X = \sup_{\|f\|_{X'} \leqslant 1} |f(x)|.$$

Let $I = [0,1]$ and denote by L^∞ the Banach space of μ-essentially bounded functions on I, where μ is Lebesgue measure. Show that $(L^\infty)'$ does not coincide with $L^1(\mu)$. [*Hint: Extend the functional* $\ell(f) = f(0)$ *from the subspace* $C(I)$ *of continuous functions on* I *to* L^∞.]

24H Riemann Surfaces

For a non-constant analytic map $f : R \to S$ between compact Riemann surfaces and a point $z \in R$, let $m_f(z)$ denote the multiplicity of f at z and $\deg(f)$ the degree of f.

State the valency theorem. For the Riemann surface $\mathbb{C}_\infty$ and a non-constant analytic function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$, which you may assume is of the form $f(z) = p(z)/q(z)$ for non-zero polynomials p, q, explain how to find $\deg(f)$. Which f are the analytic isomorphisms of $\mathbb{C}_\infty$?

If $h : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is the Möbius transformation that swaps ∞ with 1 and swaps 0 with -1, write down a formula for h, as well as a quadratic equation satisfied by the fixed points of h.

Now consider the rotational symmetry group G of a regular octahedron P. You may assume that G is realised as a group of Möbius transformations isomorphic to S_4 with the six vertices of P corresponding to the points $0, \infty, \pm 1, \pm i \in \mathbb{C}_\infty$. Write down the possible sizes of the orbits under this action of G on $\mathbb{C}_\infty$.

Consider the function $F : \mathbb{C}_\infty \to \mathbb{C}_\infty$ given by the formula

$$F(z) = \frac{(z^4+1)^2(z^4+6z^2+1)^2(z^4-6z^2+1)^2}{z^4(z^4-1)^4}.$$

Which points in $\mathbb{C}_\infty$ are mapped to ∞ by F and with what multiplicities? What is $\deg(F)$?

You may now assume that F is constant on orbits, namely that if z_1 and z_2 are in the same orbit of this action of G on $\mathbb{C}_\infty$ then $F(z_1) = F(z_2)$. By using the valency theorem, or otherwise, show that F distinguishes orbits, namely if $F(z) = F(w)$ for $z, w \in \mathbb{C}_\infty$ then z and w are in the same orbit.

25F Algebraic Geometry

In this question, all algebraic varieties are over an algebraically closed field k.

Let X be an affine variety. Define the *tangent space* of X at a point $P \in X$. Define the *dimension* of X in terms of (i) the tangent spaces of X and (ii) Krull dimension. Say what it means for the variety to be *singular* at P.

Assume the characteristic of the field k is not 2. Let $X := Z(x_1^2 - x_2^3, x_3^2 - x_4^3) \subseteq \mathbb{A}^4$. Calculate the tangent space of X at each point of X.

Consider the subset $Y \subseteq \mathbb{P}^4$ consisting of points with homogeneous coordinates $(y_0 : y_1 : y_2 : y_3 : y_4)$ such that the matrix

$$\begin{pmatrix} y_0 & y_1 & y_2 \\ y_2 & y_3 & y_4 \end{pmatrix}$$

has rank one. Show that Y is a closed subset of $\mathbb{P}^4$ in the Zariski topology. You may now assume Y is irreducible in the Zariski topology, and hence is a projective variety. What is the dimension of Y? Show that Y is non-singular.

26J Differential Geometry

Consider a smooth closed curve $\alpha : I \to \mathbb{S}^2$ on the sphere, parametrised by arc length.

(a) Define the *curvature* κ, *torsion* τ, *Frénet trihedron* $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ and *geodesic curvature* κ_g of a general curve on a general surface. Prove in the particular case of the sphere that $\alpha = -\kappa^{-1}\mathbf{n} - \tau^{-1}\kappa^{-2}\dot{\kappa}\mathbf{b}$ and $\kappa_g = -\kappa^{-1}\tau^{-1}\dot{\kappa}$.

(b) State the local Gauss–Bonnet theorem for the curve α on $\mathbb{S}^2$. Deduce that, given a fixed length $I = [0, L]$, the curve α maximising the enclosed area also minimises $\int_I \kappa_g$.

(c) Consider $\varphi : [0, L] \to \mathbb{R}$ smooth with compact support in $\{\kappa_g \neq 0\}$, and $\beta : [0, L] \to \mathbb{R}^3$ defined by $\beta = -\varphi\mathbf{t} - \kappa^{-1}\dot{\varphi}\mathbf{n} - \kappa^{-1}\kappa_g^{-1}\dot{\varphi}\mathbf{b}$. Prove that $\beta \perp \alpha$ and $\dot{\beta} \perp \dot{\alpha}$.

(d) Consider the curve $\gamma^\epsilon := (\alpha + \epsilon\beta)/|\alpha + \epsilon\beta|$ for small ϵ. You may assume that, if the value $\epsilon = 0$ is a critical point of the enclosed area, then $\int_0^L \kappa_g^{-1}\dot{\varphi} = 0$. Deduce from this equation that area-maximising curves have constant geodesic curvature.

(e) Prove that a curve α on $\mathbb{S}^2$ with constant geodesic curvature is planar by showing that the vector $\mathbf{e}(s) := \alpha(s) \times \dot{\alpha}(s) + \kappa_g\alpha(s)$ is constant.

(f) Deduce that, if a curve on $\mathbb{S}^2$ of length L encloses an area A, then $L^2 \geqslant A(4\pi - A)$ (with the convention that we always choose the smaller of the two areas enclosed by the curve).

27G Probability and Measure

(a) State the two Borel–Cantelli lemmas.

(b) Let $(X_n)_{n\in\mathbb{N}}$ be independent exponential random variables with rate 1. Let $M_n = \max_{1\leqslant m\leqslant n} X_m$. Show that

(i) $\limsup X_n/\log n = 1$ almost surely;

(ii) $\liminf M_n/\log n \geqslant 1$ almost surely.

[*Hint: You may use without proof the inequality* $e^x \geqslant 1 + x$ *for all* $x \in \mathbb{R}$.]

(c) Let μ, ν be two measures on a measurable space $(\Omega, \mathcal{F})$ such that $\mu(\Omega) < \infty$. We say $\mu \ll \nu$ if for any $A \in \mathcal{F}$, $\nu(A) = 0$ implies $\mu(A) = 0$. Show that $\mu \ll \nu$ if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any $A \in \mathcal{F}$, $\mu(A) < \varepsilon$ whenever $\nu(A) < \delta$.

28K Applied Probability

(a) Let $X = (X_t)_{t\geqslant 0}$ be a simple birth process with rate $\lambda_n = n\lambda > 0$ for all $n \geqslant 1$, and $X_0 = 1$.

(i) Show that X is non-explosive.

(ii) Let $n \geqslant 1$. Show that conditional on the event $\{X_t = n+1\}$, the times of the n births have the distribution of the order statistics of n i.i.d. random variables with probability density function

$$f(x) = \frac{\lambda e^{\lambda x}}{e^{\lambda t} - 1}, \qquad 0 \leqslant x \leqslant t\,.$$

(b) Let X_t be a birth and death process with rates $\lambda_n = n\lambda$ and $\mu_n = n\mu$ for $n \in \mathbb{N}$, $\lambda > 0$, $\mu > 0$, and assume that $X_0 = 1$. Let $h(t) = \mathbb{P}(X_t = 0)$.

(i) Show that $h(t)$ satisfies

$$h(t) = \int_0^t e^{-(\lambda+\mu)s}\{\mu + \lambda(h(t-s))^2\}\,ds\,.$$

(ii) Show that $h'(t)$ satisfies

$$h'(t) = (h(t) - 1)(\lambda h(t) - \mu)\,.$$

(iii) Find $h(t)$ for $\lambda \neq \mu$.

29L Principles of Statistics

Let $X_1, \ldots, X_n$ be i.i.d. observations from a statistical model $\{f(\cdot, \theta) : \theta \in \Theta\}$ satisfying the usual regularity conditions, where $\Theta \subseteq \mathbb{R}^p$. Suppose we wish to test

$$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta \neq \theta_0.$$

(a) Define the *Wald statistic* $W_n(\theta)$ and write down a test for H_0 based on $W_n(\theta_0)$ with asymptotic type-I error bounded by a given $\alpha \in (0, 1)$.

(b) Define the *likelihood ratio statistic* Λ_n and write down a test for H_0 based on Λ_n with asymptotic type-I error bounded by a given $\alpha \in (0, 1)$.

(c) Suppose now that $p = 1$, i.e. θ is a scalar parameter, and we are under the null H_0. By considering an appropriate Taylor expansion, show that the two test statistics above are asymptotically equivalent, in the sense that

$$\frac{\Lambda_n}{W_n(\theta_0)} \xrightarrow{P} 1.$$

[You may use, without proof, a uniform law of large numbers, as long as it is clearly stated.]

30L Stochastic Financial Models

Let $(\mathcal{F}_n)_n$ be a filtration such that $\mathcal{F}_0$ is trivial. Let $(M_n)_n$ be a martingale and let T be a stopping time with respect to the filtration.

(a) Show that the stopped process $(M_{n \wedge T})_n$ is a martingale. [Results on the martingale transform may *not* be assumed without proof.]

(b) Assuming $(M_{n \wedge T})_n$ is bounded and T is finite, show that $\mathbb{E}(M_T) = M_0$. [Versions of the optional stopping theorem may *not* be assumed without proof.]

For the rest of the problem, let $X_0 = 0$ and $X_n = \xi_1 + \cdots + \xi_n$ for $n \geqslant 1$, where $(\xi_n)_n$ are IID and generate the filtration. Suppose $\mathbb{P}(\xi_n = +1) = \mathbb{P}(\xi_n = -1) = 1/2$ for all n.

(c) Given a real number $w > 1$, show that there exists a real number $z > 0$ such that the process

$$M_n = \left(A w^{X_n} + B w^{-X_n}\right) z^n$$

is a martingale for all constants A and B.

(d) Fix positive integers a, b and define

$$T = \min\{n \geqslant 0 : X_n \in \{-a, b\}\}.$$

For any $0 < z < 1$, compute $\mathbb{E}(z^T)$. [You may use the fact that T is a finite stopping time without proof.]

31K Mathematics of Machine Learning

(a) Carefully describe the construction of the regions $\{\hat{R}_1, \ldots, \hat{R}_J\}$ of a decision tree $x \mapsto \hat{T}(x) = \sum_{j=1}^J \hat{\gamma}_j \mathbf{1}_{\hat{R}_j}(x)$ trained on data D' consisting of input–output pairs $(X_i', Y_i') \in \mathbb{R}^p \times \mathbb{R}$, $i = 1, \ldots, n$. [You need not explain how computations may be performed in a computationally efficient manner.]

(b) In the following, consider the data D' as deterministic (i.e. not random). Let data $D := (X_i, Y_i)_{i=1}^n$ consist of i.i.d. input–output pairs, and let $(X, Y) \in \mathbb{R}^p \times \mathbb{R}$ have the same distribution as (X_1, Y_1) and be independent of D. Set

$$\tilde{T}(x) := \sum_{j=1}^J \tilde{\gamma}_j \mathbf{1}_{\hat{R}_j}(x),$$

$$\text{where} \qquad \tilde{\gamma}_j := \frac{1}{N_j + 1} \sum_{i=1}^n Y_i \mathbf{1}_{\hat{R}_j}(X_i) \qquad \text{and} \qquad N_j := \sum_{i=1}^n \mathbf{1}_{\hat{R}_j}(X_i).$$

Let $\gamma_j := \mathbb{E}[\tilde{\gamma}_j \,|\, X_{1:n}]$. Show that

$$\mathbb{E}\big[(\tilde{\gamma}_j - \gamma_j)^2 \,|\, X_{1:n}\big] = \frac{1}{(1+N_j)^2} \sum_{i=1}^n \mathrm{Var}(Y_i \,|\, X_i) \mathbf{1}_{\hat{R}_j}(X_i).$$

Now suppose further that $\mathrm{Var}(Y \,|\, X = x)$ is bounded from above by σ^2 for all $x \in \mathbb{R}^p$. Show that

$$\mathbb{E}\Big[\big\{\tilde{T}(X) - \mathbb{E}\big(\tilde{T}(X) \,|\, X, X_{1:n}\big)\big\}^2\Big] \leqslant \frac{\sigma^2 J}{n}.$$

[*Hint: You may use without proof, the fact that if* $N \sim \mathrm{Binomial}(n, q)$*, for success probability* $q \in (0, 1]$*, then* $\mathbb{E}[1/(N+1)] \leqslant 1/(nq)$*.*]

32C Asymptotic Methods

(a) Suppose $f(x)$ is a real-valued function and the set of functions $\phi_n(x)$, where $n = 0, 1, 2, 3, \ldots$, forms an asymptotic sequence as $x \to x_0$. What is meant by the statement "$f(x)$ has an *asymptotic expansion* as $x \to x_0$, with respect to the $\phi_n(x)$"?

Given that

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x) \quad \text{as } x \to x_0,$$

show that

$$a_0 = \lim_{x \to x_0} \frac{f(x)}{\phi_0(x)} \qquad \text{and} \qquad a_n = \lim_{x \to x_0} \frac{f(x) - \sum_{k=0}^{n-1} a_k \phi_k(x)}{\phi_n(x)}. \qquad (\dagger)$$

(b) Consider the asymptotic sequence $\phi_n(x)$, defined by $\phi_0 = x^{-1}$ and $\phi_n(x) = x^{-n+1}\mathrm{e}^{-x}$ for $n \geqslant 1$, as $x \to \infty$, and the function

$$f(x) = \frac{1}{x} + \frac{x\mathrm{e}^{-x}}{x-1}.$$

Find the asymptotic expansion of $f(x)$ with respect to the $\phi_n(x)$ as $x \to \infty$.

Verify explicitly that your coefficients satisfy ($\dagger$) for all n.

What is the asymptotic expansion of $f(x)$ with respect to the asymptotic sequence $\psi_n(x) = x^{-n}$ as $x \to \infty$?

(c) Consider the sine-integral function,

$$\mathrm{si}(x) = \int_1^{\infty} \frac{\sin(xt)}{t}\, dt.$$

Using integration by parts, show that

$$\mathrm{si}(x) \sim \cos x \sum_{n=0}^{\infty} a_n x^{-2n-1} + \sin x \sum_{n=0}^{\infty} b_n x^{-2n-2} \qquad \text{as } x \to \infty,$$

where you should determine the coefficients a_n and b_n.

33A Dynamical Systems

(a) State the normal form for a transcritical bifurcation in terms of the time t, the dependent variable x and parameter μ. Illustrate using diagrams why this type of bifurcation is not structurally stable, making sure that your diagrams are clearly labelled.

(b) Consider the system given by

$$\begin{aligned} \dot{x} &= y - x + ax^3, \\ \dot{y} &= rx - y - zy, \\ \dot{z} &= -z + xy, \end{aligned}$$

where a and r are constants.

(i) Show that the fixed point at the origin of the system is non-hyperbolic at $r = 1$.

(ii) Find the stable, unstable and centre subspaces of the linearised system of the fixed point at the origin at $r = 1$.

(iii) Set $r = 1$ and change to new coordinates (v, w, z) where $v = (x + y)/2$, $w = (x - y)/2$ and z is unchanged. Seek the (non-extended) centre manifold by writing $w = w_c(v)$ and $z = z_c(v)$. Find w_c and z_c to fourth order in v.

[*Hint: By considering symmetries, some of this calculation can be simplified.*]

(iv) Show that the evolution equation on the centre manifold is of the form

$$\dot{v} = \frac{a-1}{2}v^3 + \frac{(3a-1)(a+3)}{8}v^5 + \dots$$

(v) For what values of a is the origin asymptotically stable when $r = 1$?

34C Integrable Systems

(a) Explain what it means to say the KdV equation $u_t + u_{xxx} - 6uu_x = 0$, where $u = u(x,t)$, has a *Lax pair formulation* in terms of the linear operators

$$L = -\partial_x^2 + u \quad \text{and} \quad A = 4\partial_x^3 - 3u\partial_x - 3\partial_x u\,.$$

(b) Consider now the case of periodic boundary conditions for the KdV equation, so that at each time t the unknown u is a real-valued function satisfying $u(x+2\pi,t) = u(x,t)$. At each fixed time t, introduce a basis φ_+, φ_- of solutions to the scattering equation $L\phi = k^2\phi$ (for $k^2 > 0$) determined by the initial conditions at $x = 0$,

$$\varphi_\pm(0) = 1, \qquad \partial_x\varphi_\pm(0) = \pm ik\,.$$

(i) Show that there exists a matrix

$$\hat{T} = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix},$$

where $a(t)$ and $b(t)$ are functions of time and $\overline{a}$, $\overline{b}$ denote the complex conjugates, such that

$$\begin{pmatrix} \varphi_+(x+2\pi) \\ \varphi_-(x+2\pi) \end{pmatrix} = \hat{T} \begin{pmatrix} \varphi_+(x) \\ \varphi_-(x) \end{pmatrix}.$$

Show further that $|a|^2 - |b|^2 = 1$.

(ii) Now as t varies let u evolve in time according to the KdV equation. Show that there exists a matrix

$$\Lambda = \begin{pmatrix} \lambda & \mu \\ \overline{\mu} & \overline{\lambda} \end{pmatrix},$$

where $\lambda(t)$ and $\mu(t)$ depend on time, such that

$$\begin{pmatrix} A\varphi_+ + \partial_t\varphi_+ \\ A\varphi_- + \partial_t\varphi_- \end{pmatrix} = \Lambda \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix}.$$

Prove that $\partial_t \hat{T} = [\Lambda, \hat{T}]$.

[*Hint: Consider* $\partial_t(\hat{T}\Psi) = (\partial_t\hat{T})\Psi + \hat{T}\partial_t\Psi$, *with* $\Psi = \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix}$.]

Deduce that $\mathrm{Re}[a] = \frac{1}{2}(a + \overline{a})$ is independent of t.

35B Principles of Quantum Mechanics

(a) State the defining properties of a *density operator* ρ in a Hilbert space of finite dimension N, and state the number of real free parameters that determine ρ. Starting from ρ_H in the Heisenberg picture, derive the time-dependent $\rho_S(t)$ in the Schrödinger picture.

(b) State the commutation relations for the spin operators $\mathbf{S}$ with each other and with $\mathbf{S} \cdot \mathbf{S}$. For the pure state $|\psi\rangle$ of a spin-$\frac{1}{2}$ qubit, you are given $\langle S_x \rangle_\psi$, $\langle S_z \rangle_\psi$, and the sign of $\langle S_y \rangle_\psi$. Determine the normalised state $|\psi\rangle$. [*Hint: Recall that the state need only be determined up to an overall phase, so that the normalised state can be parametrised by a single complex number.*]

(c) For a general mixed state of a spin-$\frac{1}{2}$ qubit, you are given $\langle S_x \rangle$, $\langle S_z \rangle$, and $\langle S_y \rangle$. Determine the mixed state. Are the expectation values of any three linearly independent Hermitian operators sufficient to fully specify a general mixed state? Present a proof or a counterexample.

36E Applications of Quantum Mechanics

Consider a Hamiltonian H that has a discrete spectrum and a ground state with energy E_0.

(a) Describe briefly how to use the variational method to provide an upper bound on E_0.

(b) Suppose that the trial wavefunction in the variational method is given by

$$|\psi\rangle = \sum_{n=1}^{N} \alpha_n |\phi_n\rangle \,,$$

where α_n are complex variational parameters, and the $|\phi_n\rangle$ form an orthonormal set, i.e., $\langle\phi_m|\phi_n\rangle = \delta_{mn}$, for m and $n = 1, 2, \ldots N$.

Apply the variational method to H with trial wavefunction $|\psi\rangle$, and show that the lowest eigenvalue of the matrix $\mathcal{H}$, which has entries $\mathcal{H}_{nm} = \langle\phi_n|H|\phi_m\rangle$, gives the optimal upper bound on the ground state energy E_0.

(c) Consider a particle of mass m in an infinite one-dimensional square well of width a, with a linear potential in the well,

$$V(x) = \begin{cases} V_0 \dfrac{x}{a} & 0 \leqslant x \leqslant a \,, \\ \infty & \text{otherwise} \,, \end{cases}$$

where $V_0 = \dfrac{9\hbar^2}{ma^2}$. Determine an upper bound for the ground state energy of this system using

$$\phi_n = \begin{cases} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) & 0 \leqslant x \leqslant a \,, \\ 0 & \text{otherwise} \,, \end{cases}$$

with $n = 1, 2,$ as trial wavefunctions.

37B Statistical Physics

(a) Explain what is meant by an *intensive quantity* and what is meant by an *extensive quantity*. Give two examples of each.

(b) A real-valued homogeneous function f of degree k satisfies

$$f(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^k f(x_1, x_2, \ldots, x_n)\,,$$

for any real λ. Show that

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = kf\,.$$

(c) Explain why the energy $E(S, V, N)$ is a homogeneous function of degree 1, where S is the entropy, V is the volume and N is the number of particles. Hence, using the first law of thermodynamics, find an expression for E in terms of S, V, N, μ, p and T, where μ is the chemical potential, p is the pressure and T is the temperature. Show that $d\mu = (Vdp - SdT)/N$.

(d) Consider a chemical reaction at constant T and p where each molecule of chemical A can change into two molecules of chemical B and one molecule of chemical C, and vice-versa, i.e. $A \leftrightarrow 2B + C$. By minimising the Gibbs free energy G, derive a relation between the chemical potentials of the three chemicals at equilibrium, where the chemical potential of chemical i is $\mu_i = \frac{\partial G}{\partial N_i}$.

38B General Relativity

(a) Define the *Einstein tensor* $G_{\mu\nu}$ in terms of the Riemann tensor $R^{\alpha}{}_{\beta\mu\nu}$ and use the Bianchi identity $\nabla_\rho R^{\alpha}{}_{\beta\mu\nu} + \nabla_\mu R^{\alpha}{}_{\beta\nu\rho} + \nabla_\nu R^{\alpha}{}_{\beta\rho\mu} = 0$ to show that $\nabla_\nu G_\mu{}^\nu = 0$. Comment briefly on the significance of this result for consistency of the Einstein equations (including a cosmological constant).

(b) For a universe described by the line element

$$ds^2 = -dt^2 + a(t)^2 \, (dx^2 + dy^2 + dz^2)\,,$$

the Einstein tensor is diagonal with $G_t{}^t = -3\dot{a}^2/a^2$ and $G_x{}^x = -2\ddot{a}/a - \dot{a}^2/a^2$, where dots denote differentiation with respect to t. Verify by direct computation that $\nabla_\nu G_\mu{}^\nu = 0$, justifying the steps that you make and computing any metric connection components $\Gamma_\nu{}^\mu{}_\rho$ that you may need.

Solve the vacuum Einstein equations with a cosmological constant $\Lambda > 0$ to obtain a result for $a(t)$ that corresponds to an expanding universe.

39C Fluid Dynamics II

A two-dimensional incompressible Stokes flow has stream function ψ such that the velocity $\mathbf{u} = \nabla\times(\psi\mathbf{k})$, where $\mathbf{k}$ is the unit vector normal to the plane of the flow. Show that

$$\nabla^4\psi = 0.$$

In plane polar coordinates (r,θ) the velocity is

$$\mathbf{u} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{e}_r - \frac{\partial\psi}{\partial r}\mathbf{e}_\theta.$$

Given that the stream function has the form $\psi = r^2 f(\theta)$, determine the rate-of-strain tensor in terms of f and its derivatives. Hence write down the corresponding deviatoric stress tensor for a fluid of dynamic viscosity μ.

Fluid with dynamic viscosity μ fills the two-dimensional region $-\alpha < \theta < 0$, $r > 0$, where $\alpha > 0$ is a constant. The boundary $\theta = -\alpha$ is rigid, while a tangential stress S is applied to the horizontal surface $\theta = 0$. Given that the stream function has the form $\psi = r^2 f(\theta)$, write down the boundary conditions that apply to $f(\theta)$. Hence, determine $f(\theta)$ and show that the surface velocity

$$U(r) = u(r,0) = \frac{Sr}{\mu}\frac{1-\cos 2\alpha - \alpha\sin 2\alpha}{\sin 2\alpha - 2\alpha\cos 2\alpha}.$$

[*Hint: In plane polar coordinates,* $\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}$.]

40D Waves

The linearized Cauchy momentum equation governing small and smooth displacements $\boldsymbol{u}(\boldsymbol{x}, t)$ in a uniform, linear isotropic elastic solid of density ρ is

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = (\lambda + \mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + \mu \nabla^2 \boldsymbol{u},$$

where the constants λ and μ are the Lamé moduli.

(a) Show that this equation supports two distinct classes of wave-like motion: P-waves for the dilatation $\vartheta = \boldsymbol{\nabla} \cdot \boldsymbol{u}$ with phase speed c_p; and S-waves for the rotation $\boldsymbol{\Omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$ with phase speed c_s. You should express c_p and c_s explicitly in terms of the Lamé moduli.

(b) Now consider a region of this solid with a horizontal plane boundary at $z = 0$ in which plane waves propagate with wave vector $\boldsymbol{k} = \kappa(\sin\theta, 0, \cos\theta)$, i.e. θ is the angle the wave vector makes with the vertical $z-$direction and κ is the magnitude of the wave vector. Explain briefly why such a domain can in general support:

(i) harmonic P-waves: $\boldsymbol{u} = \boldsymbol{A} \exp[i\kappa(x\sin\theta + z\cos\theta) - i\omega t]$;

(ii) harmonic SV-waves: $\boldsymbol{u} = \boldsymbol{B}_V \exp[i\kappa(x\sin\theta + z\cos\theta) - i\omega t]$;

(iii) and harmonic SH-waves: $\boldsymbol{u} = \boldsymbol{B}_H \exp[i\kappa(x\sin\theta + z\cos\theta) - i\omega t]$.

You should define explicitly the orientations of the complex vector amplitudes $\boldsymbol{A}$, $\boldsymbol{B}_V$ and $\boldsymbol{B}_H$.

(c) Now consider a region of the solid between a rigid plane boundary at $z = 0$ and a free surface at $z = h > 0$.

(i) Show that this region can support propagating SH-waves with wave vector in the $x-$direction (i.e. with $\theta = \pi/2$), calculating explicitly the dispersion relation.

(ii) Deduce that there is a cut-off frequency ω_n for each mode in the vertical, given by
$$\omega_n = \frac{(2n+1)\pi}{2h} c_s \,,$$
for non-negative integers $n = 0, 1, 2 \ldots$.

(iii) Express the phase velocity c and the group velocity c_g of each mode in terms of c_s, ω_n and κ.

(iv) Deduce that, for any given wave number $\kappa > 0$ and mode with $n \geqslant 0$, $c = mc_g$ with $m > 1$, where you should express m in terms of n, κ and h.

(v) Calculate m explicitly for the specific wave with horizontal wavelength h and $n = 1$.

[*Hint: You may find it useful to recall that* $\nabla^2 \boldsymbol{q} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{q}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{q})$.]

41A Numerical Analysis

(a) Let $h : \mathbb{R} \to \mathbb{R}$ be a 2-periodic function with Fourier series $h(x) = \sum_{n \in \mathbb{Z}} \widehat{h}_n e^{i\pi n x}$. Let $I(h) = \frac{1}{2} \int_{-1}^{1} h(x)\, dx$. For $N \geqslant 1$, consider the approximation

$$I_N(h) = \frac{1}{2N} \sum_{k=-N+1}^{N} h(k/N).$$

Find the error $|I_N(h) - I(h)|$ in terms of the $\widehat{h}_n$.

Assuming $|\widehat{h}_n| \leqslant M c^{|n|}$ for all $n \in \mathbb{Z}$, where $M > 0$ and $c \in (0, 1)$, show that the error decays exponentially fast with N.

(b) Let $w : \mathbb{R} \to \mathbb{R}$ be a 2-periodic function with a finite Fourier expansion

$$w(x) = \sum_{|n| \leqslant d} \widehat{w}_n e^{i\pi n x}.$$

Consider the partial differential equation for $u(x, t)$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \frac{dw}{dx} \frac{\partial u}{\partial x}$$

with initial condition $u(x, 0) = u_0(x)$ which is 2-periodic. Seek an approximate solution for $u(x, t)$ for all $t \geqslant 0$ that is 2-periodic in x with an expansion of the form

$$u(x, t) = \sum_{|n| \leqslant D} \widehat{u}_n(t) e^{i\pi n x},$$

where D is the truncation level. Write down a differential equation for the $\widehat{u}_n(t)$ of the form

$$\frac{d\widehat{u}_n(t)}{dt} = \sum_{|m| \leqslant D} B_{nm} \widehat{u}_m(t),$$

for a matrix B that you should specify.

Assume that $w(x) = \cos(\pi x)$. Show that in this case, the eigenvalues of B have non-positive real part. Is B invertible?

END OF PAPER