UNIVERSITY OF CAMBRIDGE

MAT2

MATHEMATICAL TRIPOS Part II

Monday, 03 June, 2024 1:30pm to 4:30pm

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most six questions** from Section I and from any number of questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Number Theory

Let $N \in \mathbb{N}$ be an odd composite integer, and let $b \in (\mathbb{Z}/N\mathbb{Z})^\times$. Define what it means for N to be a *Fermat pseudoprime* to the base b, and for N to be an *Euler pseudoprime* to the base b.

Now let $N = 105$. Determine the proportion of bases $b \in (\mathbb{Z}/N\mathbb{Z})^\times$ such that N is a Fermat pseudoprime to the base b. Determine the proportion of bases $b \in (\mathbb{Z}/N\mathbb{Z})^\times$ such that N is an Euler pseudoprime to the base b.

2G Topics in Analysis

(a) Define the nth *Chebychev polynomial* T_n. Show that it is indeed a polynomial and that $-1 \leqslant T_n(x) \leqslant 1$ for all $x \in [-1,1]$.

(b) Show that, if $n \geqslant 1$, the leading coefficient of T_n is 2^{n-1}.

(c) Show that $T_n(x) = T_n(-x)$ if n is even, and $T_n(x) = -T_n(-x)$ if n is odd, explaining why these results hold for all $x \in \mathbb{R}$.

(d) By looking at the roots of $T_n^{(k)}(x)$, or otherwise, show that, if $0 \leqslant r \leqslant n-1$, then $T_n^{(r)}(x)$ is increasing for $x \geqslant 1$.

(e) Compute $T_n'(1)$ and show that $T_n(x) \geqslant n(x-1)+1$ for all $x \geqslant 1$.

3K Coding and Cryptography

Briefly describe the *binary Huffman code* for encoding symbols $1, 2, \ldots, m$ occurring with probabilities $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_m > 0$.

Consider the discrete random variable X taking seven values x_i $(1 \leqslant i \leqslant 7)$ with the following probabilities:

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0.49 & 0.26 & 0.12 & 0.04 & 0.04 & 0.03 & 0.02 \end{pmatrix}.$$

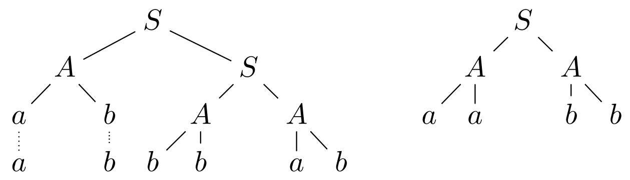
Find a binary Huffman code for X. What is its expected word length? [You do not need to simplify the expression.]

4J Automata and Formal Languages

In this question, let $\Sigma = \{a, b\}$ and $V = \{S, A\}$ be the set of terminal and non-terminal symbols, respectively.

(a) Give an example of a context-free grammar $G = (\Sigma, V, P, S)$ and a word $w \in \mathbb{W}$ such that w has a unique G-parse tree starting from S, but exactly two different G-derivations. Justify your claim.

(b) Suppose that G is a context-free grammar such that there is a G-parse tree starting from A producing $baba$ and the following two trees are G-parse trees:



For each of the following words w, prove that $w \in \mathcal{L}(G)$:

(i) $w = abbabaab$ and

(ii) $w = abbbaa$.

(c) Prove that the language $L := \{a^n b^m a^{\min(n,m)} \ : \ n, m > 1\}$ is not context-free.

[*You may use the context-free pumping lemma without proof.*]

5L Statistical Modelling

(a) What are the three main components of a *generalised linear model* for observations $(Y_1, x_1), \ldots, (Y_n, x_n)$?

(b) Assuming the model holds with the canonical link function, give expressions for the mean and variance of the Y_i as functions of the covariates x_i.

(c) Define the *Poisson generalised linear model* with the canonical link function.

6A Mathematical Biology

Consider a model for population growth in which the population $n(t)$ evolves according to

$$\frac{dn}{dt} = \alpha n - \beta n^3,$$

where $\alpha, \beta > 0$.

(a) Find and analyse the stability of the non-negative fixed points.

(b) Sketch the solution starting from a range of (non-negative) population sizes.

(c) Suppose a population either follows the model above, or it follows logistic growth. Explain briefly how these possibilities may be distinguished by observing population dynamics starting from a very small population. Illustrate your answer with a sketch showing the difference between the two models.

7D Further Complex Methods

(a) Let $f(x)$, $x \in \mathbb{R}$, be a function with a finite number of singular points $x = c_k$, $k = 1, 2, \ldots, N$, where $-\infty < a < c_1 < c_2 < \cdots < c_N < b < \infty$. Define what is meant by the *Cauchy Principal Value integral* $\mathcal{P}\int_{-\infty}^{\infty} f(x)\mathrm{d}x$. [You may assume that the improper integrals $\int_{-\infty}^{a} f(x)\mathrm{d}x$ and $\int_{b}^{\infty} f(x)\mathrm{d}x$ exist.]

(b) What is the *Hilbert transform* $\mathcal{H}(f)(y)$ of a function f?

(c) Let $f(x) = \frac{1}{x^2+1}$. Evaluate $\mathcal{H}(f)(-1)$.

8E Classical Dynamics

A rigid circular hoop of mass m and radius a hangs from a fixed point on its circumference, is constrained to lie in a vertical plane and is free to oscillate within this plane. A bead, also of mass m, can slide without friction around the hoop. [You may assume that the moment of inertia of a circular hoop of mass m and radius a about an axis through its circumference and perpendicular to the plane of the hoop is $I = 2ma^2$.]

(a) Choose a set of generalised coordinates and write down the Lagrangian for the system.

(b) Show that the frequencies for small oscillations around equilibrium are $\omega_1 = \sqrt{c_1 g/a}$ and $\omega_2 = \sqrt{c_2 g/a}$, where c_1 and c_2 are positive real numbers that you should determine.

9D Cosmology

Consider a flat (k=0) FLRW universe dominated by the potential energy of a scalar field ϕ given by

$$V(\phi) = \frac{\lambda}{n}\phi^n\,, \qquad \text{where} \qquad \lambda > 0\,,$$

and n is a positive integer. The evolution equations for the scale factor $a(t)$ and the field $\phi(t)$ in the slow-roll approximation are respectively

$$H^2 = \frac{8\pi G}{3c^2}V(\phi)\,,$$
$$3H\dot{\phi} = -c^2\frac{\mathrm{d}V}{\mathrm{d}\phi}\,,$$

where $H = \dot{a}/a$ and a dot denotes differentiation with respect to time t.

(a) By considering the chain rule $\dot{a} = (\mathrm{d}a/\mathrm{d}\phi)\dot{\phi}$, or otherwise, solve the slow-roll equations to find the scale factor as a function of $\phi(t)$,

$$a\left(\phi(t)\right) = \exp\left[\frac{4\pi G}{c^4 n}\left(\phi_i^2 - \phi(t)^2\right)\right]\,,$$

where t_i is the initial time with $a(t_i)$=1 and ϕ_i=$\phi(t_i)$, with ϕ_i assumed to be large enough to ensure inflationary expansion.

(b) By determining the Hubble parameter H, show that during inflation we have

$$\frac{1}{2c^2}\dot{\phi}^2 \;\approx\; \frac{c^4}{48\pi G}\;\frac{n^2 V(\phi)}{\phi^2}\,.$$

Deduce the approximate value of $\phi = \phi_{\mathrm{end}}$ when inflation ends, that is, when the slow-roll approximation breaks down. If $n = 6$, roughly estimate the initial value ϕ_i relative to ϕ_{end} that would be required to solve the flatness problem of the standard cosmology.

10E Quantum Information and Computation

Let U_f denote a quantum oracle which acts on $(n+1)$ qubits as follows: $\forall\, x \in \{0,1\}^n$ and $y \in \{0,1\}$,

$$U_f \left|x\right\rangle \left|y\right\rangle = \left|x\right\rangle \left|y \oplus f(x)\right\rangle ,$$

where the addition is taken modulo 2 and

$$f(x) = a.x \oplus b := \sum_{i=1}^{n} a_i x_i \oplus b \quad \text{for some } a \in \{0,1\}^n \text{ and } b \in \{0,1\}.$$

(a) (i) Write down an expression for the state $\left|\Phi_1\right\rangle$ obtained when the state $\left|0\right\rangle^{\otimes n} \left|1\right\rangle$ of $(n+1)$ qubits is acted on by $(U_f H^{\otimes(n+1)})$, where H denotes the Hadamard gate.

(ii) Let $\left|\Phi_2\right\rangle := H^{\otimes(n+1)} \left|\Phi_1\right\rangle$. Write an expression for this state.

(iii) Let $\left|\Phi_3\right\rangle := U_f \left|\Phi_2\right\rangle$. Write an expression for this state.

(b) (i) $\left|\Phi_3\right\rangle$ is a state of $n+1$ qubits. What is the probability of obtaining the n-bit string a by doing a measurement of the first n qubits in the computational basis?

(ii) What is the state of the last (i.e., the $(n+1)^{\text{th}}$) qubit after the above measurement?

(iii) Find the probability that a measurement on this qubit yields the value of b when a contains an odd number of 1s and when a contains an even number of 1s.

SECTION II

11K Coding and Cryptography

(a) Consider the use of a binary $[n, m]$-code to send one of m messages through a binary symmetric channel (BSC) with error probability p, making n uses of the channel. Define the following decoding rules: (1) *ideal observer*, (2) *maximum likelihood*, and (3) *minimum distance*. Show that if all the messages are equally likely then (1) and (2) agree. If $p < \frac{1}{2}$ show that (2) and (3) agree.

(b) Show that a BSC with error probability $p < \frac{1}{4}$ has non-zero operational capacity.

(c) State Shannon's second coding theorem. Consider a discrete memoryless channel with input X taking values over the alphabet $\{0, 1\}$. For $a, b \in \mathbb{Z}$, let Z be a random variable that is independent of X, taking values over the alphabet $\{a, b\}$ with distribution $\mathbb{P}(Z = a) = \mathbb{P}(Z = b) = \frac{1}{2}$. The output of the channel is $Y = X + Z$. What is the capacity of this discrete memoryless channel? [*Hint: The capacity depends on the value of* $b - a$.]

12J Automata and Formal Languages

(a) Let $M = (\Sigma, Q, P)$ be a register machine and $\vec{w}$ a finite sequence of words. Define the *upper register index of* M and $\{C(t, M, \vec{w}) : t \in \mathbb{N}\}$, the *computation sequence of* M *upon input* $\vec{w}$. [When defining the computation sequence, you may assume that "M transforms C into C'" is already defined.]

(b) Let $M = (\Sigma, Q, P)$ be a register machine such that $f_{M,1} = \chi_L$ for some language $L \subseteq \mathbb{W}$. Show that for every $n \in \mathbb{N}$ there is a word w such that the computation sequence of $f_{M,1}(w)$ uses at least n remove instructions of the form $-(0, q, q')$ for some $q, q' \in Q$.

A register machine with upper register index n is called an *n-register machine* (i.e. a register machine using $n + 1$ registers). A language $L \subseteq \mathbb{W}$ is called *n-computable* if its characteristic function is computable by an n-register machine. In the following, let us assume that $\Sigma = \{a, b\}$.

(c) Show that $L = \{a^n b^n : n > 0\}$ is 1-computable. [You may use constructions from the course, as long as you state them precisely and justify the scratch space that they use.]

(d) Let $M = (\Sigma, Q, P)$ be a 0-register machine such that $f_{M,1} = \chi_L$ for some language L. Show that there are natural numbers t, t', k, and ℓ and $q \in Q$ such that $k \neq \ell$ and for all $x \in \mathbb{W}$, we have

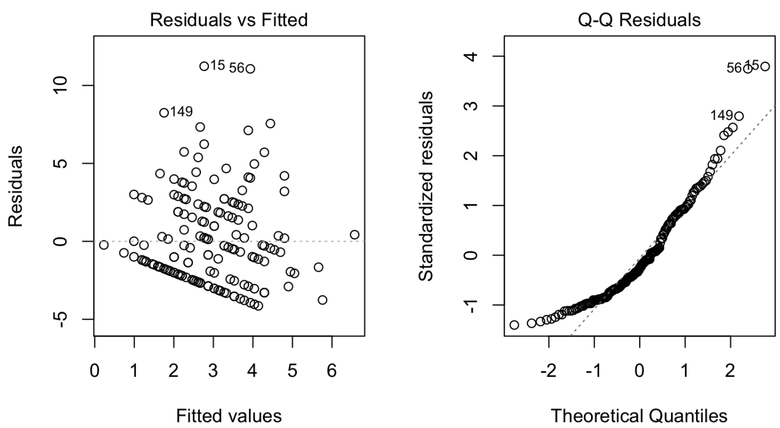
$$C(t, M, xb^k) = (q, x) = C(t', M, xb^\ell).$$

(e) Using part (d) or otherwise, show that $L = \{a^n b^n : n > 0\}$ is not 0-computable.

13L Statistical Modelling

During spawning season, female horseshoe crabs lay clusters of eggs which are fertilised externally by a number of nearby male crabs. We are given a dataset with information on $n = 173$ female crabs. It contains, among other variables, the weight and width of each crab, as well as the number `y` of male crabs in the vicinity. Consider the following (shortened) `R` output from an analysis of this dataset.

```
> head(Crabs[c("y", "weight", "width")])

    y weight width
  1 8   3.05  28.3
  2 0   1.55  22.5

> crabs.lm <- lm(y ~ weight + width, data=Crabs)
> summary(crabs.lm)

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
  (Intercept)  -4.5721     4.2132  -1.085   0.2794
  weight        1.6817     0.9015   1.865   0.0639
  width         0.1446     0.2218   0.652   0.5153

> anova(lm(y ~ 1, data=Crabs), crabs.lm)

Model 1: y ~ 1
Model 2: y ~ weight + width
  Res.Df    RSS Df Sum of Sq      F    Pr(>F)
1    172 1704.9
2    170 1477.7  2     227.2 13.069 5.252e-06 ***

> cor(Crabs$weight, Crabs$width)
[1] 0.8769373
> plot(crabs.lm, add.smooth=FALSE, which=c(1,2))
```



[QUESTION CONTINUES ON THE NEXT PAGE]

(a) Write down the statistical model fitted by `crabs.lm`.

(b) State mathematical formulas for what the columns displayed in the output of `crabs.lm` describe.

(c) State the null and alternative hypotheses for the hypothesis test performed using the `anova` command. What can we conclude from the output?

Explain this in relation to the final column of the `summary` command output by referring to the output of the `cor` command.

(d) What would you expect to see in the two diagnostic plots if the model `crabs.lm` were to fit the data well? Which model assumptions appear to be violated according to these plots?

(e) Suggest two modifications that may help to improve the quality of the fit.

14D Further Complex Methods

(a) Consider a linear input–output system

$$\mathcal{L}y(t) = f(t),$$

where $f(t)$ is the input, $y(t)$ is the output and $\mathcal{L}$ is a linear operator. Define what it means for the system to be *causal* and *stable.*

(b) Consider a linear ordinary differential equation for $t > 0$,

$$\alpha y'(t) + y(t) = f(t)\,, \tag{†}$$

with $\alpha \in \mathbb{R}$, $\alpha \neq 0$, and initial condition $y(0) = 0$.

(i) Using a Laplace transform, show that the transfer function of the linear system (†) is $G(s) = \frac{1}{\alpha s+1}$. Determine for which values of α the system is stable.

(ii) A negative feedback loop with $H(s) = k$, $k \in \mathbb{R}$, is introduced into the system for stable values of α such that the closed-loop transfer function is $G_{\mathrm{CL}}(s) = \frac{G(s)}{1+H(s)G(s)}$. By direct inspection of $G_{\mathrm{CL}}(s)$, determine the values of k for which the closed-loop system is stable. [You do not have to consider the limiting cases for k.]

Explain the *Nyquist criterion* to determine the stability of a closed-loop system. By calculating appropriate winding numbers, use the Nyquist stability criterion to determine the values of k for which the closed-loop system is stable. Compare the result with that obtained by direct inspection of $G_{\mathrm{CL}}(s)$. [*Hint: You may use without proof that the number P of poles and the number Z of zeros, counting multiplicities, of a meromorphic function $f(z)$ inside a clockwise simple closed contour γ obey the relation*

$$\frac{1}{2i\pi}\oint_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z = P - Z\,,$$

if $f(z)$ has no zeros or poles on γ.]

15D Cosmology

Consider a uniformly expanding universe with energy density $\rho(t)$ and pressure $P(t)$ which obey the continuity equation

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + P\right)\,, \qquad (\star)$$

where a dot denotes a derivative with respect to time t.

(a) Consider the conserved mass M of matter inside a uniform expanding sphere of radius $r(t) = a(t)\,x_0$, with fixed comoving radius x_0. Suppose that the radius of the sphere satisfies

$$\ddot{r} = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}\,, \qquad \text{where} \qquad \Phi(r) = -\frac{GM}{r} - \frac{1}{6}\Lambda\, r^2 c^2\,,$$

with Λ a constant. By multiplying the acceleration $\ddot{r}$ by the velocity $\dot{r}$ and integrating, show that the scale factor obeys the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\rho - \frac{kc^2}{a^2} + \frac{1}{3}\Lambda c^2\,, \qquad (\dagger)$$

where k is a constant.

(b) Now differentiate the Friedmann equation $(\dagger)$ and substitute the continuity equation $(\star)$ to find the acceleration equation for $\ddot{a}/a$ in terms of ρ, P and Λ. Briefly note two of the shortcomings of this Newtonian analysis.

(c) Consider a flat (k=0) universe with a positive cosmological constant $\Lambda > 0$ that is filled with radiation pressure $P_{\mathrm{R}} = \rho_{\mathrm{R}}/3$, measured to have energy density $\rho_{\mathrm{R}}(t_0) = \rho_{\mathrm{R}0}$ at given time $t = t_0$. Use the Friedmann equation $(\dagger)$ to show that the Hubble parameter $H = \dot{a}/a$ can be expressed as

$$H^2 = H_0^2\,\Omega_{\mathrm{R}0}\,a^{-4} + \frac{1}{3}\Lambda c^2\,, \qquad \text{where} \qquad \Omega_{\mathrm{R}0} \equiv \frac{8\pi G\,\rho_{\mathrm{R}0}}{3c^2 H_0^2}\,,$$

with $H(t_0) = H_0$ and $a(t_0) = 1$. By considering the substitution $b = a^2$ (or otherwise) find the solution for the scale factor

$$a(t) = \alpha\,[\,\sinh(\beta\,t)\,]^{1/2}\,,$$

where α and β are constants you should determine in terms of H_0 and Ω_{R0}. [You may assume that the universe started with a big bang.]

Show that the scale factor $a(t)$ gives anticipated results at early and late times. Estimate the transition time t_Λ that separates the decelerating and accelerating epochs.

[*Hint:* $\int \mathrm{d}x/\sqrt{1+\kappa^2 x^2}) = (1/\kappa)\sinh^{-1}(\kappa x) + \text{const}$, *where* $\kappa > 0$ *is a constant.*]

16I Logic and Set Theory

State the Soundness Theorem for Propositional Logic.

Let S be a consistent set of propositions. Explain briefly why the function $v\colon L \to \{0,1\}$ on the set L of all propositions defined by

$$v(t) = \begin{cases} 1 & \text{if } S \vdash t \\ 0 & \text{if } S \nvdash t \end{cases}$$

need not be a model of S. Show that S is contained in a consistent, deductively closed set $T \subset L$ such that the definition of v above with S replaced by T *is* a model of S. [You need not prove that v is a model. The set of primitive propositions here is arbitrary. You may assume Zorn's lemma.]

Show that if every finite subset of an arbitrary set S of propositions has a model, then S has a model.

Let X, Y be infinite sets such that there is an injection from X to Y. For each $x \in X$, let A_x be a non-empty, finite subset of Y. Let P be a set consisting of pairwise distinct primitive propositions $p_{x,y}$ for all $x \in X$ and $y \in Y$. For a valuation v on L, set

$$f_v = \{(x,y) \in X \times Y : v(p_{x,y}) = 1\} .$$

For each of the following statements, either write down a set $S \subset L$ that makes the statement true or prove that no such set S exists.

(i) $\{f_v : v \text{ is a model of } S\}$ is the set of all injective functions from subsets of X to Y.

(ii) $\{f_v : v \text{ is a model of } S\}$ is the set of all injective functions $g\colon X \to Y$ such that $g(x) \in A_x$ for all $x \in X$.

(iii) $\{f_v : v \text{ is a model of } S\}$ is the set of all injective functions from X to Y.

17I Graph Theory

(a) Let G be a graph on $n \geqslant 3$ vertices with minimum degree at least $\frac{n}{2}$. Prove that G is Hamiltonian.

(b) Now let G be a bipartite graph of minimum degree at least $k \geqslant 2$. Prove that G contains either a path of length $2k$ or a cycle of length $2k$. Give an example to show that G need not contain a path of length $2k$. Show also that G must contain either a path of length $4k-3$ or a 4-cycle.

18H Galois Theory

(a) Let $\alpha, \beta \in \mathbb{C}$ be algebraic over $\mathbb{Q}$. Show that if α and β have the same minimal polynomial over $\mathbb{Q}$ then $\mathbb{Q}(\alpha) \cong \mathbb{Q}(\beta)$. Let $k \geqslant 1$ be the number of such isomorphisms. Give an example where $1 < k < [\mathbb{Q}(\alpha) : \mathbb{Q}]$. Must k always divide $[\mathbb{Q}(\alpha) : \mathbb{Q}]$? Justify your answer.

(b) Let L/K be a finite extension of degree coprime to n. Clearly stating any properties of the norm that you use, show that if $\alpha \in K$ is an nth power in L then it is an nth power in K.

(c) Let $K = \mathbb{Q}(\alpha)$ where α has minimal polynomial f over $\mathbb{Q}$. Let p be a prime. Show that $f(X^p)$ is irreducible in $\mathbb{Q}[X]$ if and only if α is not a pth power in K.

19H Representation Theory

What is a *complex representation* (ρ, V) of a group G? What does it mean to say that a representation (ρ, V) of G is *faithful*? Show that every finite group has a faithful representation over the complex numbers.

Let G be a finite group, (ρ, V) be a complex representation of G, and $g \in G$. Writing $S(g)$ for the set of eigenvalues of $\rho(g)$, show that if g is conjugate to g^k in G then

$$\lambda \in S(g) \implies \lambda^k \in S(g).$$

Deduce that if p is a prime number and (ρ, V) is a faithful complex representation of S_p, then $\dim V \geqslant p - 1$.

Consider the group G of all invertible functions $\sigma \colon \mathbb{N} \to \mathbb{N}$, under composition. Does G have a faithful complex representation?

20F Number Fields

Define *algebraic integer.*

Prove that the set of algebraic integers is closed under multiplication. [You may use without proof any characterisation of algebraic integers, provided it is properly stated.]

What is the ring of integers in the number field $\mathbb{Q}(\sqrt{2})$? Prove your claim.

For a polynomial $f \in \mathbb{C}[x]$, we define

$$M(f) = |a_d| \prod_{j=1}^{d} \max\{1, |\alpha_j|\},$$

where d is the degree, a_d is the leading coefficient and $\alpha_1, \ldots, \alpha_d$ are the roots of f in $\mathbb{C}$.

Suppose $f \in \mathbb{Z}[x]$ is irreducible, $M(f) = 2$, and f has a real root $\alpha_1 > 1$.

Prove that α_1 is an algebraic integer and $|N_{\mathbb{Q}(\alpha_1)/\mathbb{Q}}(\alpha_1)| = 2$. [*Hint: Consider the number* $|N_{\mathbb{Q}(\alpha_1)/\mathbb{Q}}(\alpha_1)|/2$*, and show that it is a rational integer.*]

21J Algebraic Topology

What does it mean to say that $p : \widetilde{X} \to X$ is a *covering map*? If $\gamma : [0,1] \to X$ is a path and $\tilde{x}_0 \in \widetilde{X}$ is such that $p(\tilde{x}_0) = \gamma(0)$, prove carefully that there is a unique path $\tilde{\gamma} : [0,1] \to \widetilde{X}$ such that

(i) $\tilde{\gamma}(0) = \tilde{x}_0$, and

(ii) $p \circ \tilde{\gamma} = \gamma$.

[You may use the Lesbegue number lemma.]

Let Y be a topological space, $A \subseteq Y$ and $B \subseteq Y$ be open subspaces with disjoint closures, and $\phi : A \to B$ be a homeomorphism. Let Y/ϕ denote the quotient of Y by the equivalence relation generated by $a \sim \phi(a)$ for all $a \in A$. Show that the function

$$\begin{aligned} p : \widehat{Y/\phi} := \frac{Y \times \mathbb{Z}}{(a,i) \sim (\phi(a), i-1) \text{ for } a \in A, i \in \mathbb{Z}} &\longrightarrow Y/\phi \\ [(y,i)] &\longmapsto [y] \end{aligned}$$

is continuous and is a covering map.

Assume now that Y is path-connected. Let $a_0 \in A$ be a basepoint, which determines a basepoint $[a_0] \in Y/\phi$. Show that $\widehat{Y/\phi}$ is path-connected, that the subgroup $G \leqslant \pi_1(Y/\phi, [a_0])$ associated to the covering space $p : \widehat{Y/\phi} \to Y/\phi$ is normal, and that the quotient group $\pi_1(Y/\phi, [a_0])/G$ is isomorphic to $\mathbb{Z}$.

22G Linear Analysis

State and prove the Closest Point Theorem. Deduce that if F is a closed subspace of a Hilbert space H then H is the direct sum of F and $F^{\perp}$.

Let H be a separable Hilbert space. An operator T on H is called a *shift* if there exists an orthonormal (Hilbert) basis $(e_n)_{n=1}^{\infty}$ of H such that $T(e_n) = e_{n+1}$ for all n. Show that T is a shift if and only if T is an isometry with $\cap_{n=1}^{\infty} \mathrm{Im}\,(T^n) = \{0\}$ and $\dim\, (\mathrm{Im}\, T)^{\perp} = 1$.

23G Analysis of Functions

(a) Set $\mathbb{R}^+ = (0, \infty)$, and let $L^p = L^p(\mathbb{R}^+, dx)$, $1 < p < \infty$, where dx is Lebesgue measure on $\mathbb{R}^+$. Let $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be integrable for the product measure $dx \otimes dx$ on $\mathbb{R}^+ \times \mathbb{R}^+$. Set

$$G(y) = \int_{\mathbb{R}^+} F(x, y)dx, \; y \in \mathbb{R}^+.$$

Show that if $\|g\|_{L^q} \leqslant 1$ for $1 < q < \infty$ conjugate to p, then

$$\int_{\mathbb{R}^+} |G(y)g(y)| \, dy \leqslant \int_{\mathbb{R}^+} \left[\int_{\mathbb{R}^+} |F(x, y)|^p dy \right]^{1/p} dx.$$

(b) Now let $K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be integrable and such that

$$K(\lambda x, \lambda y) = \lambda^{-1} K(x, y), \; \lambda, x, y > 0; \quad \text{and} \quad \int_0^\infty |K(x, 1)| x^{-1/p} \, dx = 1.$$

Define $Tf(y) = \int_0^\infty K(x, y) f(x) \, dx$. Show that for $f \in L^p$ we have

$$\|Tf\|_{L^p} \leqslant \|f\|_{L^p}.$$

[*Hint: Consider* $f_z(y) = f(yz)$ *and show first that* $\|f_z\|_{L^p} = z^{-1/p} \|f\|_{L^p}$.]

[*You may use the identity*

$$\|f\|_{L^p} = \sup \left\{ \int |f(x)g(x)| dx : g \in L^q, \|g\|_{L^q} \leqslant 1 \right\}, \; q \text{ conjugate to } p,$$

without proof.]

24H Riemann Surfaces

Given two suitable topological spaces Y and X, define a *covering map* $\pi : Y \to X$. What does it mean to say that X is *simply connected*? Write down the simply connected Riemann surfaces (up to analytic isomorphism).

What is a *lattice* of $\mathbb{C}$? Prove that for any lattice L there exists a non-constant analytic function $f : \mathbb{C} \to \mathbb{C}_\infty$ where $f(z + l) = f(z)$ for all $z \in \mathbb{C}$ and $l \in L$.

Assuming now that the quotient space $\mathbb{C}/L$ is a Riemann surface where the natural projection $q : \mathbb{C} \to \mathbb{C}/L$ is analytic, deduce the existence of a unique analytic function $\overline{f} : \mathbb{C}/L \to \mathbb{C}_\infty$ such that $f = \overline{f} q$ for your function f above.

Show that neither f nor $\overline{f}$ are covering maps. Does there exist a covering map from $\mathbb{C}/L$ to $\mathbb{C}_\infty$? Justify your answer, stating clearly any results which you use.

25F Algebraic Geometry

In this question, all algebraic varieties are over an algebraically closed field k.

What does it mean for a topological space to be *irreducible*? Show that if $X \subseteq \mathbb{A}^n$ is a Zariski closed subset, then X can be written as a finite union of irreducible closed subsets of $\mathbb{A}^n$.

Write the closed subset $Z(x_3 - x_2^2, x_1^2 - x_2^2 - x_2^4 + x_3^2)$ of $\mathbb{A}^3$ as a union of irreducible closed sets.

Now take $k = \mathbb{C}$ to be the field of complex numbers. Show that the set

$$Z := \{(x, e^x) \mid x \in \mathbb{C}\} \subseteq \mathbb{A}^2$$

is dense in $\mathbb{A}^2$ in the Zariski topology.

Let X be an affine algebraic variety, and let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open cover of X. Show that $\mathcal{U}$ has a finite subcover. [*Hint: Define for any regular function f on X the distinguished open set*

$$D(f) := \{x \in X \mid f(x) \neq 0\}.$$

You may use without proof the fact that the collection of distinguished open sets form a basis for the topology on X.]

26J Differential Geometry

Let $\Sigma \subset \mathbb{R}^3$ be a smooth surface.

(a) For a point $p \in \Sigma$, define the *first fundamental form* I_p^Σ and the *second fundamental form* II_p^Σ. Give the definitions of the *shape operator*, the *mean curvature* and the *Gaussian curvature* at p. What does it mean for Σ to be *minimal*?

Now let Ω be a non-empty open subset of $\mathbb{R}^2$ and let $h : \Omega \to \mathbb{R}$ be smooth with non-degenerate differential. Let

$$\begin{aligned} \phi : \Omega &\to \mathbb{R}^3 \\ (x,y) &\mapsto (x,y,h(x,y)) \end{aligned}$$

and let $S = \phi(\Omega)$.

(b) Calculate $I^S_{\phi(x,y)}$ at an arbitrary point $(x,y) \in \Omega$ in terms of h.

(c) Write down a Gauss map $N : \Omega \to \mathbb{S}^2$ for S. Let

$$\begin{aligned} \phi_t : \Omega &\to \mathbb{R}^3 \\ (x,y) &\mapsto \phi(x,y) + tN(x,y) \end{aligned}$$

and assume that there is $\epsilon > 0$ so that $S_t := \phi_t(\Omega)$ is a smooth surface for any $t \in (-\epsilon, \epsilon)$. Prove that the second fundamental form of S satisfies

$$II^S_{\phi(x,y)} = -\frac{1}{2}\frac{d}{dt}\Big|_{t=0} I^{S_t}_{\phi_t(x,y)}$$

at any point $(x,y) \in \Omega$. Calculate $II^S_{\phi(x,y)}$ in terms of h.

(d) Derive a differential equation in h that characterises when the surface S is minimal.

(e) Calculate the area of S in terms of the height function h. Assume that for any $\eta : \Omega \to \mathbb{R}$ smooth and compactly supported in Ω, the area $A_\eta(t)$ of

$$S_t^\eta := \{(x,y,h(x,y) + t\eta(x,y)) \mid (x,y) \in \Omega\}$$

is locally minimal at $t = 0$. Use the Euler–Lagrange equation to recover the differential equation in h from part (d).

27G Probability and Measure

(a) (i) What does it mean for a measure μ on a measurable space $(\Omega, \mathcal{F})$ to be *σ-finite*? State the uniqueness of extension theorem for a σ-finite measure.

(ii) Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}$. Show that Lebesgue measure λ is translation invariant, i.e., for $x \in \mathbb{R}$ and $B \in \mathcal{B}$,

$$\lambda(B) = \lambda(B + x)$$

where $B + x = \{b + x : b \in \mathcal{B}\}$.

(iii) Show that Lebesgue measure λ is the unique translation invariant σ-finite measure on $\mathcal{B}$ such that $\lambda((0,1]) = 1$.

(b) Let X, $(X_n)_{n \in \mathbb{N}}$ be real-valued random variables with distribution functions F_X, $(F_{X_n})_{n \in \mathbb{N}}$ respectively.

(i) State what it means to say that $X_n \to X$ *in distribution* in terms of their distribution functions.

(ii) Now assume that $X_n \to X$ in distribution. Let $B_{(0,1)}$ and $\lambda|_{(0,1)}$ be the Borel σ-algebra and the Lebesgue measure on $(0,1)$ respectively. On $((0,1), \mathcal{B}_{(0,1)}, \lambda|_{(0,1)})$, define for all $\omega \in (0,1)$,

$$\tilde{X}_n(\omega) = \inf\{x \in \mathbb{R} : \omega \leqslant F_{X_n}(x)\}\,, \quad \tilde{X}(\omega) = \inf\{x \in \mathbb{R} : \omega \leqslant F_X(x)\}\,.$$

Show that $\tilde{X}$ has the same distribution as X and $\tilde{X}_n$ has the same distribution as X_n for all n, and $\tilde{X}_n \to \tilde{X}$ almost surely.

[You may use the fact that for a non-constant, right-continuous, non-decreasing function g, $f(\omega) := \inf\{x \in \mathbb{R} : \omega \leqslant g(x)\}$ is left-continuous non-decreasing and $f(\omega) \leqslant x$ if and only if $\omega \leqslant g(x)$. You may also use the fact that a non-decreasing function has at most a countable set of points of discontinuity.]

28K Applied Probability

(a) Let $(X_t)_{t\geqslant 0} \sim \text{Markov}(Q)$ be a continuous time Markov chain with generator matrix Q on a countable state space I, jump times J_n and jump chain $(Y_n)_{n\geqslant 0}$. Show that for $n \geqslant 1$ and $i_1, i_2, \ldots, i_n \in I$,

$$q_{i_n}\mathbb{P}(J_n \leqslant t < J_{n+1} | Y_0 = i_0, \ldots, Y_n = i_n) = q_{i_0}\mathbb{P}(J_n \leqslant t < J_{n+1} | Y_0 = i_n, \ldots, Y_n = i_0)\,.$$

(b) Now let $(X_t)_{t\geqslant 0}$ be irreducible. Fix any $h > 0$ and let $Z_n = X_{nh}$ for $n = 0, 1, 2, \ldots$. Show that $(Z_n)_{n\geqslant 0}$ is a discrete-time Markov chain and give its transition matrix. Show that $(X_t)_{t\geqslant 0}$ is recurrent if and only if $(Z_n)_{n\geqslant 0}$ is recurrent.

(c) Finally, let I be finite and $f : I \to \mathbb{R}$ be a function, identified with the vector $(f(x))_{x\in I}$. Show that

$$Qf(x) = \lim_{t\to 0+} \frac{\mathbb{E}_x(f(X_t)) - f(x)}{t}\,,$$

and

$$\mathbb{E}_x(f(X_t)) = f(x) + \int_0^t \mathbb{E}_x(Qf(X_s))\,ds.$$

29L Principles of Statistics

(a) Let $\hat{\theta}$ denote the maximum likelihood estimator based on i.i.d. observations $X_1, \ldots, X_n$ from a parametric statistical model $\{f(\cdot, \theta) : \theta \in \Theta\}$, where $\theta_0 \in \Theta$ is the true parameter. Write down the limiting distribution of $\sqrt{n}(\hat{\theta} - \theta_0)$ under standard regularity conditions. [You should define any quantities involved in the expression of the limiting distribution.]

Now suppose $X_1, \ldots, X_n$ are i.i.d. Uniform$[-\theta, \theta]$ random variables, for a parameter $\theta > 0$.

(b) Derive an expression for the maximum likelihood estimator $\hat{\theta}$ of θ.

(c) Let θ_0 denote the true parameter. By calculating the cumulative distribution function of $\hat{\theta}$ or otherwise, show that $n(\theta_0 - \hat{\theta}) \xrightarrow{d} Z$ for a random variable Z whose distribution you should specify.

(d) Find, with brief justification, the limiting distribution of $n(\hat{\theta}^2 - \theta_0^2)$.

(e) Do we have $\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} W$ for a random variable W with positive variance? Justify your answer.

30L Stochastic Financial Models

(a) In the context of a one-period market model, define the terms *arbitrage* and *risk-neutral measure*. State the one-period fundamental theorem of asset pricing.

(b) Given an $n \times d$ matrix P, let

$$\mathcal{A} = \{\varphi \in \mathbb{R}^d : (P\varphi)_i \geqslant 0 \text{ for all } i \text{ and } (P\varphi)_i > 0 \text{ for some } i\}$$

and

$$\mathcal{Q} = \left\{ q \in \mathbb{R}^n : P^\top q = 0, \ \sum_{i=1}^n q_i = 1, \quad q_i > 0 \text{ for all } i \right\}.$$

Prove that $\mathcal{Q} = \emptyset$ if and only if $\mathcal{A} \neq \emptyset$.

Consider a one-period market model with d risky assets, where S_t^i is the price of asset i at time $t \in \{0, 1\}$ and r is the interest rate. Assume that there exists at least one risk-neutral measure for the model, and that the random variables $\{S_1^i - (1+r)S_0^i : 1 \leqslant i \leqslant d\}$ are linearly independent.

(c) Let Y be a random variable such that $\mathbb{E}^{\mathbb{Q}}(Y) > 0$ for all risk-neutral measures $\mathbb{Q}$. Show that there exists a vector $\theta \in \mathbb{R}^d$ such that

$$Y \geqslant \theta^\top [S_1 - (1+r)S_0] \text{ almost surely,}$$

and we have strict inequality with positive probability.

(d) Let Z be a random variable such that $\mathbb{E}^{\mathbb{Q}}(Z) \geqslant 0$ for all risk-neutral measures $\mathbb{Q}$. Show that there exists a vector $\phi \in \mathbb{R}^d$ such that

$$Z \geqslant \phi^\top [S_1 - (1+r)S_0] \text{ almost surely.}$$

31K Mathematics of Machine Learning

(a) (i) Let $\mathcal{Z}$ be a non-empty set. Given $z_1, \ldots, z_n \in \mathcal{Z}$ and a class $\mathcal{F}$ of functions $f : \mathcal{Z} \to \mathbb{R}$, what is meant by the *empirical Rademacher complexity* $\hat{\mathcal{R}}(\mathcal{F}(z_{1:n}))$? Given i.i.d. random variables $Z_1, \ldots, Z_n$ taking values in $\mathcal{Z}$, what is meant by the *Rademacher complexity* $\mathcal{R}_n(\mathcal{F})$?

(ii) Suppose $\mathcal{H}$ is a class of functions $h : \mathbb{R}^p \to \{0,1\}$ with $|\mathcal{H}| \geqslant 2$. Define the *shattering coefficient* $s(\mathcal{H}, n)$ and the *VC dimension* $\mathrm{VC}(\mathcal{H})$ of $\mathcal{H}$.

(iii) Let $\mathcal{H} = \{\mathbf{1}_A : A \in \mathcal{A}\}$ where $\mathcal{A} := \left\{ \prod_{j=1}^{p} (-\infty, a_j] : a_1, \ldots, a_p \in \mathbb{R} \right\}$. Show that $\mathrm{VC}(\mathcal{H}) \leqslant p$.

(b) A new painkilling drug is tested on n patients. Let $Y_i^{(0)}$ and $Y_i^{(1)}$ be the pain levels, on a scale from 0 (no pain) to $M > 0$ (maximum pain), of the ith patient, before and after taking the painkiller respectively. Suppose the vector $X_i \in \mathbb{R}^p$ records p additional characteristics of the ith patient, such as their age, weight, height, etc. We treat $\left(Y_i^{(0)}, Y_i^{(1)}, X_i\right) \in [0, M]^2 \times \mathbb{R}^p$ for $i = 1, \ldots, n$ as independent copies of a random triple $\left(Y^{(0)}, Y^{(1)}, X\right)$. Let $\mathcal{A}$ and $\mathcal{H}$ be defined as in part (a) (iii) above. We wish to determine a region $A \in \mathcal{A}$ where if $X \in A$, we expect the drug to be effective. To this end, let h^* and $\hat{h}$ minimise

$$Q(h) := \mathbb{E}\left[\left(Y^{(1)} - Y^{(0)}\right) h(X)\right] \qquad \text{and} \qquad \hat{Q}(h) := \frac{1}{n} \sum_{i=1}^{n} \left(Y_i^{(1)} - Y_i^{(0)}\right) h(X_i)$$

respectively, over $h \in \mathcal{H}$.

(i) Using any results from the course that you need, show that

$$\mathbb{E} Q(\hat{h}) \leqslant Q(h^*) + 2\mathcal{R}_n(\mathcal{F})$$

for an appropriate class of functions $\mathcal{F}$ that you should specify.

(ii) Using any results from the course that you need, show that

$$\mathbb{E} Q(\hat{h}) \leqslant Q(h^*) + 2M \sqrt{\frac{2p \log(n+1)}{n}}.$$

32A Dynamical Systems

Let $F : I \to I$ be a continuous map of an interval $I \subset \mathbb{R}$.

(a) (i) Define what it means for F to be *chaotic*, according to Glendinning.

(ii) Define what it means for F to be *chaotic*, according to Devaney.

(b) Suppose now that F has a periodic orbit of period 3.

(i) Show that F also has periodic orbits of period n for all positive integers n.

(ii) Explain briefly why F must have at least four distinct 7-cycles.

(iii) How many distinct 8-cycles must F have?

[*Relevant theorems that you use from the course should be stated clearly.*]

33C Integrable Systems

Consider the initial boundary value problem for a function $u = u(x,t)$,

$$u_t = iu_{xx}\,, \qquad 0 < x < \infty\,, \quad t > 0,$$

$$u(x,0) = u_0(x), \qquad u(0,t) = h(t)\,.$$

Show that the equation $u_t = iu_{xx}$ has a formulation as the consistency condition for the following pair of equations for $\psi = \psi(x,t) \in \mathbb{C}$,

$$\begin{aligned}&\psi_t + ik^2\psi = iu_x - ku,\\ &\psi_x - ik\psi = u.\end{aligned}$$

By means of the integrating factor $e^{-ikx+ik^2t}$, or otherwise, deduce that

$$\hat{u}(k,t)e^{ik^2t} - \hat{u}_0(k) = \int_0^t e^{ik^2\tau}\left[ku(0,\tau) - iu_x(0,\tau)\right]d\tau\,,$$

where

$$\hat{u}(k,t) = \int_0^\infty e^{-ikx}u(x,t)dx, \qquad \text{and} \qquad \hat{u}_0(k) = \int_0^\infty e^{-ikx}u_0(x)dx\,.$$

Hence, by considering also $\hat{u}(-k,t)$, find a function $G = G(k,\tau)$ such that

$$\begin{aligned}u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} &e^{-ik^2t+ikx}\left[\hat{u}_0(k) - \hat{u}_0(-k)\right]dk\\ &+ \frac{1}{\pi}\int_{-\infty}^{+\infty}\int_0^t e^{-ik^2(t-\tau)+ikx}G(k,\tau)d\tau dk\,.\end{aligned}$$

[You may assume that u is smooth and rapidly decreasing so that $\hat{u}(k,t)$ is holomorphic for $\mathrm{Im}\{k\} < 0$, and satisfies $\lim_{|k|\to\infty}\hat{u}(k,t) = 0$.]

34B Principles of Quantum Mechanics

(a) A two-dimensional Hilbert space is spanned by two normalised vectors $|\phi\rangle$ and $|\psi\rangle$, which are not necessarily orthogonal. Consider the linear operator H defined by

$$H\,|\psi\rangle = g\,|\phi\rangle\ , \qquad\qquad H\,|\phi\rangle = g^*\,|\psi\rangle\ ,$$

where g is a complex constant. Determine the condition on $\langle\psi|\phi\rangle$ under which H is Hermitian. Henceforth assume this condition is satisfied and find the eigenvectors $|\pm\rangle$ of H and the corresponding eigenvalues. Verify that the distinct eigenvectors are orthogonal.

(b) Now assume $g = 1$. The Hamiltonian of a two-dimensional system is given by $H' = H + \Delta(t)$ where H is as in part (a) and

$$\Delta(t)\,|\phi\rangle = \Delta(t)\,|\psi\rangle = V(t)\,(|\phi\rangle + |\psi\rangle)\ ,$$

with $V(t)$ a real and time-dependent function. Working in the $|\pm\rangle$ basis, or otherwise, determine the exact probability to find the system in state $|\phi\rangle$ at time $t > 0$ if it was in state $|\psi\rangle$ at $t = 0$.

35E Applications of Quantum Mechanics

Consider the quantum mechanical scattering of a particle of mass m in three-dimensions, with Hamiltonian

$$H = \frac{|\boldsymbol{p}|^2}{2m} + V(r) \ .$$

Here the potential $V(r)$ is spherically symmetric, and it is localised in some region of space. Using a partial wave decomposition,

$$\psi(\boldsymbol{r}) = \sum_{l=0}^{\infty} \frac{u_l(r)}{r} P_l(\cos\theta) \ ,$$

where $P_l(\cos\theta)$ are Legendre polynomials, and boundary condition $u_l(0) = 0$, the time-independent Schrödinger equation for the wavefunction $\psi(\boldsymbol{r})$ of the particle reduces to

$$\left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + \frac{2m}{\hbar^2}V(r)\right) u_l(r) = k^2 u_l(r) \ , \quad \text{with} \quad E = \frac{\hbar^2 k^2}{2m} \ .$$

(a) The asymptotic behaviour, for large r, of the wavefunction can be written

$$\psi(\boldsymbol{r}) \sim \sum_{l=0}^{\infty} \frac{2l+1}{2ik} \left((-1)^{l+1} \frac{e^{-ikr}}{r} + S_l(k) \frac{e^{ikr}}{r} \right) P_l(\cos\theta) \ .$$

For real values of k, show that the coefficients $S_l(k)$ satisfy

$$S_l(k)^* \, S_l(k) = 1 \, , \qquad S_l(k) \, S_l(-k) = 1 \, .$$

Hence deduce that $S_l(k) = e^{2i\delta_l(k)}$ for some real function $\delta_l(k)$, and that $\delta_l(k) = -\delta_l(-k)$.

(b) Focus now on the low-momentum behaviour, where the $l = 0$ mode dominates. For some choice of potential $V(r)$,

$$S_0(k) = \frac{(k+3i\lambda)(k+2i\lambda)}{(k-3i\lambda)(k-2i\lambda)} \ ,$$

where λ is a real positive constant. Evaluate the scattering length a_s, and give an estimate for the total cross-section σ_T at low energies.

Briefly explain the significance of the poles of $S_0(k)$. Are there any resonances in $S_0(k)$? Why is it important that λ is positive?

36B Statistical Physics

(a) What systems are described by a *canonical ensemble*? If the energy of the ith microstate is E_i, with $i = 0, 1, 2, \ldots$, write down an expression for the partition function Z in terms of temperature T and the Boltzmann constant k_B.

(b) Calculate the partition function Z_1 for a single classical ultra-relativistic spinless particle moving in three-dimensional space in a potential $U(\mathbf{x})$. [Ultra-relativistic means that the energy-momentum relation is $E = pc$, where c is the speed of light.]

(c) A system of a large number N of identical, non-interacting particles of the type described in part (b) is in equilibrium at temperature T in a potential

$$U(\mathbf{x}) = \frac{(x^2 + y^2 + z^2)^n}{V^{2n/3}},$$

where n is a positive integer and $V > 0$ is an external parameter analogous to volume.

(i) Calculate the partition function and hence show that the Helmholtz free energy is

$$F = -Nk_BT\Big[\ln V + A\ln(k_BT) + \ln I_n + B\Big],$$

where

$$I_n = \int_0^\infty u^2\, e^{-u^{2n}}\, du,$$

and you should determine A and B.

(ii) Considering the conjugate pressure to V, $p = -\left(\frac{\partial F}{\partial V}\right)_{T,N}$, derive the equation of state.

(iii) Compute the average energy E, the variance of energy $(\Delta E)^2$ and the heat capacity C_V for the system. Comment on the behaviour of $(\Delta E)/E$ in the thermodynamic limit.

(iv) Obtain the local particle number density as a function of $\mathbf{x}$ and hence determine the most likely $|\mathbf{x}|$ to find a particle.

37D Electrodynamics

Consider a spacetime with coordinates $x^\mu = (ct, \boldsymbol{x})$ and metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$, where $\mu,\nu = 0,1,2,3$ and c is the speed of light. A 4-vector potential $A_\mu(x)$ fills spacetime and is described by the action

$$S[A_\mu] = -\frac{1}{\mu_0 c}\int \left(\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{m^2}{2}A_\mu A^\mu - \mu_0 A_\mu J^\mu\right) \mathrm{d}^4x\,,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor, $J_\mu(x)$ is a conserved 4-current density, m and μ_0 are constants and $m \geqslant 0$.

(a) Show that the equations of motion for the field,

$$\partial_\mu F^{\mu\nu} - m^2 A^\nu = -\mu_0 J^\nu\,,$$

follow from the principle of stationary action.

(b) Clearly state the conditions for the action to be invariant under Lorentz transformations and under gauge transformations of the form $A_\mu \to A_\mu + \partial_\mu \chi$, where χ is a scalar field.

(c) Show that for $m > 0$ the equations of motion imply the identity $\partial_\mu A^\mu = 0$.

(d) Writing the vector potential as $A^\mu = (\phi/c, \boldsymbol{A})$, show that for $m \geqslant 0$ the equations of motion for ϕ and $\boldsymbol{A}$ can be written as

$$\Box\phi + \frac{\partial\alpha}{\partial t} - m^2\phi = -c\mu_0 J^0\,, \qquad (\dagger)$$

$$\Box\boldsymbol{A} - \boldsymbol{\nabla}\alpha - m^2\boldsymbol{A} = -\mu_0\boldsymbol{J}\,,$$

where $\alpha = \frac{1}{c^2}\frac{\partial\phi}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{A}$ and $\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$ is the wave operator.

(e) For a point charge q at rest, the 4-current density is $J^0 = cq\delta(\boldsymbol{x})$ and $\boldsymbol{J} = 0$, where δ denotes the 3-dimensional δ function. By applying a Fourier transform in the spatial coordinates to the equation of motion ($\dagger$), show that for $m > 0$, a time-independent field solution for a point charge at $r = 0$ is given by

$$\phi = \lambda\frac{\exp(-mr)}{r}, \quad \boldsymbol{A} = 0\,,$$

where $r = |\boldsymbol{x}|$ and λ is a constant you do *not* need to determine. [*Hint: You may use without proof that the inverse Fourier transform of* $1/(|\boldsymbol{k}|^2 + m^2)$ *is*

$$\int \frac{e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}}{|\boldsymbol{k}|^2 + m^2}\frac{\mathrm{d}^3k}{(2\pi)^3} = C\frac{e^{-m|\boldsymbol{x}|}}{|\boldsymbol{x}|}\,,$$

where $\boldsymbol{k}$ *is the wave vector and* C *is a non-zero constant.*]

Provide a brief physical interpretation of this result, including the limiting case where $m \to 0$, and connect this interpretation to the result of part (b).

38B General Relativity

Consider a massive test particle moving in the Schwarzschild metric of a black hole with mass m (in units with $c = G = 1$):

$$ds^2 \;=\; -\Big(1-\frac{2m}{r}\Big)dt^2 \;+\; \Big(1-\frac{2m}{r}\Big)^{-1}dr^2 \;+\; r^2(\,d\theta^2+\sin^2\theta\, d\phi^2\,)\;.$$

(a) Assuming that the motion lies in the equatorial plane $\theta = \pi/2$, justify briefly why

$$h = r^2\dot{\phi} \qquad \text{and} \qquad \frac{1}{2}\,\dot{r}^2 - \frac{m}{r} + \frac{h^2}{2r^2} - \frac{mh^2}{r^3}$$

are constants of the motion, where dots denote derivatives with respect to the proper time of the particle.

(b) For a circular orbit with a fixed value of r, determine h and hence deduce that (i) $r > 3m$ and (ii) $(d\phi/dt)^2 = m/r^3$.

(c) Now consider a nearly circular orbit with shape given by $u(\phi) = 1/r$. Let prime denote differentiation with respect to ϕ, so that $u' = du/d\phi$. Given that $\dot{r} = -hu'$, and assuming if needed that $u' \neq 0$, show that

$$u'' \,+\, u \;=\; \frac{m}{h^2} \,+\, 3mu^2\,.$$

For $m/h \ll 1$, this equation has an approximate solution of the form

$$u \;=\; \frac{m}{h^2}(1+\alpha) \;+\; A\cos[(1+\beta)\phi]\,,$$

where the constant A obeys $|A| \ll 1$ but is otherwise arbitrary. The constants α and β are small for $m/h \ll 1$. Verify this solution by working to first order in A and determining the constants α and β to leading non-trivial order in m/h.

Comment briefly on the significance of your result for β.

39C Fluid Dynamics II

A solid cylinder of density $\rho + \Delta\rho$, length L and radius $a < L$ is placed with its axis vertical at the centre of a long, closed, vertically oriented cylindrical container of radius $a + h$, where $h \ll a$. The container is otherwise filled with an incompressible fluid of density ρ and dynamic viscosity μ, through which the solid cylinder falls with speed U. Ignoring end effects, show that the downwards velocity field in the thin gap between the solid cylinder and the container can be approximated as

$$u = -\frac{\Delta p}{2\mu L} y(h-y) + U\frac{y}{h},$$

where y is the coordinate directed inwards across the thin gap from the wall of the cylindrical container and Δp is the dynamic pressure difference between the fluid below and above the cylinder. Determine the associated volume flux along the gap and the viscous shear stress on the solid cylinder.

Use global mass conservation to show that $\Delta p \approx 6\mu a L U/h^3$. Show that the associated form drag is much larger than the viscous force on the cylinder and hence determine the speed of fall U.

40D Waves

(a) Starting from the linearized mass and momentum conservation equations governing sufficiently small and smooth perturbations of a compressible homentropic inviscid fluid at rest with constant reference density ρ_0, pressure p_0 and sound speed c_0, show that the pressure perturbation $\tilde{p}$ satisfies a wave equation. How is $\tilde{p}$ related to the velocity potential ϕ?

(b) Consider a semi-infinite straight duct of uniform cross-section, aligned along the $x-$axis for $-L \leqslant x < \infty$. There is a piston at the end of the duct which performs oscillations $\epsilon e^{i\omega t}$ about its equilibrium position at $x = -L$. The duct is filled with compressible fluid of density ρ_- and sound speed c_- in the region $-L < x < 0$ and with compressible fluid of density ρ_+ and sound speed c_+ in the region $0 < x < \infty$. The piston's oscillations are sufficiently small so that you may assume $0 < \epsilon \ll L$ and $|\epsilon\omega| \ll \min(c_-, c_+)$.

(i) Show that the complex amplitude of the velocity potential in $x > 0$ is given by

$$\epsilon c_- \frac{i\frac{\rho_+}{\rho_-}\sin\lambda - \frac{c_-}{c_+}\cos\lambda}{\left(\frac{\rho_+}{\rho_-}\right)^2 \sin^2\lambda + \left(\frac{c_-}{c_+}\right)^2 \cos^2\lambda}, \qquad \text{where} \qquad \lambda = \frac{\omega L}{c_-}.$$

(ii) Consider the two sets of frequencies of oscillation such that $\lambda = n\pi$ and $\lambda = (n + \frac{1}{2})\pi$ for integer n. Calculate the time-averaged acoustic energy flux in $x > 0$ for each set, and briefly comment on the behaviour in the case where $\rho_+ \ll \rho_-$ and $c_+ \approx c_-$.

41A Numerical Analysis

(a) The Fourier transform of an infinite sequence $(v_m)_{m\in\mathbb{Z}}$ is defined as

$$\widehat{v}(\theta) = \sum_{m\in\mathbb{Z}} v_m e^{-im\theta}, \qquad -\pi \leqslant \theta \leqslant \pi.$$

(i) Prove Parseval's identity:

$$\sum_{m\in\mathbb{Z}} |v_m|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\widehat{v}(\theta)|^2 \, d\theta.$$

(ii) Consider the following two-step recurrence in n for u_m^n with $n \in \mathbb{Z}_+$ and $m \in \mathbb{Z}$:

$$u_m^{n+1} = \frac{1}{1+\mu}\Big[(1-\mu)u_m^{n-1} + \mu(u_{m+1}^n + u_{m-1}^n)\Big],$$

with $\mu \geqslant 0$. Use Fourier analysis to determine the range of μ for which the method is stable.

(b) The linear system $d\mathbf{y}/dt = A\mathbf{y}$ is discretised by the scheme

$$\mathbf{y}^{n+1} = (I-kB)^{-1}(I-kC)^{-1}\mathbf{y}^n,$$

with $B+C=A$ and $k=\Delta t$, where $\mathbf{y}\in\mathbb{R}^M$ and A, B and C are $M\times M$ square matrices.

(i) Define the exponential of a matrix. Show that for $k \ll 1$,

$$\exp[kB]\exp[kC] = \exp[k(B+C)] + \tfrac{1}{2}k^2(BC-CB) + O(k^3).$$

(ii) Find the order of the local truncation error of the scheme.

(iii) In the special case when the matrices B and C commute, does the order of the scheme change?

END OF PAPER