UNIVERSITY OF CAMBRIDGE

MAT1

MATHEMATICAL TRIPOS Part IB

Friday, 07 June, 2024 1:30pm to 4:30pm

PAPER 4

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most four** questions from Section I and **at most six** questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Linear Algebra

State a theorem classifying $n \times n$ complex matrices up to similarity.

Let α be an endomorphism of an n-dimensional complex vector space. Define the *algebraic multiplicity* a_λ and the *geometric multiplicity* g_λ of an eigenvalue λ of α. Express a_λ and g_λ as well as the minimal polynomial of α in terms of a representation of α using the classification above.

Let α be represented by the 3×3 matrix

$$A = \begin{pmatrix} 5 & 0 & 3 \\ -1 & -1 & -1 \\ -6 & 0 & -4 \end{pmatrix} .$$

Find the eigenvalues of α and their algebraic and geometric multiplicities. Find the minimal polynomial of α.

2F Analysis and Topology

Define what it means for a subset A of a topological space (X, τ) to be *connected*.

Let $f : X \to Y$ be a continuous map between topological spaces (X, τ) and (Y, σ). Show that if X is connected, then $f(X)$ is connected.

Let $Y = \{0, 1\}$ be equipped with the discrete topology. Show that a topological space (X, τ) is connected if and only if every continuous function $h : X \to Y$ is constant.

Given a subset A of a topological space (X, τ), define the *closure* $\mathrm{Cl}(A)$ of A to be the set of all points of A together with the set of points $y \in X$ such that every open set in τ containing y contains some point of A other than y. Using the preceding part or otherwise, show that given a connected set $C \subseteq X$, $\mathrm{Cl}(C)$ is connected.

3F Complex Analysis

Define what it means for $f : U \to \mathbb{C}$ to be *holomorphic* on a domain U.

State Morera's theorem.

Deduce that the function f defined on $\mathbb{C}$ by

$$f(z) = \int_0^1 \frac{e^{tz}}{1+t^2} dt$$

is holomorphic.

Give an example to show that a holomorphic function need not possess an anti-derivative on its domain.

[*Any further results you use should be stated clearly.*]

4A Quantum Mechanics

Write down the time-independent Schrödinger equation for a particle of mass m with wavefunction $\psi(x)$ moving in a potential $V(x)$.

Consider the one-dimensional potential $V(x) = -V_0$ for $|x| < a$ and $V(x) = 0$ for $|x| > a$, for constant $V_0 > 0$.

By integrating the Schrödinger equation over a small interval around $x = a$, analyse the continuity of $\psi(x)$ and $\psi'(x)$ at $x = a$.

Show that

$$\psi(x) = \begin{cases} A\exp(-\eta|x|) & \text{for } |x| > a, \\ B\cos(kx) & \text{for } |x| < a, \end{cases}$$

is a solution of the time-independent Schrödinger equation, deriving two necessary relationships between η and k in the process.

Draw a diagram in the (k, η) plane that indicates the locus of the lowest energy level when $k < \pi/(2a)$.

5C Electromagnetism

State Faraday's law of induction, defining any terms that appear in the equation.

A circular wire loop has resistance R and lies in the $z = 0$ plane in a constant magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ with $B > 0$. The radius of the loop varies in time as $r(t)$. What is the current in the wire?

Distinguishing the two situations $\dot{r}(t) > 0$ and $\dot{r}(t) < 0$, draw a picture showing the magnetic field due to the induced current. Is the magnetic field increased or decreased inside the loop? In what direction is the Lorentz force on the wire in each case?

6D Numerical Analysis

The composite, mid-point, quadrature rule for computing the integral $I(f) = \int_0^1 f(x)\,dx$ is given by

$$I_N(f) = \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) \quad \text{with} \quad x_n = \frac{1}{N}\left(n + \tfrac{1}{2}\right).$$

Determine the order of convergence $I_N(f) \to I(f)$ of this scheme as $N \to \infty$ if f is at least twice differentiable on [0, 1].

A different function $f(x)$ is known to have a square-root singularity at $x = 0$, so that $f(x) = x^{-1/2} g(x)$, where $g(x)$ is analytic on [0,1]. Determine, with justification, a sequence y_n such that the quadrature

$$J_N(f) = \frac{2}{N} \sum_{n=0}^{N-1} y_n^{1/2} f(y_n)$$

has the same order of convergence $J_N(f) \to I(f)$ as the scheme above. [*Hint: Consider a change of variables in $I(f)$.*]

7H Markov Chains

A taxi driver moves between the airport A and two hotels B and C according to the following rules: if she is at the airport, she will proceed to one of the hotels with equal probability; if she is at a hotel, she will return to the airport with probability $\frac{3}{4}$ and travel to the other hotel with probability $\frac{1}{4}$.

(a) What is the transition matrix for the corresponding Markov chain?

(b) Suppose the driver begins at the airport at time 0.

(i) Find the probability for each of her three possible locations at time 2.

(ii) What is the probability that the driver is at the airport at time $n \geqslant 1$?

SECTION II

8G Linear Algebra

(a) Let $m, n \in \mathbb{N}$. Show that two $m \times n$ matrices A and A' over a field F are equivalent if and only if there exist vector spaces V, W, a linear map $\alpha\colon V \to W$ and bases B, B' of V and C, C' of W such that $A = [\alpha]_{B,C}$ and $A' = [\alpha]_{B',C'}$. [*You may assume the correspondence between composition of linear maps and products of matrices.*]

Define the *column rank* and the *row rank* of an $m \times n$ matrix A over F and prove that they are equal. [*You may assume the Rank-Nullity Theorem. Other results used should be proved.*]

(b) Fix $m, n \in \mathbb{N}$. Let $[m] = \{1, \ldots, m\}$ and $[n] = \{1, \ldots, n\}$. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{C}^n$. For $x = (x_j)_{j=1}^n \in \mathbb{C}^n$, let $\operatorname{supp}(x) = \{j \in [n] : x_j \neq 0\}$, and for $B \subseteq [n]$, let Bx be the vector in $\mathbb{C}^n$ with jth coordinate x_j if $j \in B$ and 0 if $j \notin B$.

Let $v_1, \ldots, v_m$ be linearly independent vectors in $\mathbb{C}^n$. Show that there is an injection $f\colon [m] \to [n]$ such that $f(i) \in \operatorname{supp}(v_i)$ for all $i \in [m]$. [*Hint: You may use the following result. If $F\colon [m] \to \mathcal{P}[n]$, where $\mathcal{P}[n]$ is the power set of $[n]$, satisfies*

$$|A| \leqslant \Big| \bigcup_{i \in A} F(i) \Big|$$

for all $A \subseteq [m]$ then there is an injection $f\colon [m] \to [n]$ such that $f(i) \in F(i)$ for all $i \in [m]$.]

Using part (a), or otherwise, show that there is a subset B of $[n]$ of size m such that $Bv_1, \ldots, Bv_m$ are linearly independent.

Deduce that there is an injection $f\colon [m] \to [n]$ such that $f(i) \in \operatorname{supp}(v_i)$ for all $i \in [m]$, and that

$$\big(\{e_j : j \in [n]\} \setminus \{e_{f(i)} : i \in [m]\}\big) \cup \{v_i : i \in [m]\}$$

is a basis of $\mathbb{C}^n$.

9E Groups, Rings and Modules

Let R be a commutative unital ring.

(a) Let M be an R-module. What does it mean for M to be *free*? Assuming R is non-zero, if $R^n \cong R^m$ as R-modules, show that $n = m$.

If P and Q are R-modules such that $P \oplus Q$ is free, must P be free? Justify your answer.

(b) (i) We say that an R-module P is *projective* if, whenever we have R-module homomorphisms $f : M \to N$ and $g : P \to N$ with f surjective, then there exists a homomorphism $h : P \to M$ with $f \circ h = g$. Show that any free module (over an arbitrary commutative unital ring) is projective.

(ii) Suppose now that R is a principal ideal domain. Prove that any submodule N of a finitely-generated free module M over R is free. [*Hint: If N is a submodule of R^n for some n, you may wish to consider the composition of maps $N \to R^n \to R$, where the first map is inclusion and the second map is projection onto the first summand.*]

Deduce that a finitely-generated projective module over a principal ideal domain is free.

10F Analysis and Topology

What does it mean for a function $f : \mathbb{R}^2 \to \mathbb{R}$ to be *differentiable* at $x \in \mathbb{R}^2$? Define the *derivative* $Df|_x$ and the *partial derivatives* $D_1 f(x)$ and $D_2 f(x)$ of f at $x \in \mathbb{R}^2$.

Show that if the partial derivatives of f exist in some open ball around $x \in \mathbb{R}^2$ and are continuous at x, then f is differentiable at x.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise.} \end{cases}$$

Find the partial derivatives of f at every point in $\mathbb{R}^2$. Are $D_1 f$ and $D_2 f$ continuous at $(0,0)$? Is f differentiable at $(0,0)$? Justify your answers.

Is it true that if f is differentiable everywhere in $\mathbb{R}^2$ then in a neighbourhood of each point at least one of the partial derivatives is bounded? Give a proof or a counterexample as appropriate.

11G Geometry

Fix real numbers $a \in (0,1]$ and $b \in (0, \sqrt{2a - a^2})$. Let $f : [-4,4] \to \mathbb{R}$ be an even function that is smooth and strictly positive on $(-4,4)$ with the properties that

$f(x) = \sqrt{1 - (3-x)^2}$ for $x \geqslant 2 + a$

$f(x) = b$ for $x \in [-2 + a, 2 - a]$

f'' has a unique zero in $(2 - a, 2 + a)$.

Let $\Sigma \subset \mathbb{R}^3$ be the smooth surface defined by

$$x \in [-4,4] \quad \text{and} \quad f(x)^2 = y^2 + z^2.$$

(a) Sketch Σ in $\mathbb{R}^3$. Sketch its orthogonal projection onto the (x,z)-plane, and mark on this diagram (without proof) the regions of Σ where its Gaussian curvature K is positive, negative and zero respectively.

(b) Compute the integral of K over the region $R \subset \Sigma$ where $x \in [2 - a, 2 + a]$. [*You may use without proof the fact that a spherical disc of spherical radius θ has area* $2\pi(1 - \cos\theta)$.]

(c) Show that the polygons obtained by cutting R along $y = 0$ are geodesic polygons only if $a = 1$.

12B Complex Methods

(i) Calculate the Laplace transform of the function defined for $0 \leqslant t < \infty$ by $f(t) = H(t - t_0)$ where H is the Heaviside function defined by $H(t) = 1$ if $t \geqslant 0$ and $H(t) = 0$ otherwise. (Here t_0 is an arbitrary positive number.)

(ii) Use the Fourier transform and contour integration to find the Green function defined by

$$-\frac{d^2G}{dx^2} + m^2 G = \delta(x), \qquad G(x) \to 0 \text{ as } |x| \to \infty,$$

where $m > 0$ and $-\infty < x < +\infty$. Explain why this Green function makes sense for $m \in \mathbb{C}$ with positive real part, and use it to write down a solution to

$$-\frac{d^2u}{dx^2} + m^2 u = f(x), \qquad u(x) \to 0 \text{ as } |x| \to \infty.$$

[*Take the Fourier transform* $\hat{G}$ *of* G *to be given by* $\hat{G}(k) = \int_{-\infty}^{+\infty} e^{-ikx} G(x) dx$.]

(iii) Use the Laplace transform to obtain an integral expression for the solution $u = u(t, x)$ of the initial value problem

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{for } -\infty < x < +\infty, \quad 0 \leqslant t < +\infty$$

$$u(0, x) = 0\,, \quad u_t(0, x) = f(x)\,.$$

[*You may assume that* $u(t, x)$ *and* $f(x)$ *vanish for* $|x|$ *sufficiently large.*]

13C Variational Principles

Three scalar fields, $\phi(\mathbf{x},t)$, $\alpha(\mathbf{x},t)$, and $\beta(\mathbf{x},t)$, are each a function of the spatial coordinates $\mathbf{x} = (x_1, x_2, x_3)$ and time t. The dynamics of these fields is governed by extremising the functional

$$S[\phi,\alpha,\beta] = \int_{-\infty}^{+\infty} \Big[-\beta\frac{\partial\alpha}{\partial t} - \frac{1}{2}\,(\nabla\phi + \beta\nabla\alpha)\cdot(\nabla\phi + \beta\nabla\alpha) \Big]\, dt\, d^3x \ .$$

Write down the Euler-Lagrange equations for ϕ, α and β.

Define the vector field

$$\mathbf{u} = \nabla\phi + \beta\nabla\alpha \ .$$

Show that the Euler-Lagrange equations can be written as

$$\nabla\cdot\mathbf{u} = 0 \quad \text{and} \quad \frac{\partial\alpha}{\partial t} + \mathbf{u}\cdot\nabla\alpha = 0 \quad \text{and} \quad \frac{\partial\beta}{\partial t} + \mathbf{u}\cdot\nabla\beta = 0 \ .$$

Hence show that the vector field $\mathbf{u}$ obeys

$$\frac{\partial u_i}{\partial t} + \mathbf{u}\cdot\nabla u_i = -\frac{\partial p}{\partial x_i} \ ,$$

where $p = -\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + f(\dot\phi,\dot\alpha,\beta)$ and $f(\dot\phi,\dot\alpha,\beta)$ is a function that you should determine, and where $\dot\phi$ and $\dot\alpha$ are the partial derivatives of ϕ and α with respect to t.

14B Methods

Let a and $\kappa \geqslant 0$ be real constants. Consider the problem

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \kappa\frac{\partial^2 u}{\partial x^2}$$

with initial condition $u(0,x) = u_0(x)$, where $u_0(x)$ is a given function. [*You may assume* u_0 *and* u *to be smooth and decreasing to zero as* $|x| \to \infty$ *as needed.*]

(i) For $a = 0, \kappa > 0$ write down an integral expression for the solution in terms of the function

$$K_t(x) = \begin{cases} (4\pi\kappa t)^{-\frac{1}{2}} \exp\left[-\frac{x^2}{4\kappa t}\right] & \text{if } t > 0 \\ 0 & \text{if } t \leqslant 0\,. \end{cases}$$

Explain briefly why your formula for $u(t,x)$ reduces to $u_0(x)$ when t tends to zero by considering the behaviour of K_t in this limit, and give a sketch to illustrate.

(ii) For $\kappa = 0$, use the method of characteristics to find the solution.

(iii) For the general case with $\kappa > 0$ and $a \in \mathbb{R}$ arbitrary, find an integral expression for the solution.

15A Quantum Mechanics

A quantum mechanical particle moves in an inverted harmonic oscillator potential. Its wavefunction $\psi(x,t)$ evolves according to

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2\psi}{\partial x^2} - \frac{1}{2}x^2\psi.$$

(i) Show that there exists a solution of the form

$$\psi(x,t) = A(t)\exp(-B(t)x^2)$$

provided that

$$\frac{dA}{dt} = -i\hbar AB$$

and

$$\frac{dB}{dt} = -\frac{i}{2\hbar} - 2i\hbar B^2.$$

(ii) Show that $B = \xi\tan(\phi+\alpha t)$ solves the equation for B, where ξ and α are constants that you should find and ϕ is a constant of integration.

(iii) Find $A(t)$ in terms of $\cos(\phi+\alpha t)$. You need not calculate its normalisation explicitly.

(iv) Compute the expectation values of $\hat{x}^2$ and $\hat{p}^2$ as functions of B.

[*Hint: You may use* $\dfrac{\int_{-\infty}^{\infty} dx\ e^{-Cx^2}x^2}{\int_{-\infty}^{\infty} dx\ e^{-Cx^2}} = \dfrac{1}{2C}$.]

16D Fluid Dynamics

A thin, horizontal layer of fluid of height $h = h_0 + \eta(x, y, t)$ flows with horizontal velocity components $\mathbf{u} = (u, v, 0)$ relative to a rotating frame of reference with Coriolis parameter $\mathbf{f} = (0, 0, f)$, in which (x, y, z) are Cartesian coordinates and where h_0 and f are constant and u and v are independent of z. When $\eta \ll h_0$, $\mathbf{u}$ and η satisfy the linearised equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{f} \times \mathbf{u} = -g\nabla\eta,$$

$$\frac{\partial \eta}{\partial t} + h_0 \nabla \cdot \mathbf{u} = 0,$$

where g is the acceleration due to gravity and $\nabla \equiv (\partial/\partial x, \partial/\partial y, 0)$ is the horizontal gradient operator.

Show that the linearised potential vorticity $\boldsymbol{\omega} - (\eta/h_0)\mathbf{f}$ is independent of time, where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the relative vorticity.

Suppose that $\eta = \eta_0$ when $\mathbf{u} \equiv 0$. Derive the evolution equation

$$\frac{\partial^2 \eta}{\partial t^2} - gh_0 \nabla^2 \eta + f^2 \eta = f^2 \eta_0.$$

Given that the fluid starts at rest with

$$\eta_0 = \begin{cases} \epsilon, & |x| < a \\ 0, & |x| > a \end{cases}$$

where ϵ is constant, determine the steady state $\eta_\infty(x)$ to which the system settles. Draw a sketch of the corresponding velocity field.

17H Statistics

Observations $\{(x_i, Y_i)\}_{i=1}^n$ are made according to the model

$$Y_i = \alpha + \beta x_i + \epsilon_i,$$

where $\{x_i\}_{i=1}^n$ are fixed constants in $\mathbb{R}$ and $\epsilon_i \overset{i.i.d.}{\sim} N(0, \sigma^2)$, for a known value of σ.

(a) Derive expressions for maximum likelihood estimators $\hat{\alpha}$ and $\hat{\beta}$ for α and β, respectively.

Now suppose the model is reparametrized as

$$Y_i = \alpha' + \beta'(x_i - \bar{x}) + \epsilon_i,$$

where $\bar{x} := \frac{1}{n}\sum_{i=1}^n x_i$. Let $\hat{\alpha}'$ and $\hat{\beta}'$ denote maximum likelihood estimators for α' and β', respectively.

(b) Show that $\hat{\beta}' = \hat{\beta}$.

(c) Show that in general, $\hat{\alpha}' \neq \hat{\alpha}$. In fact, show that $\hat{\alpha}' = \frac{1}{n}\sum_{i=1}^n Y_i$.

(d) What is the distribution of $\hat{\alpha}'$? Construct a 95% confidence interval for α' based on $\hat{\alpha}'$.

(e) Now suppose σ is unknown. Construct a 95% confidence interval for α' in this setting, and explain why it has the specified coverage.

[*Standard results can be quoted without proof.*]

18H Optimisation

(a) Describe Newton's method for minimising a function $f : \mathbb{R}^d \to \mathbb{R}$. Denote by x^* a minimiser of f, and by x_k the kth iterate in Newton's method. Stating clearly any assumptions f must satisfy, provide an upper bound on $f(x_k) - f(x^*)$.

(b) Suppose $a \geqslant 1$. Consider the following algorithm used by the ancient Babylonians to approximate $\sqrt{a}$: set $x_0 \geqslant 1$ and, for each $k \geqslant 0$, iteratively define

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right).$$

Prove that all the iterates lie in $[1, \infty)$. Derive the algorithm above as a consequence of applying Newton's method for minimising a suitable function $f : [1, \infty) \to \mathbb{R}$.

(c) For a given $a \geqslant 1$, identify a range of values for x_0 such that $x_k \to \sqrt{a}$ as $k \to \infty$. Derive an upper bound on $|x_k - \sqrt{a}|$. [*Hint: You may find the result in part (a) useful, as well as the equation* $x^3 - 3ax + 2a^{3/2} = (x - \sqrt{a})^2(x + 2\sqrt{a})$.]

END OF PAPER