UNIVERSITY OF CAMBRIDGE

MAT1

MATHEMATICAL TRIPOS   Part IB

Thursday, 06 June, 2024   1:30pm to 4:30pm

PAPER 3

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most four** questions from Section I and **at most six** questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Groups, Rings and Modules

(i) Suppose that A is a matrix over $\mathbb{Z}$. What is the *Smith normal form* for A? State the structure theorem for finitely-generated modules over $\mathbb{Z}$.

(ii) Find the Smith normal form of the matrix $\begin{pmatrix} -4 & -6 \\ 2 & 2 \end{pmatrix}$. Justify your answer.

Suppose that M is the $\mathbb{Z}$-module with generators e_1, e_2, subject to the relations $-4e_1 + 2e_2 = -6e_1 + 2e_2 = 0$. Describe M in terms of the structure theorem.

(iii) An abelian group is called *indecomposable* if it cannot be written as the direct sum of two non-trivial subgroups. Show that a finite group is indecomposable if and only if it is cyclic of prime power order.

2G Geometry

Given a smooth function $h : \mathbb{R}^2 \to \mathbb{R}$, show that the graph

$$\Gamma = \{(x, y, z) : z = h(x, y)\}$$

is a smooth surface in $\mathbb{R}^3$. Write down a parametrisation of Γ, and compute the first fundamental form and Gauss map with respect to it.

3B Complex Methods

State Cauchy's theorem.

Calculate the Fourier transform of the function

$$f_{a,b}(x) = \exp[-ax^2 + ibx],$$

where $a > 0$ and b are real numbers, making sure to justify any change of variables you use.

[*Take the Fourier transform* $\hat{f}$ *of a function* $f : \mathbb{R} \to \mathbb{C}$ *to be given by* $\hat{f}(k) = \int_{-\infty}^{+\infty} e^{-ikx} f(x)dx$.]

4C Variational Principles

Given a Lagrangian $L(\mathbf{x}, \dot{\mathbf{x}}, t)$, what is the momentum $\mathbf{p}$ conjugate to $\mathbf{x}$? What is the Hamiltonian? Under what circumstances is energy conserved?

The dynamics of a particle with position $\mathbf{x} = (x, y, z)$ is governed by the Lagrangian

$$L = mc\sqrt{\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}} - V(\mathbf{x}), \qquad (1)$$

where m and c are positive constants, and $V(\mathbf{x})$ is a potential.

Determine the momentum $\mathbf{p}$ conjugate to $\mathbf{x}$. Determine the Euler-Lagrange equation.

Show that $\mathbf{p}\cdot\mathbf{p}$ is constant. Determine the Hamiltonian. Is it possible to reconstruct the Lagrangian from the Hamiltonian?

5B Methods

Let $u(r, \theta)$ satisfy the Laplace equation

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

in the annulus $\mathcal{A}$ given in polar coordinates by

$$\mathcal{A} = \{(r, \theta) : \ a < r < b\,, 0 \leqslant \theta < 2\pi\}\,.$$

Use separation of variables to derive a general expression for u. Given boundary conditions $u(a, \theta) = 0$ and $u(b, \theta) = \cos 2\theta$, find u explicitly.

6A Quantum Mechanics

Consider a one-dimensional system described by a wavefunction ψ for which $\langle \hat{p} \rangle = 0$ and $\langle \hat{x} \rangle = 0$.

(i) Write down the commutation relation between $\hat{x}$ and $\hat{p}$.

(ii) Define the *uncertainty* ΔO of an observable $\hat{O}$ in terms of $\langle \hat{O} \rangle$ and $\langle \hat{O}^2 \rangle$.

(iii) Considering the one-parameter family of states defined by

$$\psi_s(x) = (\hat{p} - is\hat{x})\psi(x),$$

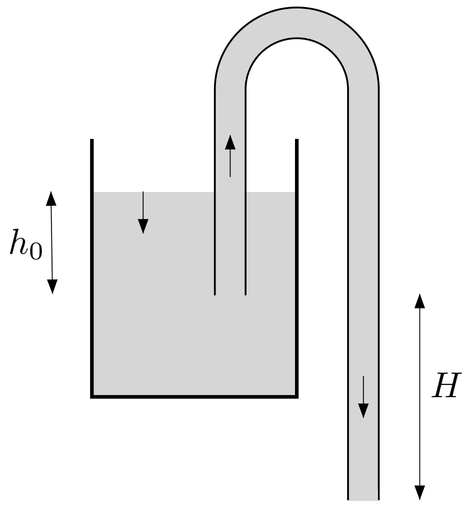
where $s \in \mathbb{R}$, derive the Heisenberg uncertainty relation between Δx and Δp.

7D Fluid Dynamics

Starting from Euler's equations for steady, inviscid flow $\mathbf{u}$ of an incompressible fluid of uniform density ρ, subject to a body force $-\nabla\chi$, prove that $\mathbf{u}\cdot\nabla H = 0$, where $H \equiv \frac{1}{2}\rho|\mathbf{u}|^2 + p + \chi$ and p is the fluid pressure. Interpret this equation physically.
[*You may use the identity* $\mathbf{u}\times(\nabla\times\mathbf{u}) = \nabla(\frac{1}{2}|\mathbf{u}|^2) - \mathbf{u}\cdot\nabla\mathbf{u}$.]

Fluid initially occupies a tank with uniform horizontal cross-sectional area A. It is syphoned out of the tank using a tube of cross-sectional area $a \ll A$ as shown in the diagram below, in which flow directions (not magnitudes) are indicated. One end of the tube is held at a distance h_0 below the initial position of the free surface of the fluid in the tank, while the other end is held at a distance H below that. The tube is full of fluid and drains freely from its lower end into the surrounding air. Assuming the flow to be quasi-steady, show that the fluid level in the tank reaches the upper end of the tube after a time

$$t = \sqrt{2}\left(\frac{A^2}{a^2} - 1\right)^{1/2} \frac{\sqrt{H+h_0} - \sqrt{H}}{\sqrt{g}}.$$



8H Markov Chains

Let $\{X_n : n \geqslant 1\}$ be independent, identically distributed, integer-valued random variables. Define

(i) $S_n = \sum_{i=1}^{n} X_i$

(ii) $L_n = \min\{X_1, X_2, \ldots, X_n\}$

(iii) $K_n = X_n + X_{n-1}$, with $X_0 = 0$.

Which of the sequences $\{X_n\}, \{S_n\}, \{L_n\}, \{K_n\}$ are necessarily Markov chains? Justify your answers.

SECTION II

9G Linear Algebra

Let V be a finite-dimensional complex inner product space with inner product $\langle \cdot, \cdot \rangle$. A map $f\colon V \to \mathbb{C}$ is *conjugate-linear* if $f(\lambda v + \mu w) = \bar{\lambda} f(v) + \bar{\mu} f(w)$ for all $v, w \in V$ and $\lambda, \mu \in \mathbb{C}$. A map $\theta\colon V \times V \to \mathbb{C}$ is a *sesquilinear form* if the map $v \mapsto \theta(v, w)$ is linear for each $w \in V$ and the map $w \mapsto \theta(v, w)$ is conjugate-linear for each $v \in V$. Show that for each such θ, there is a unique map $\beta\colon V \to V$ such that $\theta(x, y) = \langle x, \beta(y) \rangle$ for all $x, y \in V$, and moreover that β is linear.

Let $\alpha \in \mathrm{End}(V)$. Use the results above to show the existence and uniqueness of the adjoint α^* of α. Prove the following statements. [*Standard properties of adjoints can be assumed.*]

(a) For a subspace U of V, $\alpha(U) \subseteq U$ if and only if $\alpha^*(U^\perp) \subseteq U^\perp$.

(b) If $\langle \alpha(x), x \rangle = 0$ for all $x \in V$, then $\alpha = 0$. Does the same hold in a real inner product space? Justify your answer.

(c) $\alpha\alpha^* = \alpha^*\alpha$ if and only if $\|\alpha(x)\| = \|\alpha^*(x)\|$ for all $x \in V$. Does the same hold in a real inner product space? Justify your answer.

(d) If $\alpha\alpha^* = \alpha^*\alpha$, then there is an orthonormal basis of V consisting of eigenvectors of α.

10E Groups, Rings and Modules

(a) (i) Let R be a commutative unital ring. Show that an ideal I of R is prime if and only if R/I is an integral domain.

(ii) R is said to be *Boolean* if $r^2 = r$ for all $r \in R$. If R is Boolean, prove that $r + r = 0$ for all $r \in R$. Show also that if R is a non-zero integral domain and is Boolean, it is isomorphic to the field of two elements. Deduce that in a Boolean ring every prime ideal is maximal.

(b) (i) Let R be a commutative unital ring and let $R[X]$ be the ring of polynomials in X, with coefficients in R. Let I be an ideal of R and let $I[X]$ be the ideal of $R[X]$ consisting of all polynomials with coefficients in I. [*You may assume $I[X]$ is indeed an ideal.*] Show that $R[X]/I[X] \cong (R/I)[X]$. Deduce that if I is a prime ideal of R then $I[X]$ is a prime ideal of $R[X]$.

(ii) Give an example to show that if I is a maximal ideal of R then $I[X]$ need not be a maximal ideal of $R[X]$.

(c) In this part, we assume the prime number p is odd.

Let $\mathbb{F}_p$ be the field of p elements. Prove that its multiplicative group $\mathbb{F}_p^\times = \mathbb{F}_p \setminus \{0\}$ is a cyclic group.

Consider the group homomorphism $\phi : \mathbb{F}_p^\times \to \mathbb{F}_p^\times$ given by $x \mapsto x^2$, and let H be its image. Show that H has index 2 in $\mathbb{F}_p^\times$ and deduce that one of 2, 3 or 6 is a square in $\mathbb{F}_p$. Deduce that the polynomial $f(x) = x^6 - 11x^4 + 36x^2 - 36$ has a root modulo p.

11F Analysis and Topology

Let X, Y be non-empty sets.

(a) Let d_X, d_Y be metrics on X, Y, respectively. Define what it means for a function $f : X \to Y$ to be *uniformly continuous*.

We say that a sequence of functions $f_n : X \to Y$ *converges uniformly* to a function $f : X \to Y$ *with respect to* d_Y if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$ and for all $x \in X$, $d_Y(f_n(x), f(x)) < \epsilon$.

Show that a uniform limit of uniformly continuous functions $f_n : X \to Y$ is uniformly continuous. Give an example to show that the conclusion is false if convergence is pointwise but not uniform.

(b) Recall that two metrics d_1 and d_2 on X are *equivalent* if the identity map $\mathrm{id} : (X, d_1) \to (X, d_2)$ is a homeomorphism.

Show that if d_1 and d_2 are such that there exist constants $\alpha, \beta > 0$ with the property that for every $x, y \in X$,

$$\alpha d_1(x, y) \leqslant d_2(x, y) \leqslant \beta d_1(x, y),$$

then d_1 and d_2 are equivalent.

Does the reverse conclusion hold? Give a proof or a counterexample as appropriate.

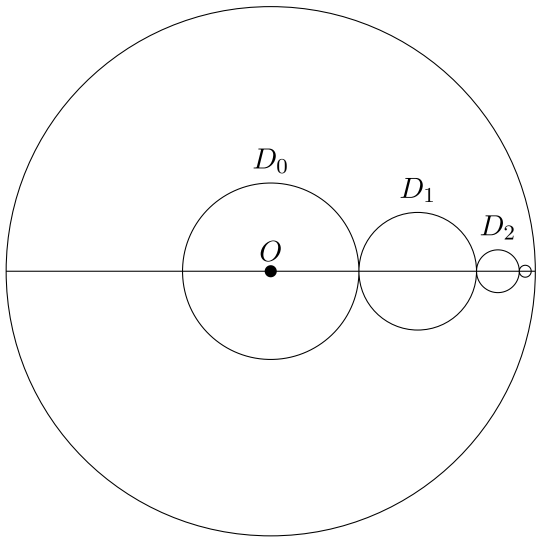
If d_3 and d_4 are equivalent metrics on Y, is it true that a sequence of functions $f_n : X \to Y$ converges uniformly with respect to d_3 if and only if it converges uniformly with respect to d_4? Give a proof or a counterexample as appropriate.

12E Geometry

Consider the Poincaré disc model of the hyperbolic plane, with Riemannian metric

$$\frac{4(\mathrm{d}x^2 + \mathrm{d}y^2)}{(1 - x^2 - y^2)^2}.$$

Let $D_0, \ldots, D_n$ be distinct closed hyperbolic discs of hyperbolic area $\frac{\pi}{2}$, such that D_0 is centred at the origin O and each successive D_i is centred on the positive real axis and tangent to D_{i-1}, as shown in the figure below. Show that the hyperbolic centre of D_n is at $(4^n - 1)/(4^n + 1)$. [If you use any formulae for hyperbolic lengths or areas then they should be proved.]



Now consider the upper-half-plane model. Let $D_0', \ldots, D_n'$ be distinct closed hyperbolic discs of hyperbolic area $\pi/2$ such that D_0' has hyperbolic centre at $(0, 1)$ and each successive D_i' has hyperbolic centre on the line $y = 1$ and is tangent to D_{i-1}'. For $n \geqslant 3$, is there an isometry from the Poincaré disc model to the upper-half-plane model which carries each D_i to D_i'? Briefly justify your answer.

13F Complex Analysis

Define the *winding number* of a closed curve $\gamma : [a,b] \to \mathbb{C}$ about a point $w \in \mathbb{C}$ which is not in the image of γ. [You do not need to justify its existence.]

State the argument principle on a domain U bounded by a closed curve γ.

Deduce Rouché's theorem, which you should state carefully.

Let f be non-constant and holomorphic on an open set containing the closed unit disc $\overline{D}$. Suppose that $|f(z)| \geqslant 1$ for all z satisfying $|z| = 1$, and that there exists z_0 in the unit disc D such that $|f(z_0)| < 1$. Show that the image of f contains D.

Let g be holomorphic and non-zero on the punctured unit disc $D^* = D \setminus \{0\}$ such that g'/g has a simple pole at 0. Show that there exists a non-zero integer k such that h'/h has a removable singularity at 0, where h is defined by $h(z) = z^{-k} g(z)$.

14B Methods

Transverse oscillations $y = y(t,x)$ of a string in a resisting medium are governed by the damped wave equation

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial t} = 0\,, \quad 0 < x < 1\,, \quad t > 0\,.$$

Assuming the string is fixed at $x = 0$ and $x = 1$ so that $y(t,0) = 0 = y(t,1)$, use separation of variables to derive an expression for the solution with given initial values

$$y(0,x) = \sum_{n=1}^{\infty} a_n \sin n\pi x\,, \qquad \frac{\partial y}{\partial t}(0,x) = 0\,. \qquad (*)$$

Calculate the Fourier coefficients $\{a_n\}$ in the particular case

$$y(0,x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \frac{1}{2} \\ 1 - x & \text{if } \frac{1}{2} \leqslant x \leqslant 1\,. \end{cases}$$

For $(*)$ in the case of general coefficients $\{a_n\}$, use the Parseval identity to calculate the energy

$$E(t) = \frac{1}{2}\int_0^1 \left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2 dx$$

at time t in terms of the coefficients $\{a_n\}$. Hence, or otherwise, show that the energy is decreasing.

15C Electromagnetism

Explain why all points of a conductor must sit on an equipotential. What does this imply for the electric field near the surface of a conductor? Derive an expression for the surface charge of a conductor in terms of the electric field.

Two spherical conducting shells, with radii R_1 and $R_2 > R_1$ are connected by a long, thin conducting rod of length $d \gg R_1 + R_2$. A charge Q is deposited on the spheres. What fraction of this charge resides on each shell? [*You may ignore the effect of the electric field from one shell on the other, and neglect the charge on the rod.*]

The same two spherical shells are are now placed concentrically around the origin. Again, they are connected by a thin conducting rod. What fraction of the charge Q sits on each shell? [*Again, you may neglect the charge on the rod.*]

A neutral conducting sphere of radius R is placed in a constant electric field $\mathbf{E} = E\hat{\mathbf{z}}$. Work in spherical polar coordinates, with $z = r\cos\theta$, and find a solution for the potential Φ outside the sphere of the form

$$\Phi = \alpha\left(r + \frac{\beta}{r^2}\right)\cos\theta$$

for some α and β that you should determine. What is the induced surface charge on the sphere? Confirm that the sphere is indeed neutral.

16D Fluid Dynamics

A solid sphere of radius a moves in a straight line with speed U through fluid of density ρ that is at rest far from the sphere. Calculate the velocity potential ϕ for inviscid, irrotational flow of the surrounding fluid. Calculate the velocity components in the frame of reference in which the fluid is at rest far from the sphere. Hence calculate the total kinetic energy of the fluid.

Now suppose that the sphere has density $\rho_s > \rho$ and falls with speed U under gravity g. Write down the rate of change of the potential energy of the system (sphere plus fluid). By considering the rate of change of the total energy of the system (potential plus kinetic) or otherwise, show that

$$\frac{dU}{dt} = \frac{\rho_s - \rho}{\rho_s + \frac{1}{2}\rho} g.$$

17A Numerical Analysis

For a function $f \in C^4[-1,2]$ consider the following approximation of $f''(-1)$:

$$f''(-1) \approx \eta(f) = a_{-1}f(-1) + a_0 f(0) + a_1 f(1) + a_2 f(2),$$

with the error

$$e(f) = f''(-1) - \eta(f).$$

We want to find the smallest constant c such that

$$|e(f)| \leqslant c \max\nolimits_{x\in[-1,2]} |f''''(x)|,$$

where $f''''(x)$ is the fourth derivative of f.

State the necessary conditions on the approximation scheme η for the inequality above to be valid with some $c < \infty$. Hence determine the coefficients a_{-1}, a_0, a_1, a_2.

State the Peano kernel theorem.

Assuming that the Peano kernel is non-negative in $x \in [-1,2]$, use the Peano kernel theorem to find the smallest constant c.

18H Statistics

Suppose $X_1, \ldots, X_n$ is an i.i.d. sample from a $N(\mu, 1)$ population, and we wish to test the hypothesis

$$H_0 : \mu = 0 \quad \text{against} \quad H_1 : \mu \neq 0.$$

(a) Define a level-α hypothesis test based on $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, and find an expression for the p-value $p(\bar{x})$ corresponding to a value $\bar{X} = \bar{x}$ in terms of the standard normal cumulative distribution function Φ.

Now consider a prior distribution on μ, defined as follows: with probability $\frac{1}{2}$, take $\mu = 0$, and with probability $\frac{1}{2}$, draw a value of μ from a $N(0, \tau^2)$ distribution, where τ is known.

(b) What is the conditional probability density function of $\bar{X}|\mu$? Derive an expression for the marginal probability density function $m(\bar{x})$ of $\bar{X}$ under the prior distribution described above.

[*Hint: The formula* $\int_{-\infty}^{\infty} e^{-ax^2-bx}dx = \sqrt{\frac{\pi}{a}}e^{b^2/(4a)}$, *for* $a > 0$, *may be helpful.*]

(c) Find an expression for the posterior probability $q(\bar{x})$ that H_0 is true, defined by $q(\bar{x}) = \mathbb{P}(\mu = 0 | \bar{X} = \bar{x})$.

(d) Suppose we fix $n = 100$ and $\tau = 1$. Compare the probabilities $p(\bar{x})$ and $q(\bar{x})$ obtained in parts (a) and (c):

(i) Which probability is larger when $\bar{x}$ is close to 0?

(ii) Which probability is larger when $|\bar{x}|$ is large?

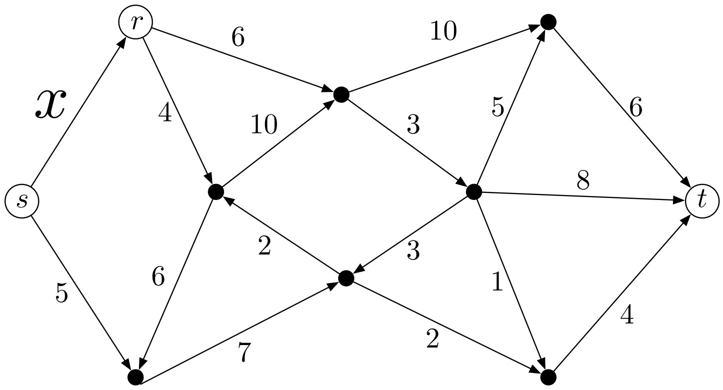
[*You may use the Taylor series approximation* $1 - \Phi(x) \approx \dfrac{e^{-x^2/2}}{\sqrt{2\pi}x}$, *which holds for large values of* x.]

[*Standard results can be quoted without proof.*]

19H Optimisation

(a) For $n > 1$, consider a directed graph G with vertices $V = \{1, 2, \ldots, n\}$ and edge set E. Following the usual convention, if $(i, j) \in E$, then G has a directed edge from vertex i to vertex j. Each edge $(i, j) \in E$ has an associated capacity $C_{ij} \geqslant 0$. Vertex 1 is designated as the source and vertex n is designated as the sink. State and prove the max-flow min-cut theorem.

(b) Consider the directed graph with edge capacities as shown in the figure below, with source s, sink t, and an intermediate vertex r:



(i) Let the capacity of the edge from s to r be x. Find the maximum flow from s to t when $x = 4$. Verify your answer by identifying a cut whose capacity equals your answer.

(ii) Let $\delta^*(x)$ be the maximum flow from s to t, as a function of x. Derive a formula for $\delta^*(x)$ when $x \geqslant 0$.

END OF PAPER