UNIVERSITY OF CAMBRIDGE

MAT1

# MATHEMATICAL TRIPOS Part IB

Wednesday, 05 June, 2024 9:00am to 12:00pm

## PAPER 2

## Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most four** questions from Section I and **at most six** questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

## At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

**Every cover sheet must also show your Blind Grade Number and desk number.**

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

**STATIONERY REQUIREMENTS**
Gold cover sheets
Green main cover sheet
Treasury tag

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

## SECTION I

### 1E Groups, Rings and Modules

State Eisenstein's irreducibility criterion.

(i) Let $n > 1$ be an integer. Prove that $X^{n-1} + X^{n-2} + \cdots + X + 1$ is irreducible in $\mathbb{Z}[X]$ if and only if $n$ is a prime number.

(ii) Show that the polynomial $X^2 + Y^2 - 1$ in $\mathbb{Q}[X, Y]$ is irreducible. Would your argument work over any field?

### 2F Analysis and Topology

Let $(X, d)$ be a metric space. Define what it means for $h : X \to X$ to be a *contraction*.

State and prove the contraction mapping theorem.

Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function, and let $r$ be a root of $f$. Suppose that on some neighbourhood $U$ of $r$, $|f'(x)| > \delta$ for some $\delta > 0$ and $|f''(x)| < M$ for some $M < \infty$. Define $g : U \to \mathbb{R}$ by $g(x) = x - f(x)/f'(x)$. Show that $g'(r) = 0$ and that $g'$ is bounded by $1/2$ in absolute value on some neighbourhood $U'$ of $r$. Deduce that $r$ is the unique fixed point of $g$ on $U'$.

### 3B Methods

For integer $n$, the Chebychev polynomials $T_n$ satisfy the equation

$$(1 - x^2)T_n'' - xT_n' + n^2 T_n = 0\,, \qquad -1 < x < 1\,.$$

Put this equation into Sturm-Liouville form and derive an orthogonality relation between $T_n$ and $T_m$ for $n \neq m$. Find a second order differential equation satisfied by the derivatives $U_n = T_n'$, and an orthogonality relation between $U_n$ and $U_m$ for $n \neq m$.

**4C Electromagnetism**

The two equations of magnetostatics are

$$\nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

Explain briefly how the current density $\mathbf{J}$ can be non-zero even though the charge density vanishes.

Explain how a vector potential $\mathbf{A}$ can be introduced to solve one of these equations. Is $\mathbf{A}$ unique?

Show that in Cartesian coordinates $(x, y, z)$ the following current density is consistent with charge conservation:

$$\mathbf{J} = J_0 \begin{pmatrix} \sin(\lambda z) \\ \cos(\lambda z) \\ 0 \end{pmatrix}$$

with $\lambda$ and $J_0$ constant. What is the resulting magnetic field? What is the vector potential?

[*Hint: Consider* $\nabla \times \mathbf{J}$.]

**5D Fluid Dynamics**

A fluid has velocity $\mathbf{u} = (y, ax)$ in Cartesian coordinates $(x, y)$, where $a$ is a real constant. Show that the flow is incompressible, determine a stream function $\psi(x, y)$ for the flow, and sketch the streamlines for $a > 0$ and for $a < 0$.

For what value of $a$ is the flow also irrotational? In this case, determine a velocity potential $\phi(x, y)$ for the flow.

**6H Statistics**

After losing a large amount of money, an unlucky gambler questions whether the game was fair and the die was really unbiased. The last 90 rolls of this die gave the following results:

| score on the die | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| number of times it occurred | 20 | 15 | 12 | 17 | 9 | 17 |

(i) Suppose the gambler wishes to test the hypothesis that the die is fair. What are the null and alternative hypotheses?

(ii) Describe Pearson's test. What is the limiting distribution of the Pearson statistic under the null hypothesis?

(iii) Compute the Pearson statistic for this test.

(iv) What is the asymptotic $p$-value of the test (written as a quantile of an appropriate distribution)?

[*Standard results can be quoted without proof, provided they are stated clearly.*]

**7H Optimisation**

Consider the following optimisation problem:

$$\begin{aligned} \text{minimise} \quad & x_1 \log x_1 - x_2 \\ \text{subject to:} \quad & x_1 + x_2 \leqslant c \\ & \sqrt{x_2} \leqslant d \\ & x_1, x_2 \geqslant 0. \end{aligned}$$

At $x_1 = 0$, the value of $x_1 \log x_1$ is defined to be equal to 0, its limiting value.

(a) Use the Lagrange method to search for a solution when $c = 3/e^2$ and $d = 2/e$, where $e$ is the base of the natural logarithm.

(b) Now use the Lagrange method to search for a solution when $c = 3/e^2$ and $d = 1/e$. Explain your observations.

## SECTION II

### 8G Linear Algebra

(a) Let $A$ be an $n \times n$ complex matrix. Define the *characteristic polynomial* of $A$. Show that $A$ is similar to an upper-triangular matrix.

Define the *minimal polynomial* $m_A$ of $A$. Prove that $m_A$ exists and is unique. Prove that $\deg(m_A) \leqslant n$.

[*You may assume properties of determinants and results about matrix representation of linear maps. Any other results used must be proved.*]

(b) Let $V$ be the real vector space of all real-valued functions on $\mathbb{R}$. For each $r \in \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$, define $D_r \colon V \to V$ by $(D_r f)(x) = f(x+r) - f(x)$ for $f \in V$, $x \in \mathbb{R}$. Find the eigenvalues of $D_r$ and show that the corresponding eigenspaces are infinite-dimensional. Show further that $D_r D_s = D_s D_r$ for all $r, s \in \mathbb{R}^\times$.

Call $f \in V$ *periodic* if $f \in \ker D_r$ for some $r \in \mathbb{R}^\times$. Show that a polynomial function in $V$ of degree $n$ cannot be written as a sum of $n$ periodic functions.

### 9E Groups, Rings and Modules

(a) If $R$ is a Noetherian ring, show that $R/I$ is Noetherian for each ideal $I$ in $R$.

State the Hilbert basis theorem.

Explain briefly why $\mathbb{Z}$ is Noetherian. Deduce from these results that the ring $\mathbb{Z}[\sqrt{d}]$ for a non-square integer $d$ is Noetherian.

(b) Let $K$ be any field. Consider the set

$$R = \left\{ f(X,Y) = \sum_{i,j} c_{ij} X^i Y^j \in K[X,Y] : c_{0j} = c_{j0} = 0 \text{ whenever } j > 0 \right\}.$$

Verify that $R$ is a subring of $K[X,Y]$ and determine, with justification, whether or not $R$ is Noetherian.

**10F Analysis and Topology**

(a) Define what it means for a topological space $(X, \tau)$ to be *compact*. Define what it means for $(X, \tau)$ to be *Hausdorff*.

Show that a closed subspace $Y$ of a compact space $(X, \tau)$ is compact.

Let $(X, \tau)$ be a compact Hausdorff space. Show that for any two disjoint closed subsets $A$ and $B$ of $X$, there exist disjoint open sets $U$ and $V$ containing $A$ and $B$, respectively.

(b) A topological space $(X, \tau)$ is called *locally compact* if for each $x \in X$ and every neighbourhood $U$ of $x$, $U$ contains a compact neighbourhood $K$ of $x$. Show that a compact Hausdorff space is locally compact.

Let $(X, \tau)$ be a locally compact Hausdorff space. Let $A \subseteq X$ be such that $A \cap K$ is closed in $K$ for every compact $K \subseteq X$. Show that $A$ is closed.

**11G Geometry**

The torus $T^2$ and Klein bottle $K$ can both be described as quotients of $\mathbb{R}^2$ by equivalence relations $\sim$ and $\simeq$ given by

$$(x, y) \sim (x + a, y + b) \quad \text{for} \quad (a, b) \in \mathbb{Z}^2$$

and

$$(x, y) \simeq (x + c, (-1)^c y + d) \quad \text{for} \quad (c, d) \in \mathbb{Z}^2 \ ,$$

respectively. Equip $T^2$ and $K$ with the standard flat Riemannian metrics induced from $\mathbb{R}^2$ by these quotient constructions.

(a) Show that the map $\widetilde{\pi} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\widetilde{\pi}(x, y) = (2x, y)$ induces a well-defined 2:1 continuous map $\pi : T^2 \to K$.

(b) Draw a fundamental domain, with arrows indicating boundary gluing directions, for each of the two quotients.

(c) On separate copies of the fundamental domain for $K$, draw the images of closed geodesics $\gamma_1$, $\gamma_2$ and $\gamma_3$ with the following properties: $\gamma_1$ cuts $K$ into a Möbius strip; $\gamma_2$ cuts $K$ into a cylinder; $\gamma_3$ intersects itself in a single point.

(d) For each $i$, how many closed geodesics $\widetilde{\gamma}_i$ are contained in the preimage of the image of $\gamma_i$ under $\pi$? For the purpose of counting, we consider two closed geodesics the same if they have the same image. On separate copies of the fundamental domain for $T^2$, draw examples of $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$ and $\widetilde{\gamma}_3$.

**12B Complex Analysis OR Complex Methods**

By considering the integral of an appropriate function on a semi-circular contour in the upper half plane, or otherwise, compute

$$\int_0^\infty \frac{(\ln x)^4}{1+x^2}\, dx\,.$$

[*Hint: You may use that* $\displaystyle\int_0^\infty \frac{(\ln x)^2}{1+x^2}\, dx \;=\; \frac{\pi^3}{8}$ .]

**13C Variational Principles**

A functional of $y(x)$ takes the form

$$I[y] = \int_0^{x_0} F(y', y, x)\, dx\ .$$

Derive the Euler-Lagrange equation and explain why solutions to this equation, assuming that they exist, extremise $I[y]$ under the assumption that $y(x)$ is fixed at each end.

The system is said to have *free boundary conditions* if $\partial F/\partial y' = 0$ at the end points. Explain why the solutions to the Euler-Lagrange equations, if they exist, also extremise $I[y]$ if free boundary conditions are imposed at each end.

Consider the functional of $y(x)$ and $z(x)$ given by

$$J[y, z] = \int_0^{x_0} \left[ y'^2 + z'^2 + 2yz \right] dx\ .$$

Find the most general solution to the Euler-Lagrange equations subject to the requirement that $y(0) = z(0) = 0$. For which values of $x_0$ are there solutions if we impose free boundary conditions at $x_0$? Find these solutions.

**14B Methods**

Define the *convolution* $f * g$ of two functions on the real line. The function $F$ is defined by

$$F(x) = \begin{cases} e^{-x} & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0. \end{cases}$$

A sequence of functions $F_1, F_2, F_3 \ldots$ is defined by $F_1 = F$, $F_n = F * F_{n-1}$ for $n \geqslant 2$ (so $F_n$ is the $n$-fold convolution of $F$ with itself). Use induction to find $F_n$, without using the Fourier transform.

The Fourier transform $\hat{f}$ of a function $f : \mathbb{R} \to \mathbb{C}$ is given by

$$\hat{f}(k) = \int_{-\infty}^{+\infty} e^{-ikx} f(x) dx.$$

Compute the Fourier transform $\hat{F}_n$.

State and prove the convolution theorem. Verify that $\hat{F}_n = (\hat{F})^n$.

Using the identity

$$\int_{-\infty}^{+\infty} e^{-ikx} dk = 2\pi\delta(x)$$

and interchanging the order of integration, show that the convolution theorem with an appropriate choice of $g$ implies the Parseval identity

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{f}(k)|^2 dk\,.$$

[*You may assume that $f$ and $\hat{f}$ are integrable and decrease rapidly at infinity, and that the order of integration in multiple integrals can be interchanged.*]

Deduce the value of the integral $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{(1+k^2)^{n+1}} dk.$

**15A Quantum Mechanics**

(i) Calculate the commutation relation between a position operator $\hat{x}$ and its associated momentum $\hat{p}_x = -i\hbar\partial/\partial x$.

(ii) Write down the time-dependent Schrödinger equation for a quantum mechanical system with Hamiltonian $\hat{H}$ and wavefunction $\psi$.

(iii) Calculate the rate of change of the expectation value of some operator $\hat{O} = \hat{O}(t)$ in terms of $\langle[\hat{O},\ \hat{H}]\rangle$ and $\langle\frac{\partial\hat{O}}{\partial t}\rangle$.

(iv) Express each of $[\hat{x},\ \hat{p}_x^2]$ and $[\hat{x}^2,\ \hat{p}_x]$ in terms of a single operator.

(v) Consider the two-dimensional harmonic oscillator whose Hamiltonian is

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + \frac{m\omega^2}{2}(\hat{x}^2 + \hat{y}^2).$$

Setting $\hat{L} = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$, calculate $[\hat{L},\ \hat{H}]$.

(vi) Changing variables to $z = x + iy$, consider the two degenerate energy eigenstates

$$\psi = Az\exp(-\beta|z|^2) \qquad \text{and} \qquad \psi^* = Az^*\exp(-\beta|z|^2)$$

where $A$ and $\beta$ are positive real constants. At time $t = 0$, a state $\frac{\sqrt{5}}{3}\psi + \frac{2}{3}\psi^*$ is prepared. What is the expectation value of $\hat{L}$ at a later time $t > 0$?

**16C Electromagnetism**

Throughout this question, use the Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.

The electromagnetic covector potential is

$$A_\mu = (-\Phi/c, A_i),$$

where $\Phi$ is the electrostatic potential, $A_i$ are the components of the vector potential and $c$ is the speed of light. The field-strength tensor is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

where $\partial_\mu$ denotes differentiation with respect to the spacetime coordinates $x^\mu = (ct, x^i)$.

Define a gauge transformation of $A_\mu$ and show that it leaves $F_{\mu\nu}$ unchanged.

Compute the components of $F_{\mu\nu}$ in terms of the electric and magnetic fields, which you should define in terms of $\Phi$ and $A_i$.

Explain how two of the four Maxwell equations follow automatically from the definition of $F_{\mu\nu}$. Show how the remaining two Maxwell equations follow from

$$\partial_\nu F^{\mu\nu} = \mu_0 j^\mu$$

for some 4-vector $j^\mu$ that you should define.

Consider the tensor

$$T^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu\rho}F^{\nu}{}_{\rho} - \frac{1}{4}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right) .$$

Compute $T^{00}$ in terms of the electric and magnetic fields and identify its physical meaning.

Show that, for any vector $k_\mu$ that is null (i.e. lightlike), if $T^{\mu\nu}k_\mu k_\nu$ is non-vanishing then it has a particular sign that you should determine. [*Hint: Consider a particular inertial frame chosen to simplify your expression.*]

**17D Numerical Analysis**

(a) Define the *linear stability domain* of a numerical method to solve the system of equations

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}).$$

What does it mean for the numerical method to be *A-stable*? Determine the linear stability domains for the forward and backward Euler methods, and deduce whether each is A-stable.

(b) What does it mean for a differential equation $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$ to be *stiff*? Illustrate your answer with the example

$$\mathbf{y}' = \mathsf{M}\mathbf{y}, \qquad \mathsf{M} = \begin{pmatrix} -10 & 10 \\ 81 & -91 \end{pmatrix},$$

determining what step size $h$ is required to maintain stability of the forward and backward Euler methods, respectively, applied to this ODE.

(c) For the ODE $\mathbf{y}' = \mathsf{M}\mathbf{y}$ with a general matrix $\mathsf{M}$, use the Milne device with the forward and backward Euler methods to determine a local error estimate, and describe how you would use that estimate to construct a stable integration scheme that achieves a desired tolerance in a reasonably small number of steps.

**18H Markov Chains**

An urn initially contains $m$ green balls and $m + 2$ red balls. A ball is picked at random: if it is green, a red ball is also removed and both are discarded; if it is red, it is replaced together with an extra red and an extra green ball. This is repeated until there are no green balls in the urn. Compute the probability that the process terminates. [Your answer should be a function of $m$.]

[*Standard results can be quoted without proof, provided they are stated clearly.*]

## END OF PAPER