UNIVERSITY OF CAMBRIDGE

MAT1

MATHEMATICAL TRIPOS Part IB

Tuesday, 04 June, 2024 9:00am to 12:00pm

PAPER 1

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most four** questions from Section I and **at most six** questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Linear Algebra

Define the *rank* and *nullity* of a linear map. State and prove the Rank-Nullity Theorem.

Let

$$W = \left\{(x_i)_{i=1}^5 \in \mathbb{R}^5 : x_1 - x_2 + 3x_4 + 5x_5 = x_1 + x_2 + 6x_3 + 4x_4 - 2x_5 = x_1 + 3x_3 + 5x_4 - 4x_5 = 0\right\}.$$

Find $\dim W$ and a basis for W.

2E Geometry

Suppose a closed orientable surface Σ of genus g is obtained by identifying pairs of edges of a $2n$-gon. By considering Euler characteristics, show that $n \geqslant 2g$.

Draw a regular hyperbolic octagon in the Poincaré disc model, indicating edge identifications to produce a genus 2 surface. What are the interior angles of the octagon?

3B Complex Analysis OR Complex Methods

Let $f(z) = \cosh \pi z$. Show that $z \mapsto \zeta = f(z)$ defines a mapping that is conformal from the complex z-plane to the complex ζ-plane, except at certain critical points which you should identify. Find the image in the ζ-plane of the strip

$$S = \{x + iy : 0 < x < \infty,\ 0 < y < 1\},$$

identifying clearly the image of each of the three line segments

$$L_1 = \{x + iy : 0 < x < \infty,\ y = 0\}, \qquad L_2 = \{x + iy : x = 0,\ 0 < y < 1\}$$

and

$$L_3 = \{x + iy : 0 < x < \infty,\ y = 1\}.$$

4C Variational Principles

Describe how to use the method of Lagrange multipliers to extremise a function $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$.

You need to make a rectangular cardboard box. The bottom of the box must be three times thicker than the front and back. The sides of the box must be twice as thick as the front and back. The box has no top. The volume of the box must be equal to three. What are the lengths of the sides and height if you wish to minimise the amount of cardboard used?

5A Numerical Analysis

Let p_n be the real polynomial of degree at most n that interpolates a continuous function $f(x)$ at the $n+1$ distinct points $\{x_0, x_1, \ldots, x_n\}$. Define the *divided difference* $f[x_0, x_1, \ldots, x_k]$.

Starting from the Lagrange formula, prove that p_n satisfies

$$p_n(x) = f(x_0) + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k] \prod_{i=0}^{k-1} (x - x_i),$$

the Newton form. [You need not write an explicit expression for the divided difference.]

Write down (without proof) the recurrence relation for the divided difference. For $n = 2$, draw a diagram that explains how the divided difference $f[x_0, x_1, \ldots, x_k]$ could be computed efficiently. Find the exact number of divisions needed for such computations with $n+1$ distinct points $\{x_0, x_1, \ldots, x_n\}$.

6H Statistics

(a) Define what it means for a statistic to be *sufficient*. State the factorization criterion.

(b) What does it mean for a sufficient statistic to be *minimal sufficient*?

Now suppose X is a single sample from a $N(0, \sigma^2)$ distribution, where σ is the parameter we wish to estimate.

(c) Prove that $|X|$ is a sufficient statistic. Is $|X|$ minimal sufficient?

(d) Suppose we instead have i.i.d. samples $X_1, X_2, \ldots, X_n \sim N(0, \sigma^2)$, where $n \geqslant 2$. Is $\sum_{i=1}^{n} |X_i|$ a sufficient statistic for estimating σ?

[*Standard results can be quoted without proof.*]

7H Optimisation

(a) Find the dual problem of the following linear program:

$$\begin{aligned} \text{minimise} \quad & \sum_{i=1}^{n} a_i x_i \\ \text{subject to:} \quad & \sum_{i=1}^{n} b_i x_i \geqslant t \\ & \sum_{i=1}^{n} c_i x_i \geqslant r \\ & x_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant n, \end{aligned}$$

where $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n, \{c_i\}_{i=1}^n, t$ and r are arbitrary real numbers.

(b) Using part (a) or otherwise, solve the following optimisation problem:

$$\begin{aligned} \text{minimise} \quad & 3x_1 + 2x_2 + 2x_3 - 2x_4 \\ \text{subject to:} \quad & x_1 + x_2 - x_4 \geqslant 7 \\ & x_1 + x_3 - 2x_4 \geqslant 11 \\ & x_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant 4. \end{aligned}$$

You should find the minimum value of the objective function and the vector (x_1, x_2, x_3, x_4) that attains this minimum value.

SECTION II

8G Linear Algebra

(a) Define the *dual space* V^* of a vector space V over a field F. Show that if V is finite-dimensional then so is V^*, and that $\dim V^* = \dim V$.

Let $U \leqslant V$. Define the *annihilator* U° of U. Provide an expression, with proof, for $\dim U^\circ$ in terms of $\dim U$ in the case when V is finite-dimensional. Deduce that if $U \neq V$ then there exists $f \in U^\circ$ such that $f \neq 0$.

Let $\alpha \colon V \to W$ be a linear map between finite-dimensional vector spaces over F. Define the *dual map* $\alpha^* \colon W^* \to V^*$. Prove that $\ker \alpha^* = (\operatorname{im} \alpha)^\circ$ and $\operatorname{im} \alpha^* = (\ker \alpha)^\circ$.

Let V be a finite-dimensional vector space over F and $U \leqslant V$. By considering the quotient map $Q \colon V \to V/U$ and the inclusion map $J \colon U \to V$, show that $\big(V/U\big)^*$ is isomorphic to U°, and that U^* is isomorphic to V^*/U°.

(b) Let V be a vector space over a field F. Let $q_1, \ldots, q_n \in V^*$ be linearly independent. Show that the linear map $Q \colon V \to F^n$ given by $Q(x) = \big(q_j(x)\big)_{j=1}^n$ is surjective. Deduce that if $f \in V^*$ and $\bigcap_{j=1}^n \ker q_j \subseteq \ker f$ then f is in the span of $q_1, \ldots, q_n$.

9E Groups, Rings and Modules

(a) State Sylow's theorems.

(i) Identify the Sylow 2-subgroups and the Sylow 3-subgroups in the symmetric group S_3.

(ii) Identify the Sylow 2-subgroups of S_4.

(iii) Identify the Sylow 2-subgroups of the alternating group A_5.

(b) Let G be a finite group that has no subgroup of index 2. Let P be a Sylow 2-subgroup of G, let H be a subgroup of index 2 in P, and let x be an element of order 2 in G.

(i) Show that x acts as an even permutation on the set of cosets of H in G. Deduce that x must fix some points of this set.

(ii) Deduce that x must be conjugate to some element of H.

10F Analysis and Topology

(a) Define what it means for a metric space (X,d) to be *complete*. Show that a closed subspace Y of a complete metric space (X,d) is complete.

(b) Let (X,d) be a metric space. For non-empty $A \subseteq X$ and $r \geqslant 0$, define the r-*expansion* $E_r(A)$ by $E_r(A) = \bigcup_{a \in A} \overline{B}(a,r)$, where $\overline{B}(a,r)$ is the closed ball of radius r centred at a. Given non-empty $A, B \subseteq X$, is it always true that $B \subseteq E_r(A)$ if and only if $A \subseteq E_r(B)$? Justify your answer.

Let $H(X)$ denote the set of non-empty closed and bounded subsets of X. Given $A, B \in H(X)$, define

$$d_H(A,B) = \inf\{r \geqslant 0 : A \subseteq E_r(B) \text{ and } B \subseteq E_r(A)\}.$$

Show that d_H is a metric on $H(X)$. Would this continue to hold if the word 'closed' were omitted from the definition of $H(X)$? [You may assume that d_H is well defined.]

Show that the function $\theta : X \to H(X)$ defined by $x \mapsto \{x\}$ is a distance-preserving map, and that its image is closed in $H(X)$. Deduce that if $(H(X), d_H)$ is complete, so is (X,d).

11E Geometry

Let $\Sigma \subset \mathbb{R}^3$ be an embedded smooth surface and $\gamma : I \to \mathbb{R}^3$ be a smooth curve contained in Σ, where $I \subset \mathbb{R}$ is an open interval.

(i) Define what it means for γ to be a *geodesic* in Σ, and what it means for a geodesic to be *complete*.

Now suppose Σ_1 and Σ_2 are embedded smooth surfaces in $\mathbb{R}^3$ that are tangent to each other along the image of a curve $\gamma : I \to \mathbb{R}^3$.

(ii) Show that γ is a geodesic in Σ_1 if and only if it is a geodesic in Σ_2.

(iii) Let Σ_1 be the surface parametrised by

$$\sigma(u,v) = \big((2+\cos v)\cos u, (2+\cos v)\sin u, u + \sin v\big).$$

Sketch Σ_1 and, by considering a suitable surface Σ_2, show that the curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ defined by

$$\gamma(t) = (\cos t, \sin t, t)$$

is a geodesic in Σ_1. [You should justify why γ is a geodesic in Σ_2.]

(iv) Show that Σ_1 contains a complete geodesic Γ that is disjoint from this γ.

12F Complex Analysis OR Complex Methods

(a) Define what it means for a holomorphic function on a domain $U \setminus \{a\}$ to have (i) a *removable singularity*, (ii) a *pole of order* k, (iii) an *essential singularity* at $z = a$.

(b) Let f be holomorphic on the punctured unit disc $D^* = D \setminus \{0\}$ such that for all $0 < r < 1$,

$$\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \leqslant 1.$$

Show that f has a removable singularity at $z = 0$.

(c) Let $h(z) = \tan z$.

(i) Classify the singularities of $h(z)$ in $\mathbb{C}$.

(ii) Find the first two terms of the Laurent expansion of $h(1/z)$ around $z_k = \frac{2}{(2k+1)\pi}$, $k \in \mathbb{Z}$.

(iii) Classify the singularities of $\exp(h(1/z))$ in $\mathbb{C}$.

13B Methods

The Green function $G(x, \xi)$ satisfies

$$G'' + \alpha(x)G' + \beta(x)G = \delta(x - \xi) \quad \text{for} \quad 0 < x < 1,$$

with $G'(0, \xi) = G'(1, \xi) = 0$, where primes denote differentiation with respect to x.

Find the function $c(\xi)$ for $0 < \xi < 1$ such that the Green function can be written as

$$G(x, \xi) = \begin{cases} c(\xi) y_1(x) y_2(\xi) & \text{for } 0 < x < \xi \\ c(\xi) y_2(x) y_1(\xi) & \text{for } \xi < x < 1 \end{cases}$$

in terms of linearly independent solutions $y_1(x)$ and $y_2(x)$ of

$$y'' + \alpha(x)y' + \beta(x)y = 0 \quad \text{for} \quad 0 < x < 1$$

that satisfy $y_1'(0) = 0$ and $y_2'(1) = 0$.

Deduce that if $\alpha(x) = 0$ for all x then $G(x, \xi) = G(\xi, x)$.

Find G explicitly for the case $\alpha(x) = 0$ and $\beta(x) = -1$ for all x. Hence solve the equation $y'' - y = x$ on the interval $[0, 1]$ with boundary conditions $y'(0) = 0 = y'(1)$.

14A Quantum Mechanics

Muonium is an atom consisting of an electron of mass m_e and charge e in the potential of an anti-muon (of opposite charge to the electron) of mass $m_\mu \gg m_e$. You may assume that the anti-muon is long-lived enough to form a bound state and that it produces the same force field as a proton.

(i) In what ways (if any) should the quantum mechanical description of the electron in muonium differ to that in the Hydrogen atom? Give reasons for your answer.

(ii) For fixed orbital angular momentum quantum number l, write down the equation satisfied by the radial part of the electron wavefunction $R(r)$. Show that it has solutions of the form

$$R(r) \propto r^l \exp\left(-\frac{r}{a(l+1)}\right),$$

where a is a constant that you should determine. Find the energy.

(iii) Atoms of muonium consistent with such solutions are prepared with energy

$$E = -\frac{e^2}{72\pi\epsilon_0 a}.$$

Show that the average of measurements of the electron-anti-muon spatial separation in a large set of such muonium atoms is equal to ta, where t is a number that you should find.

(iv) Taking one of the prepared muonium atoms with energy E, what is the numerical probability that an immediate measurement of the total orbital angular momentum yields $\hbar$?

[*Hint: the time-independent Schrödinger equation of the Hydrogen atom is*

$$-\frac{\hbar^2}{2m_e r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{2m_e r^2}\hat{L}^2\psi - \frac{e^2}{4\pi\epsilon_0 r}\psi = E\psi.$$

The normalised energy eigenstates of the Hydrogen atom have the form

$$\psi_{lm}(r,\theta,\phi) = R(r)Y_{lm}(\theta,\phi),$$

where Y_{lm} are orbital angular momentum eigenstates satisfying

$$\hat{L}^2 Y_{lm} = \hbar^2 l(l+1)Y_{lm}, \qquad \hat{L}_3 Y_{lm} = \hbar m Y_{lm},$$

where $l = 1,\ 2, \ldots$ and $m = 0,\ \pm1,\ \pm2,\ \ldots, \pm l$.

You may assume that $\int_0^\infty dt\, t^l e^{-t} = l!$ and that $\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi\ |Y_{lm}|^2 = 1$.]

15C Electromagnetism

The vector potential $\mathbf{A}$ is related to the steady current density $\mathbf{J}$ by

$$A_i(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \; \frac{J_i(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \; .$$

Show that this vector potential obeys $\nabla \cdot \mathbf{A} = 0$, stating clearly any assumption that you make.

Show that, far from a localised current, the vector potential can be written as

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{x}}{r^3} + \ldots$$

where $r = |\mathbf{x}|$ and $\mathbf{m}$ is the magnetic dipole moment, which you should define in terms of $\mathbf{J}$.

What are the dimensions of $\mathbf{J}$ and of $\mathbf{m}$? Compute the magnetic dipole $\mathbf{m}$ for:

(i) a circular thin wire, with charge per unit length η and radius R, rotating around the axis of symmetry $\hat{\mathbf{n}}$ that is normal to the plane of the hoop, with angular velocity ω;

(ii) a circular disc with charge per unit area σ and radius R, rotating around the axis of symmetry $\hat{\mathbf{n}}$ that is normal to the plane of the disc, with angular velocity ω.

16D Fluid Dynamics

A layer of fluid of uniform thickness h, density ρ and dynamic viscosity μ flows steadily down a rigid plane that is inclined at angle α to the horizontal. The surrounding air has uniform pressure p_0 but blows upslope, exerting a uniform shear stress τ on the fluid surface.

Write down the equations and boundary conditions describing parallel viscous flow in the fluid layer. Solve these to determine the pressure and velocity fields in the fluid. Hence, determine the surface velocity u_h, the shear stress τ_0 exerted by the fluid on the slope and the volume flux of fluid q per unit width across the plane. Determine how large a shear stress the blowing air must exert to cause (i) the surface velocity to be upslope (ii) the stress on the plane to be upslope (iii) the volume flux to be upslope, and order these measures of flow reversal by the magnitude of shear stress required.

17A Numerical Analysis

Suppose that the real orthogonal matrix $\Omega^{[p,q]} \in \mathbb{R}^{m\times m}$ with $1 \leqslant p < q \leqslant m$ is a Givens rotation with rotation angle θ. Write down the form of $\Omega^{[p,q]}$.

Show that for any matrix $A \in \mathbb{R}^{m\times m}$ and for any given j it is possible to choose θ such that the matrix $\Omega^{[p,q]}A$ satisfies $(\Omega^{[p,q]}A)_{qj}=0$, where $1 \leqslant j \leqslant m$.

Let

$$A = \begin{pmatrix} \sqrt{2} & 0 & \frac{1-\sqrt{3}}{\sqrt{2}} \\ 0 & \sqrt{3} & 1 \\ \sqrt{2} & \sqrt{2} & \frac{1+\sqrt{3}}{\sqrt{2}} \end{pmatrix}.$$

By applying the product $\Omega^{[p,q]}\Omega^{[p',q']}$ of two Givens rotations (picking appropriate values for p, p', q and q', with $p' < p$), find a factorisation of the matrix $A \in \mathbb{R}^{3\times 3}$ of the form $A = QR$, where $Q \in \mathbb{R}^{3\times 3}$ is an orthogonal matrix, $R \in \mathbb{R}^{3\times 3}$ is an upper triangular matrix for which the leading non-zero element in each row is positive, and every entry of Q and R is written as a rational number multiplied by the square root of an integer.

18H Statistics

Suppose X_1 and X_2 are i.i.d. from a Uniform$(\theta, \theta+1)$ distribution. For testing

$$H_0 : \theta \leqslant 0 \quad \text{against} \quad H_1 : \theta > 0,$$

consider the following two tests:

Test 1: Reject H_0 if $X_1 > 0.95$;
Test 2: Reject H_0 if $X_1 + X_2 > C$.

(a) Derive a formula for the probability density function of $X_1 + X_2$ and plot the function.

(b) Find the value of C so that Test 2 has the same size as Test 1.

(c) Calculate the power of each test as a function of θ.

(d) Is either Test 1 or Test 2 uniformly most powerful for testing the hypotheses? Justify your answer.

19H Markov Chains

Suppose $\{X_n\}_{n\geqslant 0}$ is a Markov chain such that there exists a pair (i,j) of distinct states that are "symmetric" in the sense that

$$\mathbb{P}(T_j < T_i \mid X_0 = i) = \mathbb{P}(T_i < T_j \mid X_0 = j),$$

where $T_i = \min\{n \geqslant 1 : X_n = i\}$. Denote this common conditional probability by α. Suppose $X_0 = i$, and let N denote the number of visits to j before the chain revisits i.

(a) Compute $\mathbb{E}[N]$.

(b) For each $k \geqslant 0$, compute $\mathbb{P}(N = k)$ as a function of α.

(c) Prove that if an irreducible Markov chain has an invariant distribution π, two states i and j are symmetric if and only if $\pi(i) = \pi(j)$.

[*Standard results can be quoted without proof.*]

END OF PAPER