UNIVERSITY OF CAMBRIDGE

MAT0

MATHEMATICAL TRIPOS Part IA

Wednesday, 05 June, 2024 1:30pm to 4:30pm

PAPER 4

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **all four** questions from Section I and **at most five** questions from Section II. Of the Section II questions, no more than three may be on the same course.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Numbers and Sets

State and prove Wilson's theorem. State Fermat's little theorem.

Calculate the residue of $28! \cdot 7^{29}$ modulo 31.

2E Numbers and Sets

What is a *relation* on a set S? What is an *equivalence relation*?

For each of the following relations determine whether or not $\sim$ defines an equivalence relation on $\mathbb{R}$:

(i) $x \sim y$ if and only if $xy > 0$;

(ii) $x \sim y$ if and only if $xy \geqslant 0$;

(iii) $x \sim y$ if and only if $x - y \in \mathbb{Z}$;

(iv) $x \sim y$ if and only if $x - y \geqslant 0$.

For any cases (i)-(iv) where $\sim$ is an equivalence relation, describe the set of equivalence classes.

3C Dynamics and Relativity

(a) State the *parallel axis theorem* for a rigid body. Now consider rigid bodies A and B and let C denote the rigid body obtained by connecting A to B with a massless rod. For $T = A, B, C$, denote by M_T the mass, by $\mathbf{x}_T$ the centre of mass and by $I_T(\mathbf{x})$ the moment of inertia about an axis passing through $\mathbf{x}$ in a given fixed direction $\hat{\mathbf{d}}$. Derive an expression for M_B in terms of $I_A(\mathbf{x}_A)$, $I_B(\mathbf{x}_B)$, $I_C(\mathbf{x}_C)$, $\mathbf{x}_A$, $\mathbf{x}_B$, $\mathbf{x}_C$, and M_C.

(b) For a two-dimensional rigid body lying in the (x, y) plane, prove the *perpendicular axis theorem*, namely the relation $I_z = I_x + I_y$, where I_x, I_y and I_z are the moments of inertia about the x, y and z axes. Using this relation or otherwise, compute I_x, I_y and I_z for a uniform-density ellipse of mass M and semi-major and semi-minor axes a and b.

4C Dynamics and Relativity

Consider three distinct events A, B and C in 1+1 spacetime dimensions.

(a) For each of the following statements provide an explicit algebraic proof that they always hold or a counterexample:

(i) If B is in the future lightcone of A and C is in the future lightcone of B then C is in the future lightcone of A.

(ii) If B is in the future lightcone of A and C is in the future lightcone of A then C is in the future lightcone of B.

(iii) If B is in the future lightcone of A and C is in the past lightcone of A then B is in the future lightcone of C.

(iv) If A and B are lightlike separated and A and C are lightlike separated then B and C are lightlike separated.

(b) Assume that B is in the future lightcone of A, that C is in the future lightcone of B and that $|x_C - x_A| < |x_C - x_B|$ in some inertial frame S. Prove that there exists an inertial frame S' in which $|x'_C - x'_A| > |x'_C - x'_B|$.

SECTION II

5E Numbers and Sets

State and prove the Fermat–Euler theorem.

For a $(k+1)$-digit positive integer written in base 10,

$$n = a_k a_{k-1} \cdots a_0$$

with each $a_i \in \{0, 1, 2, \ldots, 9\}$ and $a_k \neq 0$, a *cyclic permutation of* n is a number of the form

$$c_j(n) = a_{k-j} a_{k-j-1} \cdots a_0 a_k \cdots a_{k-j+1} \text{ with } j = 0, \ldots, k.$$

For example

$$c_0(120) = 120, c_1(120) = 201 \text{ and } c_2(120) = 012 = 12.$$

For each natural number $d > 1$, a number n is *d-cyclic-divisible* if every cyclic permutation of n is a multiple of d.

(i) Show that every multiple of 3 is 3-cyclic-divisible.

(ii) Show that if $d = 2$ or 5, n is d-cyclic-divisible if and only if every digit of n is a multiple of d.

(iii) Show that if n is 7-cyclic-divisible then either all its digits are equal to 7 or its number of digits is a multiple of 6.

(iv) Suppose that $p > 7$ is a prime. Show that $p - 1$ has a factor l such that, if the number of digits of n is a multiple of l, then n is p-cyclic-divisible if and only if n is a multiple of p.

6E Numbers and Sets

The *Fibonacci numbers* are defined for all non-negative integers by

$$F_0 = 0, \ \ F_1 = 1, \ \ F_{n+1} = F_n + F_{n-1} \text{ for all } n \geqslant 1.$$

Prove the following properties of the Fibonacci numbers by induction:

(i) $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ for all $n \geqslant 1$;

(ii) $F_{n+l}F_{n+m} - F_nF_{n+m+l} = (-1)^n F_m F_l$ for all $n, m, l \geqslant 0$.

Deduce that if $j \geqslant k \geqslant 0$ then

$$(F_{j+k} - F_{j-k})(F_{j+k} + F_{j-k}) = F_{2k}F_{2j}$$

and

$$F_{j+k+1}^2 + F_{j-k}^2 = F_{2k+1}F_{2j+1}.$$

7E Numbers and Sets

Given a non-empty bounded subset S of $\mathbb{R}$, what is its *supremum* $\sup S$?

What does it mean to say that a sequence (x_n) of real numbers *converges*?

(a) (i) Show that an increasing sequence of real numbers converges if and only if it is bounded.

(ii) Does the sequence $\left(12 - \frac{1000n^2+4}{1.1^n}\right)$ converge? Carefully justify your answer from first principles.

(b) Suppose that S and T are non-empty bounded subsets of $\mathbb{R}$.

(i) If $S + T = \{s + t : s \in S, t \in T\}$, then must $\sup(S + T) = \sup S + \sup T$?

(ii) If $ST = \{st : s \in S, t \in T\}$, then must $\sup ST = (\sup S)(\sup T)$?

(iii) If all the elements of S are positive and $S^T = \{s^t : s \in S, t \in T\}$, then must $\sup\left(S^T\right) = (\sup S)^{(\sup T)}$?

8E Numbers and Sets

(a) What does it mean to say a set is *countable*?

Show that a countable union of countable sets is countable. Deduce that $\mathbb{Q}$ is countable. Show that if A and B are countable then $A \times B$ is countable.

Show that $\mathbb{R}$ is not countable.

(b) A *line* in $\mathbb{R}^2$ is a subset of the form

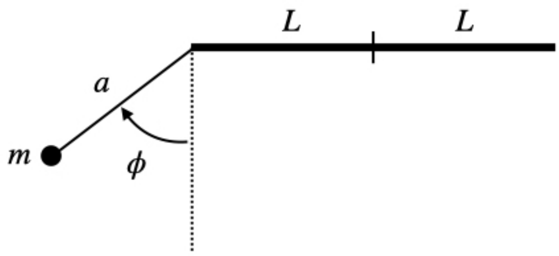
$$\{(x,y) \in \mathbb{R}^2 : ax + by + c = 0\}$$

for some $a, b, c \in \mathbb{R}$ with $(a,b) \neq (0,0)$.

(i) Must a collection of lines whose intersection is non-empty be countable? Justify your answer carefully.

(ii) Must a collection of lines each of which contains at least two distinct elements of $\mathbb{Q}^2$ be countable? Justify your answer carefully.

9C Dynamics and Relativity



(a) Consider a rigid rod of length $2L$ spinning on the horizontal plane about its centre at a constant angular velocity $\boldsymbol{\omega} = \omega \hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is an upward unit vector. A second rod of length a, which is light, connects one end of the first rod to a particle of mass m. The rods lie in the same vertical plane and the second rod makes an angle ϕ with the downward vertical direction $-\hat{\mathbf{z}}$, as in the figure. Compute the constant angular velocity ω for which ϕ takes a constant value ϕ_1. Check that your result is dimensionally correct, and briefly discuss the dependence of the result on m and L.

(b) The second rod is now substituted by a spring of rest length a, which at displacement from rest x has potential energy $V = (k/2)x^2$. Assuming again that ϕ takes a constant value ϕ_2, solve for x and hence compute the constant angular velocity ω in terms of ϕ_2 and the parameters of the problem. Contrast this result with the one obtained in part (a).

10C Dynamics and Relativity

(i) Consider a particle of mass m moving in a potential V. State which standard quantity is conserved for a central potential $V = V(r)$. Hence, starting from the radial equation of motion in polar coordinates,

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{dV}{dr} \, ,$$

derive the orbit equation for the variable $u = 1/r$.

(ii) For the potential $V = -km/r$, where k is a positive constant, solve the orbit equation and show that the integration constants can be chosen such that the orbit satisfies the equation of an ellipse in Cartesian coordinates.

(iii) Evaluate the orbit equation for the case of a potential $V = -km/(r - r_0)$, where k and r_0 are positive constants and $r > r_0$. Discuss the existence of circular orbits and their stability.

(iv) Expand the above orbit equation up to first order in small r_0. Show that this expanded equation has a solution of the form

$$u(\theta) = X\left[1 + A\cos(\omega\theta)\right] \, ,$$

for any $0 < A < 1$, where ω and X are parameters you should determine. Hence compute the angle between two successive periapses.

11C Dynamics and Relativity

(a) For a collection of particles of masses m_i and positions $\mathbf{x}_i$, define the *centre of mass* $\mathbf{R}$, the *total momentum* $\mathbf{P}$ and the *total angular momentum* $\mathbf{L}$ about the origin. Prove that

$$\mathbf{L}_{\mathrm{CoM}} = \mathbf{L} - \mathbf{R} \times \mathbf{P} \,,$$

where $\mathbf{L}_{\mathrm{CoM}}$ is the total angular momentum about the centre of mass.

(b) (i) At some initial time $t = 0$, particles with a total mass of m form a circle of radius R in the (x, y) plane at $z = 0$ and have a velocity v tangential to the circle. Compute the moment of inertia I of the system.

(ii) At later times, the particles will form a circle of radius $R(t)$. Compute $R(t)$ and the total angular momentum with respect to the origin for all times, and hence deduce the angle $\theta(t)$ by which the circle has rotated at time t.

(c) (i) A disc of radius R and negligible mass is filled uniformly with a mass m_0 of water. Compute the moment of inertia I of the disc.

(ii) As the disc spins, with angular velocity $\omega(t)$, from each point of the boundary of the disc water is sprayed out in the tangential direction to the boundary at a relative velocity u. By mimicking the derivation of the rocket equation, or otherwise, derive a first order differential equation in time for the angular velocity $\omega(t)$ of the disc that depends on R, u and the mass of water $m(t)$ that remains inside the disc at time t. [You may assume that the remaining water $m(t)$ is always uniformly distributed inside the disc.]

(d) Assume now that water is leaking out at zero relative velocity, $u = 0$, and at a constant rate $\dot{m}(t) = -\mu$. Compute the time it takes for the disc to stop spinning from an initial angular velocity ω_0.

12C Dynamics and Relativity

(a) Write down the relativistic four-momenta of a massive and a massless particle. For the decay of particle 1 into particles 2 and 3 (all massive), where particle i has mass m_i and energy E_i, use relativistic invariants to compute all energies in the frame where $E_1 = m_1 c^2$. Derive the condition on the masses for the decay to be kinematically allowed.

(b) Now consider a particle Q of mass m and $N+1$ different types of particle, $R_0, \ldots, R_N$, of masses $(nM + m_0)$, for $n = 0, 1, \ldots, N$ respectively, where $m_0 < m$ and $M = \lambda m$ with $\lambda > 1$. For each $n > 0$, particle R_n decays into particle Q and particle R_{n-1}, as long as this is kinematically allowed. If we start with one R_N particle, what is the end product after all allowed decays have happened, and how much rest mass has been converted into energy?

(c) Consider the relativistic scattering of particles A and B into C and D. Write down the relativistic kinematic invariants that can be written as bilinear functions of the 4-momenta P_a^μ for $a = A, B, C, D$. Express each kinematic invariant in terms of the masses m_a and of the following variables

$$s = (P_A + P_B)^2\,, \qquad t = (P_A - P_C)^2\,, \qquad u = (P_A - P_D)^2\,.$$

Show that the sum $s + t + u$ depends only on a certain combination of the masses.

END OF PAPER