UNIVERSITY OF CAMBRIDGE

MAT0

MATHEMATICAL TRIPOS Part IA

Monday, 3 June, 2024 9:00am to 12:00pm

PAPER 3

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **all four** questions from Section I and **at most five** questions from Section II. Of the Section II questions, no more than three may be on the same course.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1D Groups

Prove that every Möbius map sends circles and straight lines to circles and straight lines. [You may use any statement from the course about generating sets for the group of Möbius maps, provided that you state it precisely.]

Is the subgroup of the Möbius maps consisting of those that send circles to circles a normal subgroup?

2D Groups

(i) Define the groups $SO(n)$. Prove that every element of $SO(2)$ is a rotation about the origin, and that every element of $SO(3)$ is a rotation about some axis. [You may assume simple facts about orthogonal maps and rotations and reflections, but if you wish to quote any statement of the form 'these elements generate this group' then you must prove it.]

(ii) Explain why $SO(2)$ has a subgroup isomorphic to $\mathbb{Z}$. Does it have a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$? Justify your answer.

3B Vector Calculus

Let $\mathbf{F}$ and $\mathbf{G}$ be smooth vector fields in $\mathbb{R}^3$.

(i) Define the divergence $\nabla \cdot \mathbf{F}$ and the curl $\nabla \times \mathbf{F}$ in the standard Cartesian basis $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$.

(ii) Prove the following identities:

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}),$$
$$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}.$$

[You may use standard properties of the antisymmetric tensor ε_{ijk}.]

(iii) Prove that the identity $\mathbf{F} \cdot (\nabla \times \mathbf{G}) = \nabla \cdot (\mathbf{G} \times \mathbf{F})$ is true or find a counterexample.

(iv) Compute $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ in the case that $\mathbf{F} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$.

4B Vector Calculus

Define the *curvature* κ of a curve $\boldsymbol{\gamma}$ in $\mathbb{R}^3$ and describe its geometric significance. Determine the curvature for the curve

$$\boldsymbol{\gamma}(t) = (\cos(2t), \sqrt{5}\, t, \sin(2t)), \quad t \in [0, \pi]$$

as a function of its arclength (starting from the initial point $\boldsymbol{\gamma}(0)$). What is the integral of the curvature over the curve? Without performing any further computations, write down the integral of the curvature for the curve

$$\widetilde{\boldsymbol{\gamma}}(t) = (\cos(2t), 0, \sin(2t)), \quad t \in [0, \pi]$$

and explain why your result is different from the result for $\boldsymbol{\gamma}$.

SECTION II

5D Groups

(i) Define the *sign* of a permutation $\sigma \in S_n$, explaining why it is well-defined. Show also that it gives a homomorphism from S_n to $\{\pm 1\}$.

(ii) Prove that A_5 is simple. [You may assume facts about conjugacy classes in A_5, provided that you state them precisely.]

(iii) Show that there is no surjective homomorphism from A_5 to $\{\pm 1\}$.

(iv) Is there a surjective homomorphism from $A_5 \times A_5$ to $\{\pm 1\}$? Justify your answer.

6D Groups

(i) A group G is called n-*dicyclic*, where $n > 1$, if it is generated by elements a and b such that a has order $2n$ and $b^2 = a^n$ and $bab^{-1} = a^{-1}$. Prove that such a group must have order $4n$. Show that, for each n, there exists an n-dicyclic group. [Hint: find it as a subgroup of $GL_2(\mathbb{C})$, choosing a as a suitable diagonal matrix.]

(ii) Find 5 pairwise non-isomorphic groups of order 12, explaining carefully why your groups are non-isomorphic.

7D Groups

(i) State and prove Cayley's theorem.

(ii) State and prove Cauchy's theorem.

(iii) For each $n > 1$ that is a prime power, exhibit a group G of order n such that G is not a subgroup of S_{n-1}.

(iv) Let G be a finite group of order n, where $n > 1$ is *not* a prime power. Show that G is a subgroup of S_{n-1}. [Hint: for two suitable subgroups H and K of G, consider the standard actions of G on the left cosets of H and of K.]

8D Groups

(a) State and prove the orbit-stabiliser theorem for a finite group.

(b) (i) Let G be the group of all isometries of a cube in $\mathbb{R}^3$. By considering the action of G on the vertices of the cube, show that $|G| = 48$. To which standard group is the stabiliser of a vertex isomorphic?

(ii) Now consider the action of G on the edges of the cube. How large is the stabiliser of an edge? To which standard group is it isomorphic?

(iii) Now consider the action of G on the main diagonals of the cube. How large is the stabiliser of a main diagonal? To which standard group is it isomorphic? Is this action faithful?

(iv) Are every two elements of G of order 3 conjugate? Are every two elements of G of order 2 conjugate? Justify your answers.

[*Throughout this question you may assume standard properties of isometries, rotations, reflections, etc.*]

9B Vector Calculus

(a) Consider a bounded volume V in $\mathbb{R}^3$ with smooth surface $\partial V = S$. Show that if $f(x, y, z)$ and $g(x, y, z)$ are smooth functions then there exists at most one smooth function $\phi(x, y, z)$ that satisfies $\nabla^2\phi = f$ in V and the Dirichlet boundary condition $\phi = g$ on S. If we change the boundary condition to a Neumann boundary condition $\hat{\mathbf{n}} \cdot \nabla\phi = g$ on S, can there be more than one solution? (Here $\hat{\mathbf{n}}$ denotes the outward unit normal to the surface S.)

(b) Now suppose that V, f and g all have spherical symmetry. Argue briefly why any solution to the Dirichlet problem must be spherically symmetric.

(c) Compute the unique solution to the Dirichlet problem in each of the following cases:

(i) V is the volume $r \leqslant 1$, $f(r) = (r - 1)e^r$ and $g(r) = 3$.

(ii) V is the unbounded volume $r \geqslant 1$, $f(r) = -1/r^3$, $g(r) = 1$ and $\phi(r) \to 0$ as $r \to \infty$.

(iii) V is the volume $r \leqslant 1$, $f(r) = 1/(r^2 + 1)$ and $g(r) = -1$.

10B Vector Calculus

State the divergence theorem.

Let S be the surface formed by the part of the paraboloid $z = 1 - x^2 - y^2$ lying above the xy-plane and let $\mathbf{F}(x, y, z) = (x^2, -y^2, z^4)$. Calculate the flux of $\mathbf{F}$ across S (in the upward direction). Do this in two ways:

(i) By direct calculation of a surface integral.

(ii) By computing the flux over a simpler surface and applying the divergence theorem.

[*Hint: Use cylindrical polar coordinates for the surface integral.*]

11B Vector Calculus

(i) Prove that if ϕ is a smooth scalar field on $\mathbb{R}^3$ then

$$\nabla \times \nabla\phi = 0.$$

(ii) Prove that if $\mathbf{A}$ is a smooth vector field on $\mathbb{R}^3$ then

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0.$$

(iii) State Stokes' theorem. State what it means for a vector field defined on some region to be conservative. Prove that if a vector field $\mathbf{F}$ on $\mathbb{R}^3$ can be written as the gradient of a function then it is conservative.

(iv) Consider the following vector fields on $\mathbb{R}^3 \backslash \{x^2 + y^2 = 0\}$:

$$\mathbf{B}_1(x, y, z) = \left(2xe^{x^2}\cos(yz), -ze^{x^2}\sin(yz), -ye^{x^2}\sin(yz)\right)$$
$$\mathbf{B}_2(x, y, z) = \left(-y/(x^2 + y^2), x/(x^2 + y^2), 0\right)$$
$$\mathbf{B}_3(x, y, z) = \left(2x/(x^2 + y^2 + z^2), 2y/(x^2 + y^2 + z^2), 2z/(x^2 + y^2 + z^2)\right)$$

For each of these vector fields, compute the line integral around the curve C, where C is the closed curve $s \mapsto (\cos(s), \sin(s), 0)$, $s \in [0, 2\pi]$. Which of these vector fields are conservative and which can be written as gradients of functions? For those that can be written as gradients of functions, give a suitable potential function.

12B Vector Calculus

For this question, all tensors are in $\mathbb{R}^3$.

(i) Define a *rank n tensor.*

(ii) Define what it means for a tensor to be *totally antisymmetric*. For each integer $n \geqslant 2$, find all the totally antisymmetric rank n tensors.

(iii) Define what it means for a tensor to be *isotropic* and state the general form of isotropic rank 4 tensors.

(iv) Prove that

$$\varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}$$

and find $\varepsilon_{ijk}\varepsilon_{ijk}$.

(v) Find an isotropic rank 4 tensor (not identically zero) that can be written as a contraction of two antisymmetric tensors. Write down a (not identically zero) isotropic rank 5 tensor. Show that the most general isotropic rank 5 tensor must have at least ten independent components.

END OF PAPER