UNIVERSITY OF CAMBRIDGE

MAT0

MATHEMATICAL TRIPOS Part IA

Friday, 31 May, 2024 1.30pm to 4:30pm

PAPER 2

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **all four** questions from Section I and **at most five** questions from Section II. Of the Section II questions, no more than three may be on the same course.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1A Differential Equations

(a) Find all solutions of the differential equation for $y(x)$

$$xyy'' - xy'^2 = yy'.$$

[Hint: you may find the substitution $z(x) = y'(x)/y(x)$ helpful.]

(b) For $n \neq 0, 1$, show that the substitution $z = y^{1-n}$ transforms the differential equation for $y(x)$

$$y' + P(x)y = Q(x)y^n$$

into a linear differential equation that you should state explicitly.

Hence, or otherwise, solve the differential equation for $x(t) > 0$

$$\frac{1}{\sqrt{x}}\dot{x} = 2te^{-t^3} - 6t^2\sqrt{x}$$

subject to the condition $x(1) = 4$.

2A Differential Equations

A real-valued function $f(x)$ is differentiable on some interval $(-a, a)$. For any $x, y \in (-a, a)$ such that $x + y \in (-a, a)$, the equality

$$f(x+y) = \frac{f(x) + f(y)}{1 - f(x)f(y)}$$

holds.

(i) Show that $f(0) = 0$.

(ii) By considering the definition of the derivative as a limit, or otherwise, show that there exists a number C such that $f'(x) = C(1 + f^2(x))$ everywhere on the interval $(-a, a)$.

(iii) Hence find the most general form of $f(x)$. Also, find $f(x)$ that satisfies $f'(0) = 2$.

3F Probability

(a) State Markov's inequality. Prove that for any random variable X and any $t > 0$,

$$\mathbb{P}(X \geqslant x) \leqslant e^{-tx} M_X(t)\,,$$

where $M_X(t) = \mathbb{E}(e^{tX})$ is the moment generating function of X.

(b) Let $X_1, X_2, \ldots, X_n$ be i.i.d. Poisson random variables with mean 1. Let $S = X_1 + \cdots + X_n$.

(i) Compute the moment generating function of S. Find the distribution of S.

(ii) Prove that

$$\mathbb{P}(S \geqslant 2n) \leqslant (e/4)^n\,.$$

[*You may use the fact that the moment generating function $M_X(t)$ of a Poisson random variable X with mean λ is $e^{\lambda(e^t-1)}$.*]

4F Probability

Let (X_1, X_2) have a bivariate normal distribution with $\mathbb{E}(X_i) = \mu_i, \mathrm{var}(X_i) = \sigma_i^2$ for $i = 1, 2$ and $\mathrm{corr}(X_1, X_2) = \rho$.

(a) Write down the joint probability density function of (X_1, X_2).

(b) Find the conditional probability density function of $X_1 | X_2$.

(c) If $\sigma_1 = \sigma_2 = \sigma$, show that $X_1 + X_2$ and $X_1 - X_2$ are independent random variables. Find their distributions.

SECTION II

5A Differential Equations

Let $\varphi_1(x)$ and $\varphi_2(x)$ be non-trivial solutions of the equations

$$\varphi_1'' + q_1(x)\varphi_1 = 0$$

and

$$\varphi_2'' + q_2(x)\varphi_2 = 0,$$

where $q_1(x)$ and $q_2(x)$ are continuous functions such that $q_1(x) \leqslant q_2(x)$ for all x.

(i) Let x_1 and x_2 with $x_1 < x_2$ be consecutive zeroes of φ_1. By considering

$$\int_{x_1}^{x_2} (q_1(x) - q_2(x))\varphi_1(x)\varphi_2(x)\, dx$$

or otherwise, show that if both $\varphi_1(x)$ and $\varphi_2(x)$ are strictly positive on (x_1, x_2) then $q_1(x) \equiv q_2(x)$ on (x_1, x_2).

(ii) Hence prove that between any two consecutive zeroes x_1 and x_2 of $\varphi_1(x)$, there exists at least one zero of $\varphi_2(x)$, unless $q_1(x) \equiv q_2(x)$ on (x_1, x_2).

(iii) Hence show that any solution of the equation

$$y'' + (2 + \cos 3x)y = 0$$

has at least one zero on the interval $[-1, \pi - 1]$.

(iv) Show that each non-trivial solution of the equation

$$\sqrt{1 + x^3}\, y'' + y = 0$$

has at most one zero on the interval $[2, 6]$.

6A Differential Equations

Define the *generating function* $G(x)$ for a difference equation $F(u_n, u_{n-1}, \ldots, u_0) = 0$ as

$$G(x) = u_0 + u_1 x + u_2 x^2 + \cdots .$$

(a) Consider the difference equation $u_n + u_{n-1} - 6u_{n-2} = n$ for $n \geqslant 2$. Find the solution of this equation, given $u_0 = 0, u_1 = 2$.

Show that

$$(1 + x - 6x^2)G(x) = x + \frac{x}{(1-x)^2}$$

and use this expression to find the power series expansion of $G(x)$. Verify that this expansion is consistent with the u_n determined directly above.

[Hint: it may be helpful to note that $1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}$.]

(b) Find the generating function $G(x)$ for the difference equation

$$u_n - 2u_{n-1} = \left\lfloor \frac{n}{2} \right\rfloor, \quad n \geqslant 1, \quad u_0 = 1,$$

where $\lfloor \frac{n}{2} \rfloor$ is the greatest integer less than or equal to $\frac{n}{2}$. Hence solve this equation.

7A Differential Equations

Consider the following system of equations involving two functions, $x(t)$ and $y(t)$:

$$\begin{aligned}\dot{x} &= y + kx(x^2+y^2),\\ \dot{y} &= -x + ky(x^2+y^2),\end{aligned}$$

where k is a constant.

(i) Show that there exists a function $F(x(t), y(t))$ (which you should state explicitly in terms of $x(t)$ and $y(t)$) such that

$$\frac{dF}{dt} = 2kF^2.$$

Solve this equation assuming that $F = 1$ at $t = 0$.

(ii) Find the equilibrium point of this system and show that the linearised system has a centre at this point. Taking into account the nonlinear terms, deduce for which values of k this equilibrium point is stable, and why. Do the trajectories rotate clockwise or anticlockwise as t increases, and why?

(iii) By changing variables to polar coordinates, via $x(t) = r(t)\cos\theta(t)$ and $y(t) = r(t)\sin\theta(t)$, find $f(r)$ and $g(\theta)$ such that

$$\begin{aligned}\dot{r} &= f(r),\\ \dot{\theta} &= g(\theta).\end{aligned}$$

Integrate these equations to find $r(t)$ and $\theta(t)$ if $r(0) = 1$ and $\theta(0) = 0$.

(iv) Now the system is modified to:

$$\begin{aligned}\dot{x} &= y + x - 2x(x^2+y^2),\\ \dot{y} &= -x + y - 2y(x^2+y^2).\end{aligned}$$

At $t = 0$, the system is at a point on the circle $x^2 + y^2 = 1$. Determine $x^2 + y^2$ as a function of t. Find $\lim_{t\to\infty}(x^2 + y^2)$.

8A Differential Equations

The dynamics of a gas is described by a partial differential equation for the complex function $\psi(x,t) = \sqrt{\rho(x,t)}\exp[iS(x,t)]$ as

$$-2i\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} + (1 - |\psi|^2)\psi. \qquad (*)$$

Let $v(x,t) = \partial S/\partial x$.

(i) Determine the real-valued equations describing the gas dynamics in terms of ρ and v as

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= \frac{\partial A}{\partial x},\\ \frac{\partial v}{\partial t} &= \frac{\partial B}{\partial x} + \frac{\partial C}{\partial x},\end{aligned}$$

where you should specify the functions A that depends only on ρ and v, B that depends only on ρ and its derivatives in x, and C that depends only on v.

(ii) Write down the ordinary differential equation for $a(x) = \sqrt{\rho(x,t)}$ in the case of a stationary gas ($S =$ constant). What is the constant solution $a(x) = d > 0$ of this equation?

(iii) There are solutions of $(*)$ of the form $\psi(x,t) = \psi_0(\xi)$ where $\xi = x - Ut$ with U a constant satisfying $0 \leqslant U < \frac{1}{\sqrt{2}}$. Determine the ordinary differential equation that $\psi_0(\xi)$ satisfies if it is known that $\mathrm{Im}(\psi_0(\xi)) = \sqrt{2}U$ for all ξ and $|\psi_0(\xi)| \to d$ as $\xi \to \pm\infty$.

(iv) Plot $|\psi_0(\xi)|^2$ for the solutions when $U = 0$ and when $U = 1/2$ as a function of ξ, and discuss how these solutions evolve in time.

9F Probability

Let $S_1, S_2, \ldots$ be independent exponential random variables with means $\mathbb{E}(S_i) = 1/q_i$ for $i = 1, 2, \ldots$. Let $T = \min\{S_1, S_2, \ldots, S_n\}$ and let K be the value of i for which $S_i = T$.

(i) Find $\mathbb{P}(K = k, T \geqslant t)$ for $k \in \{1, 2, \ldots, n\}, t \geqslant 0$.

(ii) Find the distributions of the random variables K and T. Show that K and T are independent.

Now assume that $q_i = 1$ for all $i = 1, 2, \ldots$.

(iii) Show that for all $n \geqslant 1$, the probability density function of $X_n = \sum_{i=1}^{n} S_i$ is given by

$$f(x) = \frac{x^{n-1}}{(n-1)!} e^{-x}, \quad x > 0.$$

(iv) Let N be a geometric random variable independent of the sequence $S_1, S_2, \ldots$, with $\mathbb{P}(N = n) = p(1-p)^{n-1}$ for $n = 1, 2, \ldots$. Define

$$Y = \sum_{i=1}^{N} S_i .$$

Find $\mathbb{E}(e^{\theta Y})$ for $\theta < p$. Hence or otherwise, find the distribution of Y.

[*You may use the fact that the moment generating function $M_X(t)$ of an exponential random variable X with mean $1/\lambda$ is $\lambda/(\lambda - t)$ for $t < \lambda$.*]

10F Probability

A fair n-sided die is rolled repeatedly so that each roll is independent. We say a *match* occurs if the face i appears on the i-th roll.

(i) Find the probability p_n that at least one match occurs in the first n rolls. What is the value of $\lim_{n\to\infty} p_n$?

Now let T_n be the minimum number of rolls required until all the n faces have appeared at least once.

(ii) Show that T_n is the sum of n independent geometric random variables.

(iii) Find the expectation $\mathbb{E}(T_n)$.

(iv) Find the variance $\text{var}(T_n)$. Show that $\text{var}(T_n) \leqslant Cn^2$ where $C = \sum_{i=1}^{\infty} i^{-2}$.

[You may use the fact that the variance of a geometric random variable of parameter p is $(1-p)/p^2$.]

(v) Show that for any $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{T_n}{n\log n} - 1\right| > \varepsilon\right) = 0\,.$$

[You may use the fact that $\sum_{i=1}^{n} i^{-1}/\log n \to 1$ as $n \to \infty$. You may use standard inequalities from lectures if you state them clearly.]

11F Probability

Let $(S_n : n \geqslant 0)$ be a simple random walk on $\mathbb{Z}$ with $S_0 = 0$ and $\mathbb{P}(S_n - S_{n-1} = 1) = p$ and $\mathbb{P}(S_n - S_{n-1} = -1) = q = 1 - p$ for all $n \geqslant 1$.

(i) Find the distribution of S_n.

(ii) Find b_n, c_n so that

$$\mathbb{P}\left(\frac{S_n - b_n}{c_n} \leqslant x\right) \to \Phi(x)$$

as $n \to \infty$, where Φ is the standard normal distribution function.

[You may quote standard results from lectures.]

From now on, assume that $p = q = 1/2$.

(iii) Let T be the random number of steps taken by the random walk until it first hits $-a$ or b for some $a, b \in \mathbb{N}$. Find $\mathbb{E}(T)$.

(iv) Let V_n be the number of visits to the origin until time n, that is, $V_n = |\{0 \leqslant i \leqslant n : S_i = 0\}|$. Using Stirling's formula or otherwise, prove that there exists some $c > 0$ such that

$$\mathbb{E}(V_{2n}) \geqslant c\sqrt{n}$$

for all n.

12F Probability

A *graph* on a set V is a set of some unordered pairs of (distinct) elements of V: we call these the *edges* of the graph and the elements of V are called the *vertices*. The *degree* of a vertex is the number of edges that contain it.

We form a random graph with n vertices $v_1, v_2, \ldots, v_n$ by including the edge $v_i v_j$ with probability p for all $i \neq j$ independently.

(i) Find the distribution of the degree of the vertex v_i.

We call a vertex *isolated* if its degree is 0. Let N be the number of isolated vertices.

(ii) Find the expectation $\mathbb{E}(N)$.

(iii) Let $p = c \log n / n$. Show that if $c > 1$, then $\mathbb{P}(N = 0) \to 1$ as $n \to \infty$.

(iv) Show that if p is such that $\operatorname{var}(N)/\mathbb{E}(N)^2 \to 0$ as $n \to \infty$, then $\mathbb{P}(N = 0) \to 0$ as $n \to \infty$.

(v) Find $\mathbb{E}(N^2)$. Now let $p = c \log n / n$ with $c < 1$. Show that $\mathbb{P}(N = 0) \to 0$ as $n \to \infty$.

[*You may want to use the inequalities* $e^{-x} \geqslant 1 - x$ *for all* x*; and for any* $\alpha > 1$, $e^{-\alpha x} \leqslant 1 - x$ *for all* $x \geqslant 0$ *small enough (depending on* α*). You may use standard inequalities from lectures if you state them clearly.*]

END OF PAPER