UNIVERSITY OF CAMBRIDGE

MAT0

MATHEMATICAL TRIPOS Part IA

Thursday, 30 May, 2024 9:00am to 12:00pm

PAPER 1

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **all four** questions from Section I and **at most five** questions from Section II. Of the Section II questions, no more than three may be on the same course.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1A Vectors and Matrices

Consider a function of the complex variable z with $z \neq -1$

$$f(z) = \frac{az+b}{z+1},$$

where the coefficients a and b are complex numbers such that $a \neq b$.

(a) Find a and b such that the equation $f(z) = z$

(i) has a unique solution $z = i$;

(ii) has exactly two solutions $3i$ and $1+i$.

(b) Sketch the locus of all $z \neq 0$ satisfying

$$\arg\left(\frac{z-2}{z}\right) = \frac{\pi}{4}$$

and find its Cartesian equation.

2C Vectors and Matrices

(a) Consider a 3×3 matrix A with elements $A_{ij} = i + aj + b$, with a and b two non-zero real constants and $i, j = 1, 2, 3$.

(i) Compute the determinant of A.

(ii) Compute the kernel of A and hence its nullity.

(iii) Compute the image of A and hence its rank.

(iv) State the rank-nullity theorem and verify it for A.

(b) Now consider an $n \times m$ matrix B with $n, m > 1$ and elements $B_{ij} = i^2 + aj + b$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, where a and b are two non-zero real constants. Show that the nullity of B is $m - 2$.

3F Analysis I

Let (a_n) and (b_n) be two sequences of positive real numbers such that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge.

(a) Show that the series $\sum_{n=1}^{\infty} a_n^2$ converges.

(b) Show that the series $\sum_{n=1}^{\infty} \sqrt{a_n b_n}$ converges.

(c) Show that the series $\sum_{n=1}^{\infty} \sqrt{a_n}\, n^{-p}$ converges if $p > 1/2$. Give an example to show that this series need not converge for $p = 1/2$.

[*You may use any results from the course provided you state them clearly.*]

4D Analysis I

(a) Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients. Show that there exists $R \in [0, \infty]$, the 'radius of convergence' of the series, such that the series is convergent when $|z| < R$ and divergent when $|z| > R$.

(b) Now suppose that the a_n are real and positive, with $a_n \to \infty$. Can it happen that the two power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} a_n^2 z^n$ have the same finite non-zero radius of convergence? What about the two power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} a_n^{a_n} z^n$?

SECTION II

5A Vectors and Matrices

(a) Use the summation convention and basic properties of the Levi-Civita symbol and the Kronecker delta to prove that

(i) $\epsilon_{ijk}\epsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}$.

(ii) $|\mathbf{x}|^2|\mathbf{y}|^2 \geqslant |\mathbf{x}\cdot\mathbf{y}|^2$, for any vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.

(iii) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$, for any vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ in $\mathbb{R}^3$.

(b) Let $\mathbf{y},\mathbf{z}$ be two fixed linearly independent unit vectors in $\mathbb{R}^3$. Define a scalar function S of a unit vector $\mathbf{x}$ as

$$S = |\mathbf{x}\times\mathbf{y}|^2 + |\mathbf{x}\times\mathbf{z}|^2 + \mathbf{x}\cdot\mathbf{y} + \mathbf{z}\cdot\mathbf{x}.$$

Find $\mathbf{F}(\mathbf{y},\mathbf{z})$ such that every $\mathbf{x}$ that maximises S satisfies $\mathbf{x}\times(\mathbf{y}\times\mathbf{z}) = \mathbf{F}(\mathbf{y},\mathbf{z})$.

Another scalar function S' of a unit vector $\mathbf{x}$ is defined by

$$S' = |R(\theta_1)\mathbf{x}\times R(\theta_1)\mathbf{y}|^2 + |\mathbf{x}\times\mathbf{z}'|^2 + R(\theta_2)\mathbf{x}\cdot R(\theta_2)\mathbf{y} + \mathbf{z}'\cdot\mathbf{x},$$

where $\mathbf{z}' = 2\mathbf{z}$, $R(\theta)$ is the matrix of rotation around the $z-$axis

$$R(\theta) = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix},$$

and θ_1, θ_2 are two angles. Find $\mathbf{G}(\mathbf{y},\mathbf{z})$ such that every $\mathbf{x}$ that maximises S' satisfies $\mathbf{x}\times(\mathbf{y}\times\mathbf{z}) = \mathbf{G}(\mathbf{y},\mathbf{z})$.

6C Vectors and Matrices

(a) Find an orthogonal linear transformation that maps the triangle formed by

$$P = (0,0,0)\,, \qquad Q = (0,1,0)\,, \qquad R = (\sqrt{3}/2, 1/2, 0)\,,$$

to the triangle formed by

$$P' = (0,0,0)\,, \qquad Q' = (-1/2, 0, \sqrt{3}/2)\,, \qquad R' = (1/2, 0, \sqrt{3}/2)\,.$$

(b) Consider the linear transformation M of $\mathbb{R}^4$ representing a reflection in the hyperplane $\sum_{m=1}^{4} a_m x_m = 0$, where a_m are four real constants (not all zero). Write down the matrix of M. Hence, or otherwise, compute its determinant.

(c) Consider the linear transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $A : \mathbf{b} \mapsto \mathbf{a} \times \mathbf{b}$ for some non-zero vector $\mathbf{a}$. Write A as a 3×3 matrix and compute its trace and its determinant.

(d) A *tridiagonal* matrix is a square matrix A such that $A_{ij} = 0$ for $|i - j| \geqslant 2$. For (a_i), (b_i) and (c_i) three infinite real sequences and n any positive integer, the $n \times n$ tridiagonal matrix $A^{(n)}$ is defined by $A^{(n)}_{ii} = a_i$ for $i = 1, \ldots, n$ and $A^{(n)}_{i(i+1)} = b_i$ and $A^{(n)}_{(i+1)i} = c_i$ for $i = 1, \ldots, n-1$.

(i) Prove that the determinants $d_n = \det(A^{(n)})$ satisfy the recurrence relation

$$d_n = X_n d_{n-1} + Y_n d_{n-2}\,,$$

for each $n \geqslant 3$, where X_n and Y_n are parameters you should determine in terms of (a_i), (b_i), (c_i) and n.

(ii) In the case $a_i = -2$, $b_i = c_i = 1$ for all i, compute d_n.

7B Vectors and Matrices

(a) Define the trace, $\mathrm{Tr}(A)$, of a complex $n \times n$ matrix A. If B is another complex $n \times n$ matrix, show that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Is it always the case that $\mathrm{Tr}(ABC) = \mathrm{Tr}(ACB)$ for 2×2 matrices?

(b) Define the characteristic polynomial of a complex $n \times n$ matrix A, and define the eigenvalues and eigenvectors of A.

(c) Let A, B be real $n \times n$ matrices such that $\det(A + iB) \neq 0$. Show that there exists $\lambda \in \mathbb{R}$ such that $A + \lambda B$ is invertible. [Hint: Consider a suitable polynomial.]

(d) Prove that if C, D are real $n \times n$ matrices related by a complex similarity transformation S (i.e., an invertible complex $n \times n$ matrix S with $C = SDS^{-1}$), then they are related by a real similarity transformation (i.e., an invertible real $n \times n$ matrix T with $C = TDT^{-1}$).

(e) Show that if a complex $n \times n$ matrix A has n distinct eigenvalues then it is diagonalisable over $\mathbb{C}$. Deduce that if a real $n \times n$ matrix A has n distinct real eigenvalues then it is diagonalisable over $\mathbb{R}$.

8B Vectors and Matrices

(a) Prove that any complex $n \times n$ matrix A has at least one eigenvalue and eigenvector.

(b) (i) Define the *adjoint* of a square matrix.

(ii) Define what it means for a complex $n \times n$ matrix U to be *unitary* and prove that the eigenvalues of any such matrix have modulus 1.

A complex $n \times n$ matrix A is *normal* if it commutes with its adjoint.

(iii) Prove that if A is diagonalisable with a unitary change of basis matrix then it is normal.

(c) Let A be an $n \times n$ normal matrix and let λ be an eigenvalue of A. Show that there exists a unitary change of basis matrix U such that

$$U^*AU = \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}, \quad \text{where } B \text{ is an } (n-1) \times (n-1) \text{ normal matrix.}$$

Hence prove by induction that any normal matrix A is diagonalisable with a unitary change of basis matrix.

(d) A complex $n \times n$ matrix B is *upper triangular* if $B_{ij} = 0$ for $i > j$.

(i) Prove that given any complex $n \times n$ matrix A, there exists a unitary matrix U and an upper triangular matrix B such that $U^*AU = B$.

(ii) Find such a unitary matrix U for the matrix

$$A = \begin{pmatrix} 4 & 5 & 0 \\ 0 & 6 & 0 \\ 2 & 3 & 1 \end{pmatrix}.$$

9F Analysis I

(i) State and prove the ratio test about the convergence of series.

(ii) Let (a_n) be a sequence of positive real numbers. If $a_{n+1}/a_n \to L$ for some $L \in \mathbb{R}$, show that $a_n^{1/n} \to L$.

(iii) Show that $\sum_{n=1}^{\infty} n!/n^n$ converges.

(iv) Compute $\lim_{n\to\infty} n/(n!)^{1/n}$.

10E Analysis

Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ and a is a real number.

(a) What does it mean to say that f is *differentiable* at a? If f is differentiable at a what is the value of $f'(a)$, the derivative of f at a?

(b) Let $f \cdot g\colon \mathbb{R} \to \mathbb{R}$ be the pointwise product of f and g:

$$(f \cdot g)(x) = f(x)g(x) \text{ for all } x \in \mathbb{R}.$$

(i) Suppose that f and g are both differentiable at a. Show that $f \cdot g$ is differentiable at a. What is the value of $(f \cdot g)'(a)$ in terms of $f(a), f'(a), g(a)$ and $g'(a)$?

(ii) Suppose that f and $f \cdot g$ are both differentiable at a with $(f \cdot g)'(a) \neq 0$. Must g be differentiable at a? Justify your answer.

(c) (i) Suppose now that f is differentiable at a and g is differentiable at $f(a)$. Show that the composite $g \circ f$ is differentiable at a. What is the relationship between $(g \circ f)'(a)$, $g'(f(a))$ and $f'(a)$?

(ii) Suppose that g is differentiable at $f(a)$ and $g \circ f$ is differentiable at a with $(g \circ f)'(a) \neq 0$. Must f be differentiable at a? Justify your answer.

(iii) What does it mean to say that f is *twice differentiable* at a? Suppose now that f is twice differentiable at a and g is twice differentiable at $f(a)$. Show that $g \circ f$ is twice differentiable at a.

11E Analysis

State and prove the intermediate value theorem.

(a) Suppose that $f\colon \mathbb{C} \to \mathbb{R}$ is a continuous function that takes both positive and negative values. Show that there exists $z \in \mathbb{C}$ such that $f(z) = 0$.

(b) Let n be a positive integer and let a be a positive real number. Suppose that $g\colon [0, na] \to \mathbb{R}$ is a continuous function such that

$$g(na) = g(0) + na.$$

Show that there exists $x \in [0, (n-1)a]$ such that $g(x + a) = g(x) + a$.

12D Analysis I

(a) (i) Let $f : [0,1] \to \mathbb{R}$ be a bounded function. Define the *upper* and *lower integrals* of f, and explain what it means for f to be *Riemann integrable.*

(ii) Show that every continuous function is Riemann integrable.

(b) Let $f : [0,1] \to \mathbb{R}$ be a non-negative function that is unbounded. We say that f is *improperly integrable* with integral I (where I is a real number) if, for each positive real r, the function $f_r(x) = \min(f(x), r)$ is Riemann integrable, with $\int f_r(x)dx \to I$ as $r \to \infty$.

(i) If f is continuous at all points of $(0,1)$, must f be improperly integrable?

(ii) If there exists a countable set $A \subset [0,1]$ such that f is bounded on the set $[0,1] \setminus A$, must f be improperly integrable?

(iii) Given $x \in [0,1]$, we write x in decimal as $0.a_1a_2a_3\ldots$ (choosing say the terminating-in-9s form in case of ambiguity), and we set $f(x) = k$ if a_k is the first digit that is 5 and $f(x) = 0$ if no digit is 5. Prove that f is improperly integrable. What is the integral of f?

END OF PAPER