UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II 2024

List of Courses

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Algebraic Topology
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Paper 1, Section II

25F Algebraic Geometry

In this question, all algebraic varieties are over an algebraically closed field k.

What does it mean for a topological space to be *irreducible*? Show that if $X \subseteq \mathbb{A}^n$ is a Zariski closed subset, then X can be written as a finite union of irreducible closed subsets of $\mathbb{A}^n$.

Write the closed subset $Z(x_3 - x_2^2, x_1^2 - x_2^2 - x_2^4 + x_3^2)$ of $\mathbb{A}^3$ as a union of irreducible closed sets.

Now take $k = \mathbb{C}$ to be the field of complex numbers. Show that the set

$$Z := \{(x, e^x) \mid x \in \mathbb{C}\} \subseteq \mathbb{A}^2$$

is dense in $\mathbb{A}^2$ in the Zariski topology.

Let X be an affine algebraic variety, and let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open cover of X. Show that $\mathcal{U}$ has a finite subcover. [*Hint: Define for any regular function f on X the distinguished open set*

$$D(f) := \{x \in X \mid f(x) \neq 0\}.$$

You may use without proof the fact that the collection of distinguished open sets form a basis for the topology on X.]

Paper 2, Section II

25F Algebraic Geometry

In this question, all algebraic varieties are over an algebraically closed field k.

Let X be an affine variety. Define the *tangent space* of X at a point $P \in X$. Define the *dimension* of X in terms of (i) the tangent spaces of X and (ii) Krull dimension. Say what it means for the variety to be *singular* at P.

Assume the characteristic of the field k is not 2. Let $X := Z(x_1^2 - x_2^3, x_3^2 - x_4^3) \subseteq \mathbb{A}^4$. Calculate the tangent space of X at each point of X.

Consider the subset $Y \subseteq \mathbb{P}^4$ consisting of points with homogeneous coordinates $(y_0 : y_1 : y_2 : y_3 : y_4)$ such that the matrix

$$\begin{pmatrix} y_0 & y_1 & y_2 \\ y_2 & y_3 & y_4 \end{pmatrix}$$

has rank one. Show that Y is a closed subset of $\mathbb{P}^4$ in the Zariski topology. You may now assume Y is irreducible in the Zariski topology, and hence is a projective variety. What is the dimension of Y? Show that Y is non-singular.

Paper 3, Section II

24F Algebraic Geometry

In this question, all algebraic varieties are over an algebraically closed field k.

State the Riemann–Roch theorem.

Let X be a non-singular projective curve of genus 1, and let $P_0 \in X$ be a point. Show that if D is a divisor of degree 0 on X, then there exists a unique $P \in X$ such that D is linearly equivalent to $P - P_0$.

Use this to describe a group law on X with P_0 as the identity element. [You do not need to prove this law satisfies the group axioms.]

Consider the group law on the non-singular cubic curve

$$X = Z(y^2 z - x^3 + 4xz^2 - z^3) \subseteq \mathbb{P}^2$$

with identity element $P_0 = (0:1:0) \in X$. Let $A = (2:1:1)$ and $B = (-2:-1:1)$ be two points on X. Find $A + B$.

Let $X \subseteq \mathbb{P}^2$ be a non-singular cubic curve. A point $P_0 \in X$ is an *inflection point* of X if there exists a line $L \subseteq \mathbb{P}^2$ such that $X \cap L = \{P_0\}$. Let P_0 be an inflection point of X, and give X the group structure with identity element P_0. A point $P \in X$ is said to be 3-*torsion* if $P + P + P = P_0$. Prove that P is a 3-torsion point if and only if P is an inflection point.

Paper 4, Section II

24F Algebraic Geometry

In this question, all algebraic varieties are over the complex numbers $\mathbb{C}$.

Let $\varphi : \mathbb{P}^2 \to \mathbb{P}^1$ be a morphism. Show that φ is constant. [You may use without proof the fact that any two curves in $\mathbb{P}^2$ have non-empty intersection.]

Show that there is no closed subvariety of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ isomorphic to $\mathbb{P}^2$.

State the Riemann–Hurwitz theorem.

Consider a non-singular projective curve $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ defined by a bihomogeneous equation $f(x_0, x_1, y_0, y_1) = 0$ which is homogeneous of degree 2 in x_0, x_1 and homogeneous of degree 3 in y_0, y_1. Here x_0, x_1 are coordinates on the first $\mathbb{P}^1$ and y_0, y_1 are coordinates on the second $\mathbb{P}^1$. Compute the genus of X. Deduce that X is not isomorphic to a non-singular projective curve in $\mathbb{P}^2$. [You may use without proof the fact that a non-singular projective curve in $\mathbb{P}^2$ of degree d has genus $g = (d-1)(d-2)/2$.]

Paper 1, Section II

21J Algebraic Topology

What does it mean to say that $p : \widetilde{X} \to X$ is a *covering map*? If $\gamma : [0,1] \to X$ is a path and $\tilde{x}_0 \in \widetilde{X}$ is such that $p(\tilde{x}_0) = \gamma(0)$, prove carefully that there is a unique path $\tilde{\gamma} : [0,1] \to \widetilde{X}$ such that

(i) $\tilde{\gamma}(0) = \tilde{x}_0$, and

(ii) $p \circ \tilde{\gamma} = \gamma$.

[You may use the Lesbegue number lemma.]

Let Y be a topological space, $A \subseteq Y$ and $B \subseteq Y$ be open subspaces with disjoint closures, and $\phi : A \to B$ be a homeomorphism. Let Y/ϕ denote the quotient of Y by the equivalence relation generated by $a \sim \phi(a)$ for all $a \in A$. Show that the function

$$\begin{aligned} p : \widehat{Y/\phi} := \frac{Y \times \mathbb{Z}}{(a,i) \sim (\phi(a), i-1) \text{ for } a \in A, i \in \mathbb{Z}} &\longrightarrow Y/\phi \\ [(y,i)] &\longmapsto [y] \end{aligned}$$

is continuous and is a covering map.

Assume now that Y is path-connected. Let $a_0 \in A$ be a basepoint, which determines a basepoint $[a_0] \in Y/\phi$. Show that $\widehat{Y/\phi}$ is path-connected, that the subgroup $G \leqslant \pi_1(Y/\phi, [a_0])$ associated to the covering space $p : \widehat{Y/\phi} \to Y/\phi$ is normal, and that the quotient group $\pi_1(Y/\phi, [a_0])/G$ is isomorphic to $\mathbb{Z}$.

Paper 2, Section II

21J Algebraic Topology

State the Seifert–van Kampen theorem.

If (X, x_0) is a based topological space, and $f : (S^{n-1}, *) \to (X, x_0)$ is a map of based spaces, define the space $X \cup_f D^n$ obtained by attaching an n-cell to X along f. For $n = 2$, carefully prove a formula describing $\pi_1(X \cup_f D^2, x_0)$ in terms of the group $\pi_1(X, x_0)$ and the element $[f] \in \pi_1(X, x_0)$.

Writing $S^1 \vee S^1$ for the wedge of two circles, calculate $\pi_1(S^1 \vee S^1, *)$.

Explain how to attach 2-cells to $S^1 \vee S^1$ to obtain a space whose fundamental group is the symmetric group on 3 letters, proving carefully that this is indeed the fundamental group obtained.

[*You may use any description of the group $\pi_1(S^1, *)$, provided it is clearly stated. You should justify any presentation of the symmetric group on 3 letters that you use.*]

Paper 3, Section II

20J Algebraic Topology

Let L be a simplicial complex and $K \leqslant L$ be a subcomplex, with associated chain complexes $C_\bullet(L)$ and $C_\bullet(K)$. Setting

$$C_k(L,K) := \frac{C_k(L)}{C_k(K)}\,,$$

show that the boundary map of L descends to give $C_\bullet(L,K)$ the structure of a chain complex. Describe a long exact sequence relating $H_*(K)$, $H_*(L)$, and the homology $H_*(L,K)$ of $C_\bullet(L,K)$.

In the following, Δ^{n+1} denotes the standard $(n+1)$-simplex and $\partial\Delta^{n+1}$ denotes its boundary, i.e. the union of all its simplices of dimension $< n+1$. Suppose that $n \geqslant 2$.

Using that Δ^{n+1} has the same homology as a point, calculate the homology of $\partial\Delta^{n+1}$.

Recall that the rank $\operatorname{rk} A$ of an abelian group A is the maximal r such that $\mathbb{Z}^r \leqslant A$. If $K \leqslant \partial\Delta^{n+1}$ is a sub-simplicial complex, by considering $H_*(\partial\Delta^{n+1}, K)$ show that for each $0 < k \leqslant n$ we have

$$\operatorname{rk} H_k(K) \leqslant \binom{n+2}{k+2} - \#(k+1)\text{-simplices of } K.$$

[You may use that the rank of a subgroup or quotient of A is at most that of A.]

If $K < \partial\Delta^{n+1}$ is a proper sub-simplicial complex, show that $H_n(K) = 0$.

Paper 4, Section II

21J Algebraic Topology

State the Mayer–Vietoris theorem for a simplicial complex K which is the union of two subcomplexes M and N.

For each $k = 2, 3, 4, \ldots$, construct an explicit simplicial complex K having

$$H_i(K) \cong \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/k & i = 1 \\ 0 & \text{else.} \end{cases}$$

Justify your answer.

Paper 1, Section II

23G Analysis of Functions

(a) Set $\mathbb{R}^+ = (0,\infty)$, and let $L^p = L^p(\mathbb{R}^+, dx)$, $1 < p < \infty$, where dx is Lebesgue measure on $\mathbb{R}^+$. Let $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be integrable for the product measure $dx \otimes dx$ on $\mathbb{R}^+ \times \mathbb{R}^+$. Set

$$G(y) = \int_{\mathbb{R}^+} F(x,y)dx, \; y \in \mathbb{R}^+.$$

Show that if $\|g\|_{L^q} \leqslant 1$ for $1 < q < \infty$ conjugate to p, then

$$\int_{\mathbb{R}^+} |G(y)g(y)|\, dy \leqslant \int_{\mathbb{R}^+} \Big[\int_{\mathbb{R}^+} |F(x,y)|^p dy \Big]^{1/p} dx.$$

(b) Now let $K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be integrable and such that

$$K(\lambda x, \lambda y) = \lambda^{-1} K(x,y), \; \lambda, x, y > 0; \quad \text{and} \quad \int_0^\infty |K(x,1)| x^{-1/p}\, dx = 1.$$

Define $Tf(y) = \int_0^\infty K(x,y) f(x)\, dx$. Show that for $f \in L^p$ we have

$$\|Tf\|_{L^p} \leqslant \|f\|_{L^p}.$$

[*Hint: Consider $f_z(y) = f(yz)$ and show first that $\|f_z\|_{L^p} = z^{-1/p}\|f\|_{L^p}$.*]

[*You may use the identity*

$$\|f\|_{L^p} = \sup\Big\{ \int |f(x)g(x)|dx : g \in L^q, \|g\|_{L^q} \leqslant 1 \Big\}, \; q \textit{ conjugate to } p,$$

without proof.]

Paper 2, Section II

23G Analysis of Functions

State (without proof) the Hahn–Banach theorem for linear functionals on a normed real vector space X.

Now consider the topological dual space X'. For each $x \in X$, define $\hat{x} : X' \to \mathbb{R}$ by the action

$$\hat{x}(f) = f(x), \;\; f \in X'.$$

Show carefully that $x \mapsto \hat{x}$ defines a linear isometry from X into the bidual space X'' (the topological dual space of X'), and that

$$\|x\|_X = \sup_{\|f\|_{X'} \leqslant 1} |f(x)|.$$

Let $I = [0, 1]$ and denote by L^∞ the Banach space of μ-essentially bounded functions on I, where μ is Lebesgue measure. Show that $(L^\infty)'$ does not coincide with $L^1(\mu)$. [*Hint: Extend the functional* $\ell(f) = f(0)$ *from the subspace* $C(I)$ *of continuous functions on* I *to* L^∞*.*]

Paper 3, Section II

22G Analysis of Functions

State and prove the Sobolev embedding theorem $H^s(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$.

Show that $H^1(\mathbb{R}^3)$ contains an unbounded function.

[*You may use results from Fourier analysis without proof, provided they are carefully stated.*]

Paper 4, Section II

23G Analysis of Functions

(a) For $f \in H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, show that there exists a unique weak solution $u \in \mathcal{S}'(\mathbb{R}^n)$ to the equation

$$\Delta^2 u + u = f. \qquad (\star)$$

(b) Show that if f has compact support in $\mathbb{R}^n$ and is infinitely differentiable, then the solution u is also infinitely differentiable.

(c) Now let $n = 3$ and let $f = \delta_0$ be the Dirac measure at zero. Show that there exists a unique continuous function u solving $(\star)$.

[*You may use results about Fourier transforms and Sobolev spaces from the course without proof, provided they are clearly stated.*]

Paper 1, Section II

35E Applications of Quantum Mechanics

Consider the quantum mechanical scattering of a particle of mass m in three-dimensions, with Hamiltonian

$$H = \frac{|\boldsymbol{p}|^2}{2m} + V(r) \ .$$

Here the potential $V(r)$ is spherically symmetric, and it is localised in some region of space. Using a partial wave decomposition,

$$\psi(\boldsymbol{r}) = \sum_{l=0}^{\infty} \frac{u_l(r)}{r} P_l(\cos\theta) \ ,$$

where $P_l(\cos\theta)$ are Legendre polynomials, and boundary condition $u_l(0) = 0$, the time-independent Schrödinger equation for the wavefunction $\psi(\boldsymbol{r})$ of the particle reduces to

$$\left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + \frac{2m}{\hbar^2}V(r)\right) u_l(r) = k^2 u_l(r) \ , \quad \text{with} \quad E = \frac{\hbar^2 k^2}{2m} \ .$$

(a) The asymptotic behaviour, for large r, of the wavefunction can be written

$$\psi(\boldsymbol{r}) \sim \sum_{l=0}^{\infty} \frac{2l+1}{2ik} \left((-1)^{l+1} \frac{e^{-ikr}}{r} + S_l(k) \frac{e^{ikr}}{r} \right) P_l(\cos\theta) \ .$$

For real values of k, show that the coefficients $S_l(k)$ satisfy

$$S_l(k)^* \, S_l(k) = 1 \, , \qquad S_l(k) \, S_l(-k) = 1 \, .$$

Hence deduce that $S_l(k) = e^{2i\delta_l(k)}$ for some real function $\delta_l(k)$, and that $\delta_l(k) = -\delta_l(-k)$.

(b) Focus now on the low-momentum behaviour, where the $l = 0$ mode dominates. For some choice of potential $V(r)$,

$$S_0(k) = \frac{(k+3i\lambda)(k+2i\lambda)}{(k-3i\lambda)(k-2i\lambda)} \ ,$$

where λ is a real positive constant. Evaluate the scattering length a_s, and give an estimate for the total cross-section σ_T at low energies.

Briefly explain the significance of the poles of $S_0(k)$. Are there any resonances in $S_0(k)$? Why is it important that λ is positive?

Paper 2, Section II

36E Applications of Quantum Mechanics

Consider a Hamiltonian H that has a discrete spectrum and a ground state with energy E_0.

(a) Describe briefly how to use the variational method to provide an upper bound on E_0.

(b) Suppose that the trial wavefunction in the variational method is given by

$$|\psi\rangle = \sum_{n=1}^{N} \alpha_n |\phi_n\rangle \,,$$

where α_n are complex variational parameters, and the $|\phi_n\rangle$ form an orthonormal set, i.e., $\langle\phi_m|\phi_n\rangle = \delta_{mn}$, for m and $n = 1, 2, \ldots N$.

Apply the variational method to H with trial wavefunction $|\psi\rangle$, and show that the lowest eigenvalue of the matrix $\mathcal{H}$, which has entries $\mathcal{H}_{nm} = \langle\phi_n|H|\phi_m\rangle$, gives the optimal upper bound on the ground state energy E_0.

(c) Consider a particle of mass m in an infinite one-dimensional square well of width a, with a linear potential in the well,

$$V(x) = \begin{cases} V_0 \, \dfrac{x}{a} & 0 \leqslant x \leqslant a \,, \\ \infty & \text{otherwise} \,, \end{cases}$$

where $V_0 = \dfrac{9\hbar^2}{ma^2}$. Determine an upper bound for the ground state energy of this system using

$$\phi_n = \begin{cases} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) & 0 \leqslant x \leqslant a \,, \\ 0 & \text{otherwise} \,, \end{cases}$$

with $n = 1, 2$, as trial wavefunctions.

Paper 3, Section II

34E Applications of Quantum Mechanics

(a) Consider a one-dimensional system with a periodic potential $V(x) = V(x + a)$, where a is a positive constant. The Schrödinger equation for the system is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\,\psi(x)\ , \qquad (\star)$$

where E is the energy of the system. The Floquet matrix $F(E)$ is given by

$$\begin{pmatrix} \psi_1(x+a) \\ \psi_2(x+a) \end{pmatrix} = F(E) \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}\ ,$$

where ψ_1 and ψ_2 are two linearly independent solutions to $(\star)$.

(i) Show that $\det(F) = 1$.

(ii) Explain why the trace $\mathrm{Tr}F$ is real.

(iii) Explain what the implications on the band structure are when $(\mathrm{Tr}F)^2 < 4$ and when $(\mathrm{Tr}F)^2 > 4$.

(b) Consider a tight-binding Hamiltonian that acts upon a single band of localised states in one dimension,

$$H_t = t \sum_{j\in\mathbb{Z}} \Big(|j\rangle\langle j+1| + |j\rangle\langle j-1| + (-1)^j |j\rangle\langle j| \Big)\ ,$$

where t is a positive constant. The integer j should be thought of as indexing sites along a chain of atoms separated from each other by a distance a; the state $|j\rangle$ locates an electron on atom j.

(i) Consider the translation operator T_ℓ which acts on the states $|j\rangle$ by shifting them to another site, $T_\ell|j\rangle = |j+\ell\rangle$, with $\ell \in \mathbb{Z}$. Determine the values of ℓ for which T_ℓ commutes with the Hamiltonian H_t. Hence determine the lattice spacing of the system.

(ii) Using Bloch's theorem, show that the Hamiltonian of the system can be reduced to a 2×2 matrix. Find the eigenvalues of this matrix.

(iii) What is the range of the first Brillouin zone? Plot the energy bands for the first Brillouin zone. In your plot you should clearly label the axes and indicate the values of the energies at the boundaries of the Brillouin zone. What is the bandwidth of each of the bands?

Paper 4, Section II

34E Applications of Quantum Mechanics

Consider a particle of charge $-e$ and mass m moving in three dimensions with an electric field $\boldsymbol{E}$ and a magnetic field $\boldsymbol{B}$. The time-dependent Schrödinger equation for the particle's wave function $\psi(\boldsymbol{x}, t)$ is

$$i\hbar\left(\frac{\partial}{\partial t} - \frac{ie}{\hbar}\phi\right)\psi = -\frac{\hbar^2}{2m}\left(\boldsymbol{\nabla} + \frac{ie}{\hbar}\boldsymbol{A}\right)^2\psi\ , \qquad (\star)$$

where $\boldsymbol{A}$ is the vector potential and ϕ is the electric potential, with $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ and $\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \boldsymbol{A}}{\partial t}$.

(a) The Schrödinger equation $(\star)$ should be invariant under the gauge transformations

$$\begin{aligned}\boldsymbol{A}(\boldsymbol{x}, t) \quad &\rightarrow \quad \boldsymbol{A}(\boldsymbol{x}, t) + \boldsymbol{\nabla} f(\boldsymbol{x}, t)\ ,\\ \phi(\boldsymbol{x}, t) \quad &\rightarrow \quad \phi(\boldsymbol{x}, t) - \frac{\partial}{\partial t} f(\boldsymbol{x}, t)\ ,\end{aligned}$$

where f is an arbitrary smooth function, together with a suitable transformation of $\psi(\boldsymbol{x}, t)$. State such a suitable transformation of $\psi(\boldsymbol{x}, t)$ and check explicitly that $(\star)$ is invariant under these transformations.

(b) In the presence of electric and magnetic fields, the probability current is modified to

$$\boldsymbol{J}(\boldsymbol{x}, t) = -\frac{i\hbar}{2m}\left(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\right) + \frac{e}{m}\psi^*\psi\boldsymbol{A}\ .$$

Show that $\boldsymbol{J}$ is gauge invariant and that it satisfies the conservation equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0\ ,$$

where $\psi(\boldsymbol{x}, t)$ satisfies the Schrödinger equation and $\rho = \psi^*\psi$.

(c) Consider the situation with constant electromagnetic fields, where $\boldsymbol{B} = (0, 0, B)$, $\boldsymbol{E} = (0, \mathcal{E}, 0)$, and B and $\mathcal{E}$ are constants. Verify that the vector and electric potential can be chosen to have the form

$$\boldsymbol{A} = B(-y, 0, 0)\ , \qquad \phi = -\mathcal{E} y\ .$$

Restrict the motion to the (x, y) plane. Show that stationary states are given by

$$\psi(\boldsymbol{x}, t) = e^{ikx} e^{-iEt/\hbar} \varphi(y)\ ,$$

where $\varphi(y)$ is a harmonic oscillator wavefunction (i.e. it satisfies the time-independent Schrödinger equation with a harmonic oscillator potential), E is the energy and k is a real constant. At what position is the oscillator centered?

Find the allowed energies, and show that they can be written as

$$E = \hbar\omega_1\left(n + \frac{1}{2}\right) + \mathcal{W}\ ,$$

where n is a non-negative integer, and you should determine both ω_1 and $\mathcal{W}$ explicitly. Briefly discuss what happens to the Landau levels in the presence of this electric field.

Paper 1, Section II

28K Applied Probability

(a) Let $(X_t)_{t\geqslant 0} \sim \text{Markov}(Q)$ be a continuous time Markov chain with generator matrix Q on a countable state space I, jump times J_n and jump chain $(Y_n)_{n\geqslant 0}$. Show that for $n \geqslant 1$ and $i_1, i_2, \ldots, i_n \in I$,

$$q_{i_n}\mathbb{P}(J_n \leqslant t < J_{n+1} | Y_0 = i_0, \ldots, Y_n = i_n) = q_{i_0}\mathbb{P}(J_n \leqslant t < J_{n+1} | Y_0 = i_n, \ldots, Y_n = i_0)\,.$$

(b) Now let $(X_t)_{t\geqslant 0}$ be irreducible. Fix any $h > 0$ and let $Z_n = X_{nh}$ for $n = 0, 1, 2, \ldots$. Show that $(Z_n)_{n\geqslant 0}$ is a discrete-time Markov chain and give its transition matrix. Show that $(X_t)_{t\geqslant 0}$ is recurrent if and only if $(Z_n)_{n\geqslant 0}$ is recurrent.

(c) Finally, let I be finite and $f : I \to \mathbb{R}$ be a function, identified with the vector $(f(x))_{x\in I}$. Show that

$$Qf(x) = \lim_{t\to 0+} \frac{\mathbb{E}_x(f(X_t)) - f(x)}{t}\,,$$

and

$$\mathbb{E}_x(f(X_t)) = f(x) + \int_0^t \mathbb{E}_x(Qf(X_s))\,ds.$$

Paper 2, Section II

28K Applied Probability

(a) Let $X = (X_t)_{t \geqslant 0}$ be a simple birth process with rate $\lambda_n = n\lambda > 0$ for all $n \geqslant 1$, and $X_0 = 1$.

(i) Show that X is non-explosive.

(ii) Let $n \geqslant 1$. Show that conditional on the event $\{X_t = n+1\}$, the times of the n births have the distribution of the order statistics of n i.i.d. random variables with probability density function

$$f(x) = \frac{\lambda e^{\lambda x}}{e^{\lambda t} - 1}, \qquad 0 \leqslant x \leqslant t\,.$$

(b) Let X_t be a birth and death process with rates $\lambda_n = n\lambda$ and $\mu_n = n\mu$ for $n \in \mathbb{N}$, $\lambda > 0$, $\mu > 0$, and assume that $X_0 = 1$. Let $h(t) = \mathbb{P}(X_t = 0)$.

(i) Show that $h(t)$ satisfies

$$h(t) = \int_0^t e^{-(\lambda+\mu)s} \{\mu + \lambda (h(t-s))^2\}\, ds\,.$$

(ii) Show that $h'(t)$ satisfies

$$h'(t) = (h(t) - 1)(\lambda h(t) - \mu)\,.$$

(iii) Find $h(t)$ for $\lambda \neq \mu$.

Paper 3, Section II

27K Applied Probability

(a) State and prove Little's formula for a regenerative process.

(b) What is an $M/G/1$ *queue*? Find the expected length of the busy period for an $M/G/1$ queue.

(c) Suppose customers arrive at a single server at rate $\lambda > 0$ and require an exponential amount of service time with rate $\mu > 0$. Customers not being served are impatient and will leave at rate $\delta > 0$, independently of their position in the queue. Let X_t denote the length of the queue at time t.

(i) Show that the system $X = (X_t)_{t \geqslant 0}$ has an invariant distribution.

(ii) Is X positive recurrent? Justify your answer.

Paper 4, Section II

27K Applied Probability

(a) A train station has a platform where passengers arrive according to a renewal process with rate $\mu > 0$ (i.e. inter-arrival times have mean $1/\mu$). As soon as N passengers arrive, a train dispatches with all N passengers on board. The process continues. For $c > 0$, the company incurs a cost at the rate of nc per unit time when exactly n passengers are waiting, and a fixed cost K each time a train departs. What is the optimum value of N?

(b) A factory manufactures bulbs with independent lifetimes that are uniformly distributed in $[0, 3]$ (months). Whenever a bulb from this factory dies, it is immediately replaced by a new bulb from the same factory. You observe a bulb from this factory at the instant t. Let A_t be the time the bulb has been running (until the instant t) and let E_t be the remaining time the bulb will last (after instant t). For large t, find $\mathbb{P}(A_t \leqslant x)$, $\mathbb{P}(E_t \leqslant x)$ and $\mathbb{P}(A_t + E_t \leqslant x)$ for $x \in [0, 3]$.

(c) Let Π be the points of a non-homogeneous Poisson process on $\mathbb{R}^d$ with intensity function λ. Let $S = \sum_{x \in \Pi} g(x)$ where g is a smooth non-negative function. Show that $\mathbb{E}(S) = \int_{\mathbb{R}^d} g(u)\lambda(u)\, du$ and $\operatorname{var}(S) = \int_{\mathbb{R}^d} g(u)^2 \lambda(u)\, du$, provided these integrals converge.

Paper 2, Section II

32C Asymptotic Methods

(a) Suppose $f(x)$ is a real-valued function and the set of functions $\phi_n(x)$, where $n = 0, 1, 2, 3, \ldots,$ forms an asymptotic sequence as $x \to x_0$. What is meant by the statement "$f(x)$ has an *asymptotic expansion* as $x \to x_0$, with respect to the $\phi_n(x)$"?

Given that

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x) \quad \text{as } x \to x_0,$$

show that

$$a_0 = \lim_{x \to x_0} \frac{f(x)}{\phi_0(x)} \qquad \text{and} \qquad a_n = \lim_{x \to x_0} \frac{f(x) - \sum_{k=0}^{n-1} a_k \phi_k(x)}{\phi_n(x)}. \tag{$\dagger$}$$

(b) Consider the asymptotic sequence $\phi_n(x)$, defined by $\phi_0 = x^{-1}$ and $\phi_n(x) = x^{-n+1}\mathrm{e}^{-x}$ for $n \geqslant 1$, as $x \to \infty$, and the function

$$f(x) = \frac{1}{x} + \frac{x\mathrm{e}^{-x}}{x-1}.$$

Find the asymptotic expansion of $f(x)$ with respect to the $\phi_n(x)$ as $x \to \infty$.

Verify explicitly that your coefficients satisfy ($\dagger$) for all n.

What is the asymptotic expansion of $f(x)$ with respect to the asymptotic sequence $\psi_n(x) = x^{-n}$ as $x \to \infty$?

(c) Consider the sine-integral function,

$$\mathrm{si}(x) = \int_1^{\infty} \frac{\sin(xt)}{t}\, dt.$$

Using integration by parts, show that

$$\mathrm{si}(x) \sim \cos x \sum_{n=0}^{\infty} a_n x^{-2n-1} + \sin x \sum_{n=0}^{\infty} b_n x^{-2n-2} \qquad \text{as } x \to \infty,$$

where you should determine the coefficients a_n and b_n.

Paper 3, Section II

30C Asymptotic Methods

(a) State *Watson's lemma.*

(b) Consider the integral

$$I(x) = \int_0^1 \sqrt{t}\, \mathrm{e}^{ixt}\, dt\,,$$

where x is real and positive. Use the method of steepest descent to show that

$$I(x) \sim \frac{\sqrt{\pi}}{2}\,(-ix)^{-3/2} - \mathrm{e}^{ix}\sum_{n=0}^{\infty} a_n\,(-ix)^{-n-1}\,,$$

as $x \to \infty$, where the a_n are coefficients that you should determine.

[Hint: $\int_0^\infty \sqrt{u}e^{-u}du = \sqrt{\pi}/2$.*]*

Paper 4, Section II

31C Asymptotic Methods

Consider the eigenvalue problem

$$\epsilon^2 \frac{d^2 y}{dx^2} - \left(1 - \frac{\lambda^2}{x^2}\right) y = 0, \tag{$\star$}$$

defined for $x \in [1, \infty)$, with eigenvalue $\lambda > 1$ and small parameter $0 < \epsilon \ll 1$, and with boundary conditions $y(1) = 0$ and $y \to 0$ as $x \to \infty$.

(a) Find the single turning point of this problem, $x = x_{\text{tp}}$.

Write down the WKBJ form of the inner solution y_1, valid in the region $x_{\text{tp}} - x \gg \epsilon$, and the outer solution y_2, valid in the region $x - x_{\text{tp}} \gg \epsilon$. You may quote relevant WKBJ results from the lectures.

(b) Derive an approximate solution to ($\star$) near the turning point. By matching it to y_1, obtain the quantisation condition

$$\int_1^{\lambda} \left(\frac{\lambda^2}{x^2} - 1\right)^{1/2} dx = \epsilon \left(n + \frac{3}{4}\right) \pi, \tag{$\dagger$}$$

for integer n. [You may quote the following asymptotic behaviour of the Airy function:

$$\text{Ai}(t) \sim \pi^{-1/2} (-t)^{-1/4} \cos\left[\tfrac{2}{3}(-t)^{3/2} - \tfrac{1}{4}\pi\right] \quad \text{as} \quad t \to -\infty. \text{]}$$

(c) We seek approximations to the smallest eigenvalues λ, which are close to 1. Using the Taylor-expansion method in ($\dagger$), or otherwise, show that $\lambda \approx 1 + \alpha\epsilon^{2/3}$, where you should determine α. [*Hint: Consider the substitution* $x = \lambda(1 - u)$.]

(d) Show that the largest eigenvalues, $\lambda \gg 1$, can be approximated by solutions to the transcendental equation

$$\lambda \ln \lambda = \epsilon \left(n + \frac{3}{4}\right) \pi.$$

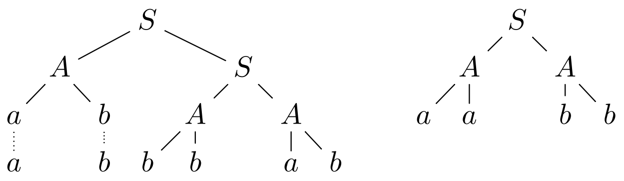
Paper 1, Section I

4J Automata and Formal Languages

In this question, let $\Sigma = \{a, b\}$ and $V = \{S, A\}$ be the set of terminal and non-terminal symbols, respectively.

(a) Give an example of a context-free grammar $G = (\Sigma, V, P, S)$ and a word $w \in \mathbb{W}$ such that w has a unique G-parse tree starting from S, but exactly two different G-derivations. Justify your claim.

(b) Suppose that G is a context-free grammar such that there is a G-parse tree starting from A producing $baba$ and the following two trees are G-parse trees:



For each of the following words w, prove that $w \in \mathcal{L}(G)$:

(i) $w = abbabaab$ and

(ii) $w = abbbaa$.

(c) Prove that the language $L := \{a^n b^m a^{\min(n,m)} : n, m > 1\}$ is not context-free.

[*You may use the context-free pumping lemma without proof.*]

Paper 2, Section I

4J Automata and Formal Languages

(a) Define what an *index set* is and when it is called *non-trivial*.

(b) Define the index set $\mathbf{Inf}$.

(c) State Rice's theorem.

(d) Let $X \subseteq \mathbb{W}$ and let $\mathbf{Inf}_X := \{w \in \mathbf{Inf} : \mathrm{ran}(f_{w,1}) \subseteq X\}$. Show, by modifying the proof of Rice's theorem or otherwise, that for each nonempty X, the set $\mathbf{Inf}_X$ is not computable.

Paper 3, Section I

4J Automata and Formal Languages

(a) Provide an example of a grammar $G = (\Sigma, V, P, S)$ such that for

$$Q := P \cup \{S \to \varepsilon\} \text{ and}$$
$$H := (\Sigma, V, Q, S),$$

we have $\mathcal{L}(G) \cup \{\varepsilon\} \neq \mathcal{L}(H)$. Justify your claim.

(b) Let $G = (\Sigma, V, P, S)$ be an arbitrary grammar. Let $S^* \notin V$ and $V^* := V \cup \{S^*\}$. If $\alpha \to \beta \in P$, let $\alpha^* \to \beta^*$ be the rule where all instances of S are replaced by S^* (both on the left and on the right of the arrow). Let

$$P^* := \{\alpha^* \to \beta^* \,:\, \alpha \to \beta \in P\} \text{ and}$$
$$P^+ := P^* \cup \{S \to S^*\}.$$

Show that for $G^+ := (\Sigma, V^*, P^+, S)$, we have that $\mathcal{L}(G) = \mathcal{L}(G^+)$. [You may use without proof that isomorphic grammars produce the same language.]

(c) Again, let $G = (\Sigma, V, P, S)$ be an arbitrary grammar and use the notation from (b). Let

$$Q^+ := P^+ \cup \{S \to \varepsilon\} \text{ and}$$
$$H^+ := (\Sigma, V^*, Q^+, S)$$

and prove that $\mathcal{L}(G) \cup \{\varepsilon\} = \mathcal{L}(H^+)$.

Paper 4, Section I

4J Automata and Formal Languages

(a) Let $D = (\Sigma, Q, \delta, q_0, F)$ and $D' = (\Sigma, Q', \delta', q_0', F')$ be two deterministic automata. Define what it means that f is a *homomorphism* from D to D'.

(b) Prove that if f is a homomorphism from D to D', then $\mathcal{L}(D) = \mathcal{L}(D')$.

(c) Define the following partial order on deterministic automata: we write $D \leqslant D'$ if and only if there is an injective homomorphism from D to D'. Check whether the following statements are true or false. Justify your answers.

(i) There is a deterministic automaton D such that, up to isomorphism, there are only finitely many D' such that $D \leqslant D'$.

(ii) There is a deterministic automaton D such that, up to isomorphism, there are only finitely many D' such that $D' \leqslant D$.

Paper 1, Section II

12J Automata and Formal Languages

(a) Let $M = (\Sigma, Q, P)$ be a register machine and $\vec{w}$ a finite sequence of words. Define the *upper register index of* M and $\{C(t, M, \vec{w}) : t \in \mathbb{N}\}$, the *computation sequence of* M *upon input* $\vec{w}$. [When defining the computation sequence, you may assume that "M transforms C into C'" is already defined.]

(b) Let $M = (\Sigma, Q, P)$ be a register machine such that $f_{M,1} = \chi_L$ for some language $L \subseteq \mathbb{W}$. Show that for every $n \in \mathbb{N}$ there is a word w such that the computation sequence of $f_{M,1}(w)$ uses at least n remove instructions of the form $-(0, q, q')$ for some $q, q' \in Q$.

A register machine with upper register index n is called an *n-register machine* (i.e. a register machine using $n + 1$ registers). A language $L \subseteq \mathbb{W}$ is called *n-computable* if its characteristic function is computable by an n-register machine. In the following, let us assume that $\Sigma = \{a, b\}$.

(c) Show that $L = \{a^n b^n : n > 0\}$ is 1-computable. [You may use constructions from the course, as long as you state them precisely and justify the scratch space that they use.]

(d) Let $M = (\Sigma, Q, P)$ be a 0-register machine such that $f_{M,1} = \chi_L$ for some language L. Show that there are natural numbers t, t', k, and ℓ and $q \in Q$ such that $k \neq \ell$ and for all $x \in \mathbb{W}$, we have

$$C(t, M, xb^k) = (q, x) = C(t', M, xb^\ell).$$

(e) Using part (d) or otherwise, show that $L = \{a^n b^n : n > 0\}$ is not 0-computable.

Paper 3, Section II

12J Automata and Formal Languages

(a) Define what it means for a language $L \subseteq \mathbb{W}$ to *satisfy the regular pumping lemma*.

(b) Prove that every regular language satisfies the regular pumping lemma. [You may assume that "regular" is equivalent to "accepted by a deterministic automaton" without proof.]

(c) Let L be a finite language such that there is a $w \in L$ with $|w| = 100$. Show that there can be no deterministic automaton D with at most 100 states such that $L = \mathcal{L}(D)$.

(d) Let $\Sigma = \{0, 1, 2\}$ and let $L \subseteq \{0, 1\}^*$ be an arbitrary language. Show that

$$\widehat{L} := \{u2v : u \in \mathbb{W}, v \in L\} \cup \{0, 1\}^*$$

satisfies the regular pumping lemma.

(e) Using part (d) or otherwise, provide an example of a language L such that L satisfies the regular pumping lemma, but for any grammar G, we have that $L \neq \mathcal{L}(G)$.

Paper 1, Section I

8E Classical Dynamics

A rigid circular hoop of mass m and radius a hangs from a fixed point on its circumference, is constrained to lie in a vertical plane and is free to oscillate within this plane. A bead, also of mass m, can slide without friction around the hoop. [You may assume that the moment of inertia of a circular hoop of mass m and radius a about an axis through its circumference and perpendicular to the plane of the hoop is $I = 2ma^2$.]

(a) Choose a set of generalised coordinates and write down the Lagrangian for the system.

(b) Show that the frequencies for small oscillations around equilibrium are $\omega_1 = \sqrt{c_1 g/a}$ and $\omega_2 = \sqrt{c_2 g/a}$, where c_1 and c_2 are positive real numbers that you should determine.

Paper 2, Section I

8E Classical Dynamics

(a) In Lagrangian mechanics, explain what is meant by the *generalised momentum* associated to a generalised coordinate q.

(b) What does it mean for a generalised coordinate to be *ignorable*? Show that the generalised momentum associated to an ignorable coordinate is conserved. [You may state the Euler–Lagrange equations without proof.]

(c) A certain system has generalised coordinates (q_1, q_2, q_3) and Lagrangian

$$L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2\right) - \frac{1}{2}\left(q_1^2 + q_2^2 + q_3^2\right) - \alpha\left(q_1 q_2 + q_2 q_3 + q_3 q_1\right) ,$$

where α is constant. Show that L is invariant under rotations around the $(1,1,1)$ axis in q-space. Hence find two conserved quantities.

Paper 3, Section I

8E Classical Dynamics

(a) Using spherical polar coordinates, (r, θ, ϕ), write down the Hamiltonian for a particle of mass m moving in a spherically symmetric potential.

(b) Show that p_ϕ and $p_\theta^2 + (p_\phi^2/\sin^2\theta)$ are each conserved. Interpret them physically. [You may state Hamilton's equations without proof.]

(c) The particle executes circular motion in a plane through the origin inclined at angle ψ to the plane $\theta = \pi/2$. Evaluate $p_\theta(\theta)$ and show that it vanishes when $\sin\theta = \pm\cos\psi$. Interpret this result physically.

Paper 4, Section I

8E Classical Dynamics

(a) What is meant by the *phase space* of a mechanical system with n degrees of freedom?

(b) Write down Hamilton's equations in terms of a Poisson bracket on phase space.

(c) A particle of charge e and mass m moves through a magnetic field $\mathbf{B}$ in three dimensions with electric field $\mathbf{E} = \mathbf{0}$. Show that its equations of motion can be obtained using the non-standard Poisson brackets

$$\{q_i, q_j\} = 0\,, \qquad \{q_i, p_j\} = \delta_{ij}\,, \qquad \{p_i, p_j\} = e\,\epsilon_{ijk} B_k\,,$$

but with a free Hamiltonian, $H = \frac{|\mathbf{p}|^2}{2m}$, where q_i and p_i are generalised coordinates and momenta, respectively.

Paper 2, Section II

14E Classical Dynamics

A mass m_1 is suspended from a fixed point with coordinates $(x, y, z) = (0, 0, 0)$ by a spring with spring constant k_1. A second mass m_2 is suspended from the first mass by a spring with spring constant k_2. Each spring has natural length ℓ. The motion of the masses is restricted to the (x, y)-plane, with gravity acting in the $-y$ direction. The position of the first mass is $(x_1, y_1, 0)$ and the position of the second mass is $(x_2, y_2, 0)$.

(a) Write down the Lagrangian of the system and hence determine the equations of motion.

(b) Find the equilibrium position of each mass that has $y_2 < y_1 < 0$.

(c) For the remainder of this question suppose that the x-coordinate of mass m_2 is held fixed at its equilibrium value, and consider the case $m_1 = m_2 = m$ and $k_1 = k_2 = k$. One of the system's normal modes has the first mass moving in the x direction with no motion in the y direction and the other mass stationary. Show that this mode's frequency ω_1 satisfies

$$\omega_1^2 = \frac{k}{m}\left(2 - \frac{1}{\frac{2mg}{k\ell} + 1} - \frac{1}{\frac{mg}{k\ell} + 1}\right).$$

Find the other normal modes and corresponding frequencies, showing that they are independent of the strength of gravity.

Paper 4, Section II

15E Classical Dynamics

Torque free rotation of a rigid body with principle moments of inertia (I_1, I_2, I_3) is described by Euler's equations

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3\,, \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1\,, \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2\,. \end{aligned}$$

(a) Write down expressions for the kinetic energy T and total angular momentum L, and show that they are each conserved.

(b) Suppose that $I_1 < I_2$ and $I_3 = I_1 + I_2$, and that initially $\omega_2(0) = 0$ while $\omega_1(0)\sqrt{I_2 - I_1} = \omega_3(0)\sqrt{I_2 + I_1}$. Show that subsequently

$$\dot{\omega}_1^2 = \left(\frac{2T}{I_2} - \omega_1^2\right)\left(\frac{I_2 - I_1}{I_2 + I_1}\right)\omega_1^2\,.$$

(c) Hence show that

$$\omega_1(t) = A\ \mathrm{sech}(\Omega t)$$

for constants A and Ω which you should find in terms of the kinetic energy and moments of inertia.

(d) Describe the motion as $t \to \infty$.

Paper 1, Section I

3K Coding and Cryptography

Briefly describe the *binary Huffman code* for encoding symbols $1, 2, \ldots, m$ occurring with probabilities $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_m > 0$.

Consider the discrete random variable X taking seven values x_i $(1 \leqslant i \leqslant 7)$ with the following probabilities:

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0.49 & 0.26 & 0.12 & 0.04 & 0.04 & 0.03 & 0.02 \end{pmatrix}.$$

Find a binary Huffman code for X. What is its expected word length? [You do not need to simplify the expression.]

Paper 2, Section I

3K Coding and Cryptography

(a) Show that Hamming's original code is perfect.

(b) Consider the code obtained by using Hamming's original code for the first 7 bits and the final bit as a check digit, so that

$$x_1 + x_2 + \cdots + x_8 \equiv 0 \pmod 2.$$

Find the minimum distance for this code. How many errors can it detect? How many errors can it correct?

(c) Given a code of length n which corrects e errors, can you always construct a code of length $n+1$ which detects $2e+1$ errors? Give a brief justification of your answer.

Paper 3, Section I

3K Coding and Cryptography

In the following, equivalent definitions of the Reed–Muller code can be used without justification.

(a) Let $n = 2^d$ $(d \geqslant 1)$. For $0 \leqslant r \leqslant d$, state and prove a formula for the rank of the Reed–Muller code $\mathrm{RM}(d,r)$ of length n. State its minimum distance.

(b) Consider the Mariner 9 code RM(5,1). What is its information rate? What proportion of errors can it correct in a single codeword? How do these two properties compare to the Hamming code of length 31?

(c) Show that all but two codewords in $\mathrm{RM}(d,1)$ have the same weight.

Paper 4, Section I

3K Coding and Cryptography

(a) (i) Consider a cryptosystem $(\mathcal{K}, \mathcal{M}, \mathcal{C})$. Let e, d be the respective encryption and decryption functions. Model the key and messages as independent random variables K, M taking values in $\mathcal{K}, \mathcal{M}$, respectively and such that $M = d(C, K) \in \mathcal{M}$ and $C = e(K, M) \in \mathcal{C}$. Show that $H(M|C) \leqslant H(K|C)$.

(ii) Let $\mathcal{M} = \mathcal{C} = \mathcal{A}$, where $\mathcal{A}$ is a finite alphabet. Suppose we send n messages $(M_1, \ldots, M_n)$ encrypted as $(C_1, \ldots, C_n)$ using the same key. Define the *unicity distance*. By making some reasonable assumptions, give a closed formula for the unicity distance as a function of $|\mathcal{K}|$, $|\mathcal{A}|$ and a certain constant.

(b) Suppose we model English text by a sequence of random variables $(X_n)_{n \geqslant 1}$ taking values in $\mathcal{A} = \{A, B, \ldots, Z, \texttt{space}\}$. We define the entropy of English to be

$$H_E = \lim_{n \to \infty} H(X_1, \ldots, X_n)/n.$$

Assuming H_E exists, show that $0 \leqslant H_E \leqslant \log_2 27$. [You may assume Gibbs' inequality.]

Paper 1, Section II

11K Coding and Cryptography

(a) Consider the use of a binary $[n, m]$-code to send one of m messages through a binary symmetric channel (BSC) with error probability p, making n uses of the channel. Define the following decoding rules: (1) *ideal observer*, (2) *maximum likelihood*, and (3) *minimum distance*. Show that if all the messages are equally likely then (1) and (2) agree. If $p < \frac{1}{2}$ show that (2) and (3) agree.

(b) Show that a BSC with error probability $p < \frac{1}{4}$ has non-zero operational capacity.

(c) State Shannon's second coding theorem. Consider a discrete memoryless channel with input X taking values over the alphabet $\{0, 1\}$. For $a, b \in \mathbb{Z}$, let Z be a random variable that is independent of X, taking values over the alphabet $\{a, b\}$ with distribution $\mathbb{P}(Z = a) = \mathbb{P}(Z = b) = \frac{1}{2}$. The output of the channel is $Y = X + Z$. What is the capacity of this discrete memoryless channel? [*Hint: The capacity depends on the value of* $b - a$.]

Paper 2, Section II

12K Coding and Cryptography

(a) (i) Describe briefly the *Rabin cryptosystem*, including how to encrypt and decrypt messages. Show that breaking the Rabin cryptosystem is essentially as difficult as factoring the public modulus, N.

(ii) Criticise the following authentication procedure:

Alice chooses N as the public modulus for the Rabin cryptosystem. To be sure you are in communication with Alice, you send her a "random item" $r = m^2 \pmod{N}$. On receiving r, Alice proceeds to decode using her knowledge of the factorisation of N, and finds a square root m_1 of r. She returns m_1 to you and you check that $r = m_1^2 \pmod{N}$.

(b) (i) Describe briefly the *RSA cryptosystem* with public modulus N.

A budget internet company decides to provide each of its customers with their own RSA ciphers using a common modulus N. Customer j is given the public key (N, e_j) and sent secretly their decrypting exponent d_j. The company then sends out the same message, suitably encrypted, to each of its customers. You intercept two of these messages to customers i and j where e_i and e_j are coprime. Explain how you would decipher the message.

You are one of the customers, and so also know your own decrypting exponent. Can you decipher any message sent to another customer?

(ii) Explain why it might be a bad idea to use RSA with public modulus $N = pq$ with $|p - q|$ small.

A user of RSA accidentally chooses the public modulus N to be a large prime number. Explain why this system is not secure.

Paper 1, Section I

9D Cosmology

Consider a flat (k=0) FLRW universe dominated by the potential energy of a scalar field ϕ given by

$$V(\phi) = \frac{\lambda}{n}\phi^n\,, \qquad \text{where} \qquad \lambda > 0\,,$$

and n is a positive integer. The evolution equations for the scale factor $a(t)$ and the field $\phi(t)$ in the slow-roll approximation are respectively

$$H^2 = \frac{8\pi G}{3c^2}V(\phi)\,,$$
$$3H\dot{\phi} = -c^2\frac{\mathrm{d}V}{\mathrm{d}\phi}\,,$$

where $H = \dot{a}/a$ and a dot denotes differentiation with respect to time t.

(a) By considering the chain rule $\dot{a} = (\mathrm{d}a/\mathrm{d}\phi)\dot{\phi}$, or otherwise, solve the slow-roll equations to find the scale factor as a function of $\phi(t)$,

$$a\left(\phi(t)\right) = \exp\left[\frac{4\pi G}{c^4 n}\left(\phi_i^2 - \phi(t)^2\right)\right]\,,$$

where t_i is the initial time with $a(t_i)$=1 and ϕ_i=$\phi(t_i)$, with ϕ_i assumed to be large enough to ensure inflationary expansion.

(b) By determining the Hubble parameter H, show that during inflation we have

$$\frac{1}{2c^2}\dot{\phi}^2 \;\approx\; \frac{c^4}{48\pi G}\,\frac{n^2 V(\phi)}{\phi^2}\,.$$

Deduce the approximate value of $\phi = \phi_{\text{end}}$ when inflation ends, that is, when the slow-roll approximation breaks down. If $n = 6$, roughly estimate the initial value ϕ_i relative to ϕ_{end} that would be required to solve the flatness problem of the standard cosmology.

Paper 2, Section I

9D Cosmology

A non-relativistic particle species in equilibrium with mass m, temperature T and chemical potential μ satisfying $k_{\mathrm{B}}T \ll mc^2$ and $\mu \ll mc^2$, is described by the Maxwell-Boltzmann distribution. This can be integrated over momenta to give the total number density

$$n = g_s \left(\frac{2\pi m k_{\mathrm{B}} T}{h^2}\right)^{3/2} \exp\left[(\mu - mc^2)/(k_{\mathrm{B}}T)\right] ,$$

where g_s is the degeneracy.

(a) Deuterium is in equilibrium with non-relativistic protons and neutrons at around $t \approx 100$ seconds ($k_{\mathrm{B}}T \approx 0.1\,\mathrm{MeV}$) through the interaction $D \longleftrightarrow n + p$. Show that the ratio of the number densities can be expressed as

$$\frac{n_D}{n_p n_n} \approx \left(\frac{\pi m_p k_{\mathrm{B}} T}{h^2}\right)^{-3/2} e^{B_D/(k_{\mathrm{B}}T)} ,$$

where the deuterium binding energy is $B_D = (m_p + m_n - m_D)c^2 = 2.2\,\mathrm{MeV}$, and m_p, m_n and m_D are the proton, neutron and deuterium masses, respectively. [*Hint: The degeneracy factor for Deuterium is* $g_D = 4$.]

(b) Now use fractional densities relative to the baryon number density n_B (e.g. $X_p = n_p/n_B$) to find an expression for $X_D/(X_p X_n)$. In this case, replace $n_B = \eta\, n_\gamma$ where η is the baryon-to-photon ratio and the photon number is

$$n_\gamma = \frac{16\pi\zeta(3)}{(hc)^3}(k_{\mathrm{B}}T)^3 ,$$

where ζ is the Riemann zeta function. Briefly explain how the fractional density ratio $X_D/(X_p X_n)$ offers insight into the "deuterium bottleneck", that is, the delay in forming deuterium nuclei to temperatures well below the binding energy, $k_{\mathrm{B}}T \ll B_D$?

(c) In an alternative cosmology, the baryon-to-photon ratio η is larger. Assuming that the decoupling of neutrons and protons is unaffected by this change, would the helium abundance Y_{He} be larger or smaller in this scenario than the standard result $Y_{\mathrm{He}} \approx 0.25$? Explain your reasoning.

Paper 3, Section I

9D Cosmology

The Friedmann and acceleration equations for a universe without a cosmological constant ($\Lambda = 0$) are given by

$$\mathcal{H}^2 + kc^2 = \frac{8\pi G}{3c^2}\rho a^2\,, \qquad \mathcal{H}' = -\frac{4\pi G}{3c^2}(\rho + 3P)a^2\,,$$

where ρ is the energy density, P is the pressure, k is the curvature and primes denote differentiation with respect to conformal time τ (defined by $\mathrm{d}\tau = \mathrm{d}t/a(t)$). Here, a is the scale factor and $\mathcal{H}$ is the conformal Hubble parameter defined by $\mathcal{H} \equiv a'/a$.

(a) Assume that the universe is filled with a single-component fluid with equation of state $P = w\rho$, where w is a constant. By introducing the density parameter $\Omega = 8\pi G a^2 \rho/(3c^2\mathcal{H}^2)$, show that the two evolution equations can be recast as

$$\frac{kc^2}{\mathcal{H}^2} = \Omega - 1\,, \qquad 2\mathcal{H}' + (1+3w)\left(\mathcal{H}^2 + kc^2\right) = 0\,.$$

Hence, find the following evolution equation for the density parameter,

$$\Omega' = (1+3w)\mathcal{H}\Omega(\Omega - 1)\,. \qquad (\dagger)$$

(b) Using the time evolution of Ω from equation ($\dagger$), qualitatively describe the flatness problem of an expanding universe ($\mathcal{H} > 0$) for models with an equation of state parameter in the range $0 \leqslant w \leqslant 1$. In particular, roughly sketch the time evolution of Ω in a radiation-filled universe with $w = 1/3$ taking initial values for both $\Omega < 1$ and $\Omega > 1$.

(c) Briefly discuss how the flatness problem can be alleviated by an early epoch of inflation during which $w \approx -1$.

Paper 4, Section I

9D Cosmology

The energy density of photons ρ is given by the Stefan–Boltzmann law,

$$\rho = \frac{4\sigma}{c} T^4 \,,$$

where T is the temperature and σ a constant. As the volume V of the Universe slowly expands, the first law of thermodynamics relates the photon entropy S to the energy $E = \rho V$ and the pressure $P = \rho/3$ (for vanishing chemical potential $\mu = 0$),

$$\mathrm{d}E = T\mathrm{d}S - P\mathrm{d}V \,.$$

(a) By substituting the Stefan–Boltzmann law, show that the entropy differential becomes

$$\mathrm{d}S = \frac{16\sigma}{3c}\Big(T^3\mathrm{d}V + 3T^2V\mathrm{d}T\Big) \,,$$

which should be integrated to find an expression for the photon entropy density $s = S/V$.

(b) If the interaction rate Γ maintaining the photons in equilibrium is much greater than the Hubble expansion rate H (i.e. $\Gamma \gg H$), briefly give the key reason why the photon number and entropy are conserved as the Universe expands (provided the effective number of degrees of freedom g_* of particle species in equilibrium also does not change). Why does the photon temperature fall as $T \propto 1/a$, where a is the scale factor?

(c) Electrons and positrons annihilate and fall out of equilibrium after neutrino decoupling at around $k_BT \approx 1\,\mathrm{MeV}$. By counting the effective number of degrees of freedom g_* in equilibrium before and after this process, provide a brief explanation for why the photon temperature T_γ and neutrino temperature T_ν are related today by

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} \,.$$

Paper 1, Section II

15D Cosmology

Consider a uniformly expanding universe with energy density $\rho(t)$ and pressure $P(t)$ which obey the continuity equation

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P) , \qquad (\star)$$

where a dot denotes a derivative with respect to time t.

(a) Consider the conserved mass M of matter inside a uniform expanding sphere of radius $r(t) = a(t)\, x_0$, with fixed comoving radius x_0. Suppose that the radius of the sphere satisfies

$$\ddot{r} = -\frac{\mathrm{d}\Phi}{\mathrm{d}r} , \qquad \text{where} \qquad \Phi(r) = -\frac{GM}{r} - \frac{1}{6}\Lambda\, r^2 c^2 ,$$

with Λ a constant. By multiplying the acceleration $\ddot{r}$ by the velocity $\dot{r}$ and integrating, show that the scale factor obeys the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\rho - \frac{kc^2}{a^2} + \frac{1}{3}\Lambda c^2 , \qquad (\dagger)$$

where k is a constant.

(b) Now differentiate the Friedmann equation ($\dagger$) and substitute the continuity equation ($\star$) to find the acceleration equation for $\ddot{a}/a$ in terms of ρ, P and Λ. Briefly note two of the shortcomings of this Newtonian analysis.

(c) Consider a flat (k=0) universe with a positive cosmological constant $\Lambda > 0$ that is filled with radiation pressure $P_{\mathrm{R}} = \rho_{\mathrm{R}}/3$, measured to have energy density $\rho_{\mathrm{R}}(t_0) = \rho_{\mathrm{R0}}$ at given time $t = t_0$. Use the Friedmann equation ($\dagger$) to show that the Hubble parameter $H = \dot{a}/a$ can be expressed as

$$H^2 = H_0^2\, \Omega_{\mathrm{R0}}\, a^{-4} + \frac{1}{3}\Lambda c^2 , \qquad \text{where} \qquad \Omega_{\mathrm{R0}} \equiv \frac{8\pi G\, \rho_{\mathrm{R0}}}{3c^2 H_0^2} ,$$

with $H(t_0) = H_0$ and $a(t_0) = 1$. By considering the substitution $b = a^2$ (or otherwise) find the solution for the scale factor

$$a(t) = \alpha\, [\sinh(\beta\, t)]^{1/2} ,$$

where α and β are constants you should determine in terms of H_0 and Ω_{R0}. [You may assume that the universe started with a big bang.]

Show that the scale factor $a(t)$ gives anticipated results at early and late times. Estimate the transition time t_Λ that separates the decelerating and accelerating epochs.

[*Hint:* $\int \mathrm{d}x/\sqrt{1+\kappa^2 x^2}) = (1/\kappa)\sinh^{-1}(\kappa x) + \mathrm{const}$, *where* $\kappa > 0$ *is a constant.*]

Paper 3, Section II

14D Cosmology

In a flat expanding universe dominated by non-relativistic matter with energy density $\rho(\mathbf{x},t)$, we represent small density inhomogeneities $\delta(\mathbf{x},t)$, $|\delta(\mathbf{x},t)| \ll 1$, relative to the homogeneous background $\bar{\rho}(t) = \bar{\rho}_0/a(t)^3$ by the relation $\rho = \bar{\rho}(1+\delta)$, with scale factor $a(t)$ (take $a(t_0)$=1 today). The continuity, Euler and Poisson equations are respectively

$$\begin{aligned}
&\dot{\delta} + \boldsymbol{\nabla}\cdot[\mathbf{v}(1+\delta)] = 0\,,\\
&\dot{\mathbf{v}} + 2\frac{\dot{a}}{a}\mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla}\phi - \frac{c^2}{\rho}\boldsymbol{\nabla}P\,,\\
&\nabla^2\phi = \frac{4\pi G}{c^2}\bar{\rho}\,\delta\,,
\end{aligned}$$

where $P(\mathbf{x},t)$ is the pressure and ϕ the perturbation of the Newtonian potential. Here, we will neglect rotational modes and assume the velocity $\mathbf{v}$ is solely compressional, that is, the divergence is the only non-vanishing part and $\theta \equiv \boldsymbol{\nabla}\cdot\mathbf{v}$. We also assume that pressure effects can be described using the sound speed c_s^2 defined by $c_s^2 \equiv c^2(\mathrm{d}P/\mathrm{d}\rho)$.

(a) Show that for an equation of state given by $P = P(\rho)$, the pressure term in the linearised Euler equation can be written as $(c^2/\rho)\boldsymbol{\nabla}P \approx c_s^2\boldsymbol{\nabla}\delta$. Linearise both the continuity equation and the divergence of the Euler equation (using $\theta = \boldsymbol{\nabla}\cdot\mathbf{v}$) and substitute the Poisson equation in the latter. Transforming to Fourier space with comoving wavemodes $\mathbf{k}$ (i.e. $\boldsymbol{\nabla} \to i\mathbf{k}/a$), combine these to obtain the evolution equation for density perturbations in an expanding universe,

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - \left(\frac{4\pi G}{c^2}\bar{\rho} - \frac{c_s^2 k^2}{a^2}\right)\delta = 0\,, \tag{$\dagger$}$$

where $k = |\mathbf{k}|$. Briefly explain the qualitative implications of each term in this equation. Define the Jeans length λ_J and discuss its significance.

(b) Suppose that the non-relativistic pressure is given by an equation of state $P = \alpha\rho^{4/3}$ where $0 < \alpha \ll 1$; assume that this does not influence the background evolution of a matter-dominated FLRW universe with $\bar{\rho}/c^2 = 1/(6\pi G t^2)$. Show that the varying sound speed has the following time-dependence,

$$\frac{c_s^2(t)}{a^2} \approx \frac{L_0^2}{t^2}\,,$$

where L_0^2 is a constant. Using this sound speed, seek power law solutions $\delta \propto t^\gamma$ of the density perturbation equation ($\dagger$) to find the general solution

$$\delta(\mathbf{k},t) = A_{\mathbf{k}}t^{n_+} + B_{\mathbf{k}}t^{n_-}\,, \qquad \text{where} \qquad n_\pm = -\frac{1}{6} \pm \left[\left(\frac{5}{6}\right)^2 - L_0^2k^2\right]^{1/2}\,.$$

Given $\tilde{k}_\mathrm{J} = 5/(6L_0)$, describe these solutions at late times in the asymptotic limits where $k \ll \tilde{k}_\mathrm{J}$ and where $k \gg \tilde{k}_\mathrm{J}$. What would be the consequence of choosing α such that $L_0 \approx 50\,\mathrm{Mpc}$ today?

Paper 1, Section II

26J Differential Geometry

Let $\Sigma \subset \mathbb{R}^3$ be a smooth surface.

(a) For a point $p \in \Sigma$, define the *first fundamental form* I_p^Σ and the *second fundamental form* II_p^Σ. Give the definitions of the *shape operator*, the *mean curvature* and the *Gaussian curvature* at p. What does it mean for Σ to be *minimal*?

Now let Ω be a non-empty open subset of $\mathbb{R}^2$ and let $h : \Omega \to \mathbb{R}$ be smooth with non-degenerate differential. Let

$$\begin{aligned} \phi : \Omega &\to \mathbb{R}^3 \\ (x, y) &\mapsto (x, y, h(x, y)) \end{aligned}$$

and let $S = \phi(\Omega)$.

(b) Calculate $I^S_{\phi(x,y)}$ at an arbitrary point $(x, y) \in \Omega$ in terms of h.

(c) Write down a Gauss map $N : \Omega \to \mathbb{S}^2$ for S. Let

$$\begin{aligned} \phi_t : \Omega &\to \mathbb{R}^3 \\ (x, y) &\mapsto \phi(x, y) + tN(x, y) \end{aligned}$$

and assume that there is $\epsilon > 0$ so that $S_t := \phi_t(\Omega)$ is a smooth surface for any $t \in (-\epsilon, \epsilon)$. Prove that the second fundamental form of S satisfies

$$II^S_{\phi(x,y)} = -\frac{1}{2}\frac{d}{dt}\Big|_{t=0} I^{S_t}_{\phi_t(x,y)}$$

at any point $(x, y) \in \Omega$. Calculate $II^S_{\phi(x,y)}$ in terms of h.

(d) Derive a differential equation in h that characterises when the surface S is minimal.

(e) Calculate the area of S in terms of the height function h. Assume that for any $\eta : \Omega \to \mathbb{R}$ smooth and compactly supported in Ω, the area $A_\eta(t)$ of

$$S_t^\eta := \{(x, y, h(x, y) + t\eta(x, y)) \mid (x, y) \in \Omega\}$$

is locally minimal at $t = 0$. Use the Euler–Lagrange equation to recover the differential equation in h from part (d).

Paper 2, Section II

26J Differential Geometry

Consider a smooth closed curve $\alpha : I \to \mathbb{S}^2$ on the sphere, parametrised by arc length.

(a) Define the *curvature* κ, *torsion* τ, *Frénet trihedron* $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ and *geodesic curvature* κ_g of a general curve on a general surface. Prove in the particular case of the sphere that $\alpha = -\kappa^{-1}\mathbf{n} - \tau^{-1}\kappa^{-2}\dot{\kappa}\mathbf{b}$ and $\kappa_g = -\kappa^{-1}\tau^{-1}\dot{\kappa}$.

(b) State the local Gauss–Bonnet theorem for the curve α on $\mathbb{S}^2$. Deduce that, given a fixed length $I = [0, L]$, the curve α maximising the enclosed area also minimises $\int_I \kappa_g$.

(c) Consider $\varphi : [0, L] \to \mathbb{R}$ smooth with compact support in $\{\kappa_g \neq 0\}$, and $\beta : [0, L] \to \mathbb{R}^3$ defined by $\beta = -\varphi\mathbf{t} - \kappa^{-1}\dot{\varphi}\mathbf{n} - \kappa^{-1}\kappa_g^{-1}\dot{\varphi}\mathbf{b}$. Prove that $\beta \perp \alpha$ and $\dot{\beta} \perp \dot{\alpha}$.

(d) Consider the curve $\gamma^\epsilon := (\alpha + \epsilon\beta)/|\alpha + \epsilon\beta|$ for small ϵ. You may assume that, if the value $\epsilon = 0$ is a critical point of the enclosed area, then $\int_0^L \kappa_g^{-1}\dot{\varphi} = 0$. Deduce from this equation that area-maximising curves have constant geodesic curvature.

(e) Prove that a curve α on $\mathbb{S}^2$ with constant geodesic curvature is planar by showing that the vector $\mathbf{e}(s) := \alpha(s) \times \dot{\alpha}(s) + \kappa_g \alpha(s)$ is constant.

(f) Deduce that, if a curve on $\mathbb{S}^2$ of length L encloses an area A, then $L^2 \geqslant A(4\pi - A)$ (with the convention that we always choose the smaller of the two areas enclosed by the curve).

Paper 3, Section II

25J Differential Geometry

Let $n \geqslant 1$ and $1 \leqslant k \leqslant N$ be integers. Let $\mathbf{Id}$ denote the identity matrix.

(a) State the definition of a *k-dimensional smooth manifold* $X \subset \mathbb{R}^N$. Define the *tangent space* $T_x X$ at a point $x \in X$ in terms of a parametrisation about x, and prove that the tangent space is independent of the parametrisation.

(b) Let $GL(n) \subset \mathbb{R}^{n^2}$ be the set of real $n \times n$ invertible matrices. Prove that it is a manifold, give its dimension, and compute $T_{\mathbf{Id}} GL(n)$.

(c) Let $SL(n) \subset \mathbb{R}^{n^2}$ be the set of real $n \times n$ matrices with determinant 1. Prove that it is a manifold, give its dimension, and compute $T_{\mathbf{Id}} SL(n)$.

(d) Let $SO(n) \subset \mathbb{R}^{n^2}$ be the set of real $n \times n$ matrices with determinant 1 and orthonormal column vectors. Prove that it is a manifold, give its dimension, and compute $T_{\mathbf{Id}} SO(n)$.

Paper 4, Section II

25J Differential Geometry

(a) Let X be a smooth, compact, connected manifold. State the homotopy lemma for smooth maps $f, g : X \to X$ and the homogeneity lemma for X. Prove that the number of pre-images modulo 2 of a smooth map $f : X \to X$ is the same at every regular value. Deduce a proof of the smooth Brouwer fixed-point theorem.

(b) Let $n \geqslant 1$ and $A := \{1 \leqslant \|x\| \leqslant 2\} \subset \mathbb{R}^{n+1}$. Consider a smooth map $\varphi : \mathbb{S}^n \to \mathbb{S}^n$ such that $\varphi(x) \perp x$ for all $x \in \mathbb{S}^n$, and extend it to $\tilde{\varphi} : A \to A$ by $\tilde{\varphi}(x) := \varphi(x/\|x\|)\|x\|$. For $\epsilon > 0$, define $\psi_\epsilon : A \to A$ by

$$\psi_\epsilon(x) := \frac{x + \epsilon\tilde{\varphi}(x)}{\sqrt{1+\epsilon^2}}\,.$$

Prove that, for ϵ small enough, ψ_ϵ is a diffeomorphism from A to A with $\det D\psi_\epsilon > 0$.

(c) Prove $\int_A \det(\mathrm{Id} + \epsilon D\tilde{\varphi}) = \mathrm{vol}(A)(1+\epsilon^2)^{(n+1)/2}$, where $\mathrm{vol}(A)$ is the volume of A.

(d) Deduce that, when n is even, there is no smooth map $\varphi : \mathbb{S}^n \to \mathbb{S}^n$ such that $\varphi(x) \perp x$ for all $x \in \mathbb{S}^n$.

Paper 1, Section II

32A Dynamical Systems

Let $F : I \to I$ be a continuous map of an interval $I \subset \mathbb{R}$.

(a) (i) Define what it means for F to be *chaotic*, according to Glendinning.

(ii) Define what it means for F to be *chaotic*, according to Devaney.

(b) Suppose now that F has a periodic orbit of period 3.

(i) Show that F also has periodic orbits of period n for all positive integers n.

(ii) Explain briefly why F must have at least four distinct 7-cycles.

(iii) How many distinct 8-cycles must F have?

[*Relevant theorems that you use from the course should be stated clearly.*]

Paper 2, Section II

33A Dynamical Systems

(a) State the normal form for a transcritical bifurcation in terms of the time t, the dependent variable x and parameter μ. Illustrate using diagrams why this type of bifurcation is not structurally stable, making sure that your diagrams are clearly labelled.

(b) Consider the system given by

$$\begin{aligned}\dot{x} &= y - x + ax^3,\\ \dot{y} &= rx - y - zy,\\ \dot{z} &= -z + xy,\end{aligned}$$

where a and r are constants.

(i) Show that the fixed point at the origin of the system is non-hyperbolic at $r = 1$.

(ii) Find the stable, unstable and centre subspaces of the linearised system of the fixed point at the origin at $r = 1$.

(iii) Set $r = 1$ and change to new coordinates (v, w, z) where $v = (x + y)/2$, $w = (x - y)/2$ and z is unchanged. Seek the (non-extended) centre manifold by writing $w = w_c(v)$ and $z = z_c(v)$. Find w_c and z_c to fourth order in v.

[*Hint: By considering symmetries, some of this calculation can be simplified.*]

(iv) Show that the evolution equation on the centre manifold is of the form

$$\dot{v} = \frac{a-1}{2}v^3 + \frac{(3a-1)(a+3)}{8}v^5 + \ldots$$

(v) For what values of a is the origin asymptotically stable when $r = 1$?

Paper 3, Section II

31A Dynamical Systems

(a) A dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^2$ has a periodic orbit $\mathbf{x} = \mathbf{X}(t)$ with period T. The linearised evolution of a small perturbation $\mathbf{x} = \mathbf{X}(t) + \boldsymbol{\eta}(t)$ is given by $\eta_i(t) = \Phi_{ij}(t)\eta_j(0)$. Obtain the differential equation and initial condition satisfied by the matrix $\boldsymbol{\Phi}(t)$.

Explain how the stability of the orbit $\mathbf{X}(t)$ is connected to the quantity

$$\exp\left[\int_0^T \boldsymbol{\nabla}\cdot\mathbf{f}\left(\mathbf{X}(t)\right)\,dt\right].$$

(b) Using the energy-balance method for nearly Hamiltonian systems, find a condition on the parameter a for a limit cycle to exist in the system given by

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x + \epsilon(1 - x^2 + ay^2)y,\end{aligned}$$

where $0 < \epsilon \ll 1$. Determine the stability of the limit cycle.

[*Hint:* $\int_0^{2\pi} \sin^2\theta\cos^2\theta\,d\theta = \pi/4$ *and* $\int_0^{2\pi}\sin^4\theta\,d\theta = 3\pi/4.$]

Paper 4, Section II

32A Dynamical Systems

(a) Consider a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^2$.

(i) State the Poincaré–Bendixson Theorem.

(ii) Explain why every periodic orbit must enclose at least one fixed point.

(b) Consider the system in $\mathbb{R}^2$ given by

$$\begin{aligned} \dot{x} &= -x + ay + x^2 y, \\ \dot{y} &= \quad b - ay - x^2 y, \end{aligned}$$

where a and b are real, positive parameters.

(i) Find the fixed point of the system. On a sketch of the (a, b)-plane (for the quadrant where $a, b > 0$), show where the fixed point is stable.

(ii) A closed region $\mathcal{D}$ of the (x, y)-plane is given by the filled polygon with vertices at $(0, 0)$, $(0, b/a)$, $(b, b/a)$, (x^*, y^*) and $(x^*, 0)$, where (x^*, y^*) is such that $\dot{x} = 0$ at that point and the line segment between $(b, b/a)$ and (x^*, y^*) has gradient -1.

Show that trajectories do not leave $\mathcal{D}$.

[There is no need to determine (x^*, y^*) explicitly, but you may use that $b < x^*$.]

(iii) Use your results above to give conditions on the parameters a and b for the system to have a periodic orbit.

Paper 1, Section II

37D Electrodynamics

Consider a spacetime with coordinates $x^\mu = (ct, \boldsymbol{x})$ and metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$, where $\mu,\nu = 0,1,2,3$ and c is the speed of light. A 4-vector potential $A_\mu(x)$ fills spacetime and is described by the action

$$S[A_\mu] = -\frac{1}{\mu_0 c}\int \left(\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{m^2}{2}A_\mu A^\mu - \mu_0 A_\mu J^\mu\right) \mathrm{d}^4x\,,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor, $J_\mu(x)$ is a conserved 4-current density, m and μ_0 are constants and $m \geqslant 0$.

(a) Show that the equations of motion for the field,

$$\partial_\mu F^{\mu\nu} - m^2 A^\nu = -\mu_0 J^\nu\,,$$

follow from the principle of stationary action.

(b) Clearly state the conditions for the action to be invariant under Lorentz transformations and under gauge transformations of the form $A_\mu \to A_\mu + \partial_\mu \chi$, where χ is a scalar field.

(c) Show that for $m > 0$ the equations of motion imply the identity $\partial_\mu A^\mu = 0$.

(d) Writing the vector potential as $A^\mu = (\phi/c, \boldsymbol{A})$, show that for $m \geqslant 0$ the equations of motion for ϕ and $\boldsymbol{A}$ can be written as

$$\Box\phi + \frac{\partial \alpha}{\partial t} - m^2\phi = -c\mu_0 J^0\,, \qquad (\dagger)$$

$$\Box\boldsymbol{A} - \boldsymbol{\nabla}\alpha - m^2\boldsymbol{A} = -\mu_0\boldsymbol{J}\,,$$

where $\alpha = \frac{1}{c^2}\frac{\partial\phi}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{A}$ and $\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$ is the wave operator.

(e) For a point charge q at rest, the 4-current density is $J^0 = cq\delta(\boldsymbol{x})$ and $\boldsymbol{J} = 0$, where δ denotes the 3-dimensional δ function. By applying a Fourier transform in the spatial coordinates to the equation of motion ($\dagger$), show that for $m > 0$, a time-independent field solution for a point charge at $r = 0$ is given by

$$\phi = \lambda\frac{\exp(-mr)}{r}, \quad \boldsymbol{A} = 0\,,$$

where $r = |\boldsymbol{x}|$ and λ is a constant you do *not* need to determine. [*Hint: You may use without proof that the inverse Fourier transform of* $1/(|\boldsymbol{k}|^2 + m^2)$ *is*

$$\int \frac{e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}}{|\boldsymbol{k}|^2 + m^2}\frac{\mathrm{d}^3k}{(2\pi)^3} = C\frac{e^{-m|\boldsymbol{x}|}}{|\boldsymbol{x}|}\,,$$

where $\boldsymbol{k}$ *is the wave vector and* C *is a non-zero constant.*]

Provide a brief physical interpretation of this result, including the limiting case where $m \to 0$, and connect this interpretation to the result of part (b).

Paper 3, Section II

36D Electrodynamics

The Maxwell equations for charge and current densities $\rho(\boldsymbol{x},t)$ and $\boldsymbol{J}(\boldsymbol{x},t)$ are

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=\frac{\rho}{\epsilon_0}\,,\qquad \boldsymbol{\nabla}\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t}\,,$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}=0\,,\qquad \boldsymbol{\nabla}\times\boldsymbol{B}=\mu_0\boldsymbol{J}+\mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\,.$$

The energy and momentum density of the electromagnetic field are defined by

$$\mathcal{E}=\frac{1}{2}\left(\epsilon_0\boldsymbol{E}^2+\frac{\boldsymbol{B}^2}{\mu_0}\right)\qquad\text{and}\qquad \boldsymbol{g}=\varepsilon_0\boldsymbol{E}\times\boldsymbol{B}\,.$$

(a) Use the Maxwell equations to show that $\boldsymbol{g}$ obeys the local conservation law

$$\partial_t g_j+\nabla_i\sigma_{ij}=-(\rho\boldsymbol{E}+\boldsymbol{J}\times\boldsymbol{B})_j\,,$$

with

$$\sigma_{ij}=\tilde{a}\,E_iE_j+\tilde{b}\,\boldsymbol{E}^2\delta_{ij}+\tilde{c}\,B_iB_j+\tilde{d}\,\boldsymbol{B}^2\delta_{ij}\,,$$

where δ_{ij} is the Kronecker delta and $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$ are constants you should determine. [*Hint: A vector field* $\boldsymbol{a}$ *satisfies* $(\boldsymbol{\nabla}\times\boldsymbol{a})\times\boldsymbol{a}=(\boldsymbol{a}\cdot\boldsymbol{\nabla})\boldsymbol{a}-\frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{a}^2)$]

(b) Provide a brief physical interpretation of the tensor σ_{ij} and, in particular, its diagonal and off-diagonal components.

(c) The electric field of a homogeneously charged sphere with total charge q and radius R centered on the origin is given by

$$\boldsymbol{E}=E(r)\boldsymbol{e}_r\quad\text{with}\quad E(r)=\begin{cases}\frac{q}{4\pi\epsilon_0}\frac{r}{R^3} & \text{for } r\leqslant R\\ \frac{q}{4\pi\epsilon_0}\frac{1}{r^2} & \text{for } r>R\,,\end{cases}$$

where $r=|\boldsymbol{x}|$ and $\boldsymbol{e}_r$ is the unit vector in the radial direction. The magnetic field $\boldsymbol{B}$ is zero. Calculate the energy density $\mathcal{E}$ and, thus, the total energy W_{em} contained in the electromagnetic field.

Under a Lorentz transformation with velocity $\boldsymbol{v}=(0,0,v)$ in the z direction and Lorentz factor $\gamma=1/\sqrt{1-(v^2/c^2)}$, the electromagnetic field changes to $\boldsymbol{E}'$ and $\boldsymbol{B}'$ given by

$$\boldsymbol{E}'_{||}=\boldsymbol{E}_{||}\,,\qquad \boldsymbol{E}'_{\perp}=\gamma(\boldsymbol{E}_{\perp}+\boldsymbol{v}\times\boldsymbol{B})\,,$$

$$\boldsymbol{B}'_{||}=\boldsymbol{B}_{||}\,,\qquad \boldsymbol{B}'_{\perp}=\gamma\left(\boldsymbol{B}_{\perp}-\frac{\boldsymbol{v}}{c}\times\frac{\boldsymbol{E}}{c}\right)\,,$$

where the subscripts $||$ and $\perp$, respectively, denote field components parallel and perpendicular to $\boldsymbol{v}$. Compute the linear momentum $\boldsymbol{P}=(0,0,P_z)$ contained in the electromagnetic field of a homogeneously charged sphere moving with $\boldsymbol{v}$ in the slow-velocity limit, i.e. ignoring terms of order $(v/c)^2$ or higher. [*Hint: Vector fields* $\boldsymbol{a}$, $\boldsymbol{b}$ *and* $\boldsymbol{c}$ *satisfy the relation* $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$ *and* $\int_0^\pi\sin^3\theta\,\mathrm{d}\theta=\frac{4}{3}$.]

Paper 4, Section II

36D Electrodynamics

(a) Define a *dielectric* medium and qualitatively explain how the polarization $\boldsymbol{P}$ and magnetization $\boldsymbol{M}$ result in bound charge and bound current densities

$$\rho_{\mathrm{bd}} = -\boldsymbol{\nabla}\cdot\boldsymbol{P}\,, \qquad \boldsymbol{J}_{\mathrm{bd}} = \frac{\partial \boldsymbol{P}}{\partial t} + \boldsymbol{\nabla}\times\boldsymbol{M}\,.$$

(b) Starting from the *microscopic* Maxwell equations for the electric and magnetic fields $\boldsymbol{E}$, $\boldsymbol{B}$,

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{E} &= \frac{\rho}{\epsilon_0}\,, & \boldsymbol{\nabla}\times\boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t}\,, \\ \boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0\,, & \boldsymbol{\nabla}\times\boldsymbol{B} &= \mu_0\boldsymbol{J} + \mu_0\epsilon_0\frac{\partial \boldsymbol{E}}{\partial t}\,, \end{aligned}$$

derive the *macroscopic* Maxwell equations for a dielectric medium in terms of the electric displacement $\boldsymbol{D}$ and the magnetising field $\boldsymbol{H}$, which you should define, as well as the free charge and current densities.

(c) Consider a spherical shell with radial extent $R_1 < r < R_2$, that consists of a linear magnetic medium with constant permeability μ such that $\boldsymbol{B} = \mu\boldsymbol{H}$. This shell is placed inside a magnetic field that approaches $\boldsymbol{B} = B_0\boldsymbol{e}_z$, $B_0 = \text{const}$, as $r\to\infty$. The electric field, the polarization, and the free charge and current densities are zero. Using the ansatz $\boldsymbol{B} = \boldsymbol{\nabla}\psi$, show that a solution for the magnetic field in all space is given by

$$\psi(r,\theta) = \begin{cases} (a_1 r + b_1/r^2)\cos\theta & \text{for } r < R_1 \\ (a_2 r + b_2/r^2)\cos\theta & \text{for } R_1 < r < R_2 \\ (a_3 r + b_3/r^2)\cos\theta & \text{for } r > R_2\,. \end{cases}$$

Using the boundary conditions at infinity, the origin, $r = R_1$ and $r = R_2$, derive six conditions for the free parameters a_1, b_1, a_2, b_2, a_3 and b_3. Express the parameters b_1, a_2, b_2 and a_3 in terms of B_0, μ, R_1, R_2 and a_1. Briefly describe how you would calculate the remaining parameters a_1 and b_3 (you do not need to compute a_1 and b_3).

[*Hint: In polar coordinates, the Laplace operator and the derivative in the Cartesian z direction are*

$$\begin{aligned} \nabla^2 f &= \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}f\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}f\,, \\ \frac{\partial}{\partial z}f &= \cos\theta\frac{\partial}{\partial r}f - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}f\,.] \end{aligned}$$

Paper 1, Section II

39C Fluid Dynamics II

A solid cylinder of density $\rho + \Delta\rho$, length L and radius $a < L$ is placed with its axis vertical at the centre of a long, closed, vertically oriented cylindrical container of radius $a + h$, where $h \ll a$. The container is otherwise filled with an incompressible fluid of density ρ and dynamic viscosity μ, through which the solid cylinder falls with speed U. Ignoring end effects, show that the downwards velocity field in the thin gap between the solid cylinder and the container can be approximated as

$$u = -\frac{\Delta p}{2\mu L} y(h-y) + U\frac{y}{h},$$

where y is the coordinate directed inwards across the thin gap from the wall of the cylindrical container and Δp is the dynamic pressure difference between the fluid below and above the cylinder. Determine the associated volume flux along the gap and the viscous shear stress on the solid cylinder.

Use global mass conservation to show that $\Delta p \approx 6\mu a L U/h^3$. Show that the associated form drag is much larger than the viscous force on the cylinder and hence determine the speed of fall U.

Paper 2, Section II

39C Fluid Dynamics II

A two-dimensional incompressible Stokes flow has stream function ψ such that the velocity $\mathbf{u} = \nabla\times(\psi\mathbf{k})$, where $\mathbf{k}$ is the unit vector normal to the plane of the flow. Show that

$$\nabla^4\psi = 0.$$

In plane polar coordinates (r, θ) the velocity is

$$\mathbf{u} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{e}_r - \frac{\partial\psi}{\partial r}\mathbf{e}_\theta.$$

Given that the stream function has the form $\psi = r^2 f(\theta)$, determine the rate-of-strain tensor in terms of f and its derivatives. Hence write down the corresponding deviatoric stress tensor for a fluid of dynamic viscosity μ.

Fluid with dynamic viscosity μ fills the two-dimensional region $-\alpha < \theta < 0$, $r > 0$, where $\alpha > 0$ is a constant. The boundary $\theta = -\alpha$ is rigid, while a tangential stress S is applied to the horizontal surface $\theta = 0$. Given that the stream function has the form $\psi = r^2 f(\theta)$, write down the boundary conditions that apply to $f(\theta)$. Hence, determine $f(\theta)$ and show that the surface velocity

$$U(r) = u(r, 0) = \frac{Sr}{\mu}\frac{1 - \cos 2\alpha - \alpha\sin 2\alpha}{\sin 2\alpha - 2\alpha\cos 2\alpha}.$$

[*Hint: In plane polar coordinates,* $\nabla^2\psi = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial\psi}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2\psi}{\partial\theta^2}$.]

Paper 3, Section II

38C Fluid Dynamics II

A long, thin organism can be modelled as a body that occupies the region $y = 0$, $x > 0$ in two dimensions, surrounded by an incompressible fluid. The organism is at rest relative to the fluid that is far away from it, and it exerts a tangential stress $Sx^{-1/2}$, with $S > 0$ a constant, on the surrounding fluid.

(a) Write down the boundary-layer equations describing steady high-Reynolds-number flow around the organism. What boundary conditions apply to this flow?

(b) Use scaling to show that the width δ of the boundary layer in the y direction satisfies $\delta \sim (\rho\nu^2/S)^{1/3}x^{1/2}$, where ρ and ν are the density and kinematic viscosity of the fluid, respectively. What is the associated velocity scale in the direction parallel to the organism?

(c) Determine the form of a similarity solution to the boundary-layer equations in which the stream function is proportional to a dimensionless function $f(\eta)$ of a suitable similarity variable η. Determine the differential equation and boundary conditions satisfied by $f(\eta)$. Explain briefly how one could solve the differential equation numerically to determine the velocity field and, specifically, the velocity of the fluid adjacent to the organism.

Paper 4, Section II

38C Fluid Dynamics II

Given a material interface $r = R(\theta, t)$ between two fluid regions in plane polar coordinates (r, θ), explain why

$$\frac{\partial R}{\partial t} + \frac{v}{r}\frac{\partial R}{\partial \theta} = u,$$

where $u\boldsymbol{e}_r + v\boldsymbol{e}_\theta$ is the fluid velocity at the interface.

An ocean gyre is modelled as a two-dimensional, circular patch of water of radius a in solid-body motion with angular velocity ω, surrounded by stationary water. Consider small, sinusoidal perturbations to the edge of the gyre $r = a + \eta(\theta, t)$, where $\eta = \epsilon \exp(ik\theta + \sigma t)$ and $\epsilon \ll ka$. Assuming potential flow, determine the relationship between the growth rate σ and the wave number k.

Briefly describe the subsequent motion of the gyre. Do disturbances propagate upstream or downstream relative to the original motion of the gyre?

[*Hint: In plane-polar coordinates,* $\nabla^2\phi = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial \phi}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2}$.]

Paper 1, Section I

7D Further Complex Methods

(a) Let $f(x)$, $x \in \mathbb{R}$, be a function with a finite number of singular points $x = c_k$, $k = 1, 2, \ldots, N$, where $-\infty < a < c_1 < c_2 < \cdots < c_N < b < \infty$. Define what is meant by the *Cauchy Principal Value integral* $\mathcal{P}\int_{-\infty}^{\infty} f(x)\mathrm{d}x$. [You may assume that the improper integrals $\int_{-\infty}^{a} f(x)\mathrm{d}x$ and $\int_{b}^{\infty} f(x)\mathrm{d}x$ exist.]

(b) What is the *Hilbert transform* $\mathcal{H}(f)(y)$ of a function f?

(c) Let $f(x) = \frac{1}{x^2+1}$. Evaluate $\mathcal{H}(f)(-1)$.

Paper 2, Section I

7D Further Complex Methods

Consider the differential equation

$$x\frac{\mathrm{d}^3 y(x)}{\mathrm{d}x^3} + 2y(x) = 0 \qquad (\dagger)$$

on the domain $x \in (0, \infty)$.

(a) Write

$$y(z) = \int_\gamma e^{zt} f(t)\,\mathrm{d}t,$$

where γ is a contour in the complex plane, and substitute this expression for y into the differential equation ($\dagger$). Explain how the resulting integral equation can be solved by finding an appropriate function $f(t)$ and contour γ. Determine this function $f(t)$ and clearly state any required conditions on γ.

(b) Express the solution $y(x)$ in integral form. [You do *not* have to evaluate this integral, but you should simplify it as far as possible.] [*Hint: Consider a subset of the real axis for your choice of the contour γ.*]

Paper 3, Section I

7D Further Complex Methods

Let $F(z)$ be defined on the half plane $H = \{z \in \mathbb{C} \;:\; \mathsf{Re}(z) > 0\}$ such that

1. $F(z)$ is analytic on H.
2. $F(z+1) = zF(z)$ for all $z \in H$,
3. $F(z)$ is bounded in the strip $\{z \in \mathbb{C} \;:\; 1 \leqslant \mathsf{Re}(z) \leqslant 2\}$,
4. $F(1) = 1$.

(a) By using $F(z+1) = zF(z)$ for all $z \in \mathbb{C}$, extend F meromorphically to $z \in \mathbb{C}$ with $\mathsf{Re}(z) \leqslant 0$. Identify and characterize the singular points of the extended function.

(b) Now consider a function $\Gamma : H \to \mathbb{C}$ that also satisfies the four conditions listed above. By meromorphically extending Γ in the same way as done for F in part (a), show that the function $f(z) = F(z) - \Gamma(z)$ is entire.

(c) Assuming additionally that $f(z)$ is bounded on the strip $0 \leqslant \mathsf{Re}(z) \leqslant 1$, show that the function $S(z) = f(z)f(1-z)$ is entire and bounded on $\mathbb{C}$. You may conclude without further proof that $S(z)$ is therefore constant. Use this result to prove Wielandt's theorem, i.e. that $F(z) = \Gamma(z)$ for $z \in \mathbb{C}$.

Paper 4, Section I

7D Further Complex Methods

The Papperitz equation is a second-order linear differential equation given by

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \left(\frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c}\right)\frac{\mathrm{d}w}{\mathrm{d}z} \qquad (\dagger)$$
$$- \frac{(b-c)(c-a)(a-b)}{(z-a)(z-b)(z-c)}\left(\frac{\alpha\alpha'}{(z-a)(b-c)} + \frac{\beta\beta'}{(z-b)(c-a)} + \frac{\gamma\gamma'}{(z-c)(a-b)}\right) w = 0$$

with the constraint $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma' = 1$ where $\alpha, \alpha', \beta, \beta', \gamma, \gamma', a, b, c \in \mathbb{C}$ and a, b, c are pairwise distinct.

(a) State the general conditions for $z_0 \in \mathbb{C}$ to be a *regular singular point* for a second-order linear differential equation. Determine the regular singular points of the Papperitz equation.

(b) Consider the Papperitz equation with non-integer $\alpha - \alpha'$, $\beta - \beta'$ and $\gamma - \gamma'$. Derive the leading-order exponents of two linearly independent solutions of this equation near each of its regular singular points.

(c) What is the *Papperitz symbol* for equation ($\dagger$)?

(d) Consider now the hypergeometric equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \left(\frac{C}{z} + \frac{1+A+B-C}{z-1}\right)\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{AB}{z(z-1)} w = 0\,,$$

where A, B, $C \in \mathbb{C}$. This equation can be obtained as a special case of the Papperitz equation by setting $a = 0$, $b = 1$ and taking the limit $c \to \infty$. In the following you may set $c = \infty$ without further proof or derivation. By comparing the coefficients of the hypergeometric and the Papperitz equations, and setting $\alpha = 0$, $\beta = 0$, establish a Papperitz symbol for the hypergeometric equation.

Paper 1, Section II

14D Further Complex Methods

(a) Consider a linear input–output system

$$\mathcal{L}y(t) = f(t),$$

where $f(t)$ is the input, $y(t)$ is the output and $\mathcal{L}$ is a linear operator. Define what it means for the system to be *causal* and *stable*.

(b) Consider a linear ordinary differential equation for $t > 0$,

$$\alpha y'(t) + y(t) = f(t)\,, \qquad (\dagger)$$

with $\alpha \in \mathbb{R}$, $\alpha \neq 0$, and initial condition $y(0) = 0$.

(i) Using a Laplace transform, show that the transfer function of the linear system ($\dagger$) is $G(s) = \frac{1}{\alpha s+1}$. Determine for which values of α the system is stable.

(ii) A negative feedback loop with $H(s) = k$, $k \in \mathbb{R}$, is introduced into the system for stable values of α such that the closed-loop transfer function is $G_{\mathrm{CL}}(s) = \frac{G(s)}{1+H(s)G(s)}$. By direct inspection of $G_{\mathrm{CL}}(s)$, determine the values of k for which the closed-loop system is stable. [You do not have to consider the limiting cases for k.]

Explain the *Nyquist criterion* to determine the stability of a closed-loop system. By calculating appropriate winding numbers, use the Nyquist stability criterion to determine the values of k for which the closed-loop system is stable. Compare the result with that obtained by direct inspection of $G_{\mathrm{CL}}(s)$. [*Hint: You may use without proof that the number P of poles and the number Z of zeros, counting multiplicities, of a meromorphic function $f(z)$ inside a clockwise simple closed contour γ obey the relation*

$$\frac{1}{2i\pi}\oint_{\gamma}\frac{f'(z)}{f(z)}\mathrm{d}z = P - Z\,,$$

if $f(z)$ has no zeros or poles on γ.]

Paper 2, Section II

13D Further Complex Methods

(a) When is a function $f : \mathbb{C} \to \mathbb{C}$ called *elliptic*? Define a *fundamental cell* explicitly and state the property of an elliptic function regarding the number of its zeros and poles in a fundamental cell.

(b) Show that an elliptic function without poles is constant. [You may use without proof that an entire and bounded function is constant.]

(c) Let z_j denote the poles of an elliptic function f in a fundamental cell. Show that

$$\sum_j \mathrm{Res}(f; z_j) = 0.$$

Can there exist an elliptic function with a single pole of order one in a fundamental cell?

(d) If h is a meromorphic function on and inside a simple closed clockwise contour γ and h has no zeros or poles on γ, then

$$\frac{1}{2\pi i} \oint_\gamma \frac{h'(z)}{h(z)} \mathrm{d}z = P - Z\,,$$

where P and Z denote respectively the number of poles and zeros, counting multiplicities, of $h(z)$ inside the contour γ. Using this relation, show that a non-constant elliptic function f takes each value the same number of times in a cell, counting multiplicities.

(e) An example of an elliptic function is the Weierstrass function

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{(m,n)} \left[\frac{1}{(z - w_{m,n})^2} - \frac{1}{w_{m,n}^2} \right] ,$$

where $w_1,\ w_2 \in \mathbb{C} \setminus \{0\}$ with $\frac{w_1}{w_2} \notin \mathbb{R}$, $w_{m,n} = m w_1 + n w_2$ and the sum extends over $(m, n) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$. Identify and characterize the singularities of the Weierstrass function.

(f) The Laurent series of the Weierstrass function about $z_0 = 0$ can be written as

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{k=0}^{\infty} a_{2k} z^{2k}.$$

Calculate the coefficients a_{2k}.

Paper 1, Section II

18H Galois Theory

(a) Let $\alpha, \beta \in \mathbb{C}$ be algebraic over $\mathbb{Q}$. Show that if α and β have the same minimal polynomial over $\mathbb{Q}$ then $\mathbb{Q}(\alpha) \cong \mathbb{Q}(\beta)$. Let $k \geqslant 1$ be the number of such isomorphisms. Give an example where $1 < k < [\mathbb{Q}(\alpha) : \mathbb{Q}]$. Must k always divide $[\mathbb{Q}(\alpha) : \mathbb{Q}]$? Justify your answer.

(b) Let L/K be a finite extension of degree coprime to n. Clearly stating any properties of the norm that you use, show that if $\alpha \in K$ is an nth power in L then it is an nth power in K.

(c) Let $K = \mathbb{Q}(\alpha)$ where α has minimal polynomial f over $\mathbb{Q}$. Let p be a prime. Show that $f(X^p)$ is irreducible in $\mathbb{Q}[X]$ if and only if α is not a pth power in K.

Paper 2, Section II

18H Galois Theory

(a) Let L/K be a field extension. Explain what it means to say that

(i) L/K is *finite*;

(ii) L/K is *separable*;

(iii) L/K is *simple*.

Which pairs of these properties together imply the third? In each case give a proof or counterexample.

(b) Let L be the splitting field of $f(X) = X^3 - X - 1$ over $\mathbb{Q}$. Compute $\mathrm{Gal}(L/\mathbb{Q})$. Show that L has a unique quadratic subfield, and write it in the form $\mathbb{Q}(\sqrt{d})$ for d a squarefree integer. Show also that if α is a root of f then $L = \mathbb{Q}(\alpha + \sqrt{d})$.

Paper 3, Section II

18H Galois Theory

(a) Let $M/L_1/K$ and $M/L_2/K$ be finite extensions of fields. Define the composite L_1L_2. Show that if L_1/K is Galois then L_1L_2/L_2 is Galois, and that there is an injective group homomorphism $\mathrm{Gal}(L_1L_2/L_2) \to \mathrm{Gal}(L_1/K)$.

(b) Let K be a field of characteristic not dividing n, and with algebraic closure $\overline{K}$. Let $\zeta_n \in \overline{K}$ be a primitive nth root of unity. Prove that $[K(\zeta_n) : K]$ divides $\phi(n)$, and that equality holds if $K = \mathbb{Q}$.

(c) Let $L_1 = \mathbb{Q}(\zeta_9)$ and $L_2 = \mathbb{Q}(\zeta_{15})$. Write each of the fields $L_1 \cap L_2$ and L_1L_2 in the form $\mathbb{Q}(\zeta_n)$ for suitable n. Justify your answers.

Paper 4, Section II

18H Galois Theory

(a) Give an example of a pair of monic quartic polynomials $f, g \in \mathbb{Z}[X]$ whose splitting fields over $\mathbb{Q}$ are isomorphic, but whose Galois groups over $\mathbb{Q}$ are not conjugate as subgroups of S_4.

(b) Let p be a prime and $q = p^d$ with $d \geqslant 1$. Show that there exists a field with q elements, and that it is unique up to isomorphism. [Standard results about splitting fields may be quoted without proof.] Show also that any irreducible factor of $X^q - X \in \mathbb{F}_p[X]$ has degree at most d.

(c) State a theorem which explains how reduction mod p may be used to help compute Galois groups. The polynomial $f(X) = X^4 - 21X^2 + 3X + 100$ has discriminant 547^2. Compute $\mathrm{Gal}(f/\mathbb{Q})$.

Paper 1, Section II

38B General Relativity

Consider a massive test particle moving in the Schwarzschild metric of a black hole with mass m (in units with $c = G = 1$):

$$ds^2 = -\Big(1 - \frac{2m}{r}\Big)dt^2 + \Big(1 - \frac{2m}{r}\Big)^{-1}dr^2 + r^2(\,d\theta^2 + \sin^2\theta\,d\phi^2\,)\;.$$

(a) Assuming that the motion lies in the equatorial plane $\theta = \pi/2$, justify briefly why

$$h = r^2\dot{\phi} \qquad \text{and} \qquad \frac{1}{2}\,\dot{r}^2 - \frac{m}{r} + \frac{h^2}{2r^2} - \frac{mh^2}{r^3}$$

are constants of the motion, where dots denote derivatives with respect to the proper time of the particle.

(b) For a circular orbit with a fixed value of r, determine h and hence deduce that (i) $r > 3m$ and (ii) $(d\phi/dt)^2 = m/r^3$.

(c) Now consider a nearly circular orbit with shape given by $u(\phi) = 1/r$. Let prime denote differentiation with respect to ϕ, so that $u' = du/d\phi$. Given that $\dot{r} = -hu'$, and assuming if needed that $u' \neq 0$, show that

$$u'' + u = \frac{m}{h^2} + 3mu^2\,.$$

For $m/h \ll 1$, this equation has an approximate solution of the form

$$u = \frac{m}{h^2}(1+\alpha) + A\cos[(1+\beta)\phi]\,,$$

where the constant A obeys $|A| \ll 1$ but is otherwise arbitrary. The constants α and β are small for $m/h \ll 1$. Verify this solution by working to first order in A and determining the constants α and β to leading non-trivial order in m/h.

Comment briefly on the significance of your result for β.

Paper 2, Section II

38B General Relativity

(a) Define the *Einstein tensor* $G_{\mu\nu}$ in terms of the Riemann tensor $R^{\alpha}{}_{\beta\mu\nu}$ and use the Bianchi identity $\nabla_\rho R^{\alpha}{}_{\beta\mu\nu} + \nabla_\mu R^{\alpha}{}_{\beta\nu\rho} + \nabla_\nu R^{\alpha}{}_{\beta\rho\mu} = 0$ to show that $\nabla_\nu G_\mu{}^\nu = 0$. Comment briefly on the significance of this result for consistency of the Einstein equations (including a cosmological constant).

(b) For a universe described by the line element

$$ds^2 = -dt^2 + a(t)^2 \left(dx^2 + dy^2 + dz^2\right),$$

the Einstein tensor is diagonal with $G_t{}^t = -3\dot{a}^2/a^2$ and $G_x{}^x = -2\ddot{a}/a - \dot{a}^2/a^2$, where dots denote differentiation with respect to t. Verify by direct computation that $\nabla_\nu G_\mu{}^\nu = 0$, justifying the steps that you make and computing any metric connection components $\Gamma_\nu{}^\mu{}_\rho$ that you may need.

Solve the vacuum Einstein equations with a cosmological constant $\Lambda > 0$ to obtain a result for $a(t)$ that corresponds to an expanding universe.

Paper 3, Section II

37B General Relativity

(a) Consider a curve $x^\alpha(\lambda)$ with tangent vector $T^\alpha = dx^\alpha/d\lambda$, in a spacetime with metric $g_{\mu\nu}$. For any vector fields U^α and W^α, show that, on the curve,

$$\frac{d}{d\lambda}\left(g_{\mu\nu}U^\mu W^\nu \right) \;=\; (\nabla_T U)^\alpha W_\alpha \;+\; U_\alpha (\nabla_T W)^\alpha\,,$$

where $\nabla_V = V^\alpha \nabla_\alpha$ denotes the covariant directional derivative along a vector V^α.

(b) Now consider a one-parameter family of geodesics defined by the functions with two arguments, $x^\alpha(\tau,\sigma)$. Fixing a value of σ in these functions gives a timelike geodesic parametrised by proper time τ, with tangent vector $T^\alpha = \partial x^\alpha/\partial\tau$ satisfying $T^\alpha T_\alpha = -1$, while fixing a value of τ gives a curve parametrised by σ, with tangent vector $S^\alpha = \partial x^\alpha/\partial\sigma$.

Show that the commutator $[T,S]^\alpha = (\nabla_T S)^\alpha - (\nabla_S T)^\alpha = 0$ and hence derive the equation of geodesic deviation in the form

$$(\nabla_T \nabla_T S)^\alpha \;+\; E^\alpha{}_\beta S^\beta \;=\; 0\,,$$

where $E^\alpha{}_\beta$ is a tensor to be defined in terms of T^α and the Riemann tensor $R^\alpha{}_{\beta\mu\nu}$.

(c) Suppose that in part (b),

$$R_{\alpha\beta\mu\nu} \;=\; K\left(g_{\alpha\mu}\, g_{\beta\nu} \;-\; g_{\alpha\nu}\, g_{\beta\mu} \right),$$

where K is a scalar. Show that, if $T^\alpha S_\alpha = 0$ at some point on a particular geodesic, then

$$(\nabla_T \nabla_T S)^\alpha \;-\; K S^\alpha \;=\; 0$$

at all points on that geodesic.

[*Hint: You may use, without proof, standard properties of the metric connection, symmetries of the Riemann tensor, and the Ricci identity in the form*

$$(\nabla_T \nabla_S V)^\alpha \;-\; (\nabla_S \nabla_T V)^\alpha \;=\; (\nabla_{[T,S]} V)^\alpha \;+\; R^\alpha{}_{\beta\mu\nu} V^\beta T^\mu S^\nu$$

for vector fields T^α, S^α and V^α.]

Paper 4, Section II

37B General Relativity

(a) For a spacetime with metric $g_{\alpha\beta}$, write down an explicit expression for the metric-preserving Levi-Civita connection $\Gamma^{\,\alpha}_{\beta\,\gamma}$.

For a spacetime that is nearly flat, the metric can be expressed in the form

$$g_{\alpha\beta} \,=\, \eta_{\alpha\beta} + h_{\alpha\beta}\,,$$

where $\eta_{\alpha\beta}$ is a flat metric with constant components (though not necessarily diagonal in the coordinates used) and the components $h_{\alpha\beta}$ and their derivatives are small. Show that to leading order in these small quantities,

$$2R_{\alpha\beta} \,= A\, h_{\alpha}{}^{\gamma}{}_{,\gamma\beta} \,+ B\, h_{\beta}{}^{\gamma}{}_{,\gamma\alpha} \,+ C\, h^{\gamma}{}_{\gamma,\alpha\beta} \,+ D\, h_{\alpha\beta,\gamma\rho}\,\eta^{\gamma\rho}\,,$$

for constants A, B, C, D which you should determine. Indices are raised and lowered using $\eta_{\alpha\beta}$.

(b) Consider the following metric

$$ds^2 \,=\, 2\,du\,dv \,+\, dx^2 \,+\, dy^2 \,+\, H(u,x,y)\,du^2\,,$$

where $H(u,x,y)$ is a smooth function that is not necessarily small. You may assume that the only nonzero connection coefficients are $\Gamma^{\,v}_{x\,u}, \Gamma^{\,v}_{y\,u}, \Gamma^{\,x}_{u\,u}, \Gamma^{\,y}_{u\,u}, \Gamma^{\,v}_{u\,u}$, and the coefficients related to these by symmetry. You may also assume that R_{uu} is the only coefficient of the Ricci tensor that is not trivially zero. Compute R_{uu} and hence obtain the Ricci scalar. Show that the full nonlinear vacuum field equations, with no cosmological constant, reduce to a partial differential equation for H, that you should determine.

[*You may assume in both parts of the question that*

$$R^{\alpha}{}_{\beta\,\mu\,\nu} \,=\, \Gamma^{\,\alpha}_{\nu\,\beta,\,\mu} - \Gamma^{\,\alpha}_{\mu\,\beta,\,\nu} + \Gamma^{\,\alpha}_{\mu\,\gamma}\Gamma^{\,\gamma}_{\nu\,\beta} - \Gamma^{\,\alpha}_{\nu\,\gamma}\Gamma^{\,\gamma}_{\mu\,\beta}\,. \qquad]$$

Paper 1, Section II

17I Graph Theory

(a) Let G be a graph on $n \geqslant 3$ vertices with minimum degree at least $\frac{n}{2}$. Prove that G is Hamiltonian.

(b) Now let G be a bipartite graph of minimum degree at least $k \geqslant 2$. Prove that G contains either a path of length $2k$ or a cycle of length $2k$. Give an example to show that G need not contain a path of length $2k$. Show also that G must contain either a path of length $4k-3$ or a 4-cycle.

Paper 2, Section II

17I Graph Theory

State and prove Turán's theorem.

A *rhombus* is the graph formed by two triangles sharing an edge. Prove that if G is a graph on $n \geqslant 4$ vertices that has more edges than $T_2(n)$, then G contains a rhombus.

Find a graph G on 6 vertices such that $e(G) = e(T_2(6))$ and G does not contain a rhombus, but G is not isomorphic to $T_2(6)$.

Paper 3, Section II

17I Graph Theory

(a) Show that every graph of average degree at least d contains a subgraph of minimum degree at least $d/2$.

(b) Let $\delta, g \geqslant 3$ be fixed. Using random graph methods, or otherwise, show that there exists a graph G on at most $n = (100\delta)^g$ vertices such that the minimum degree of G is at least δ and the girth of G is at least g.

Paper 4, Section II

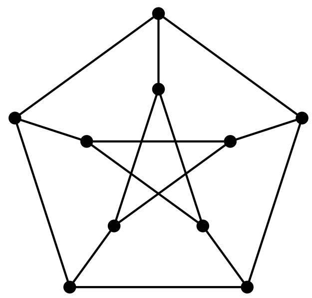
17I Graph Theory

What does it mean to say that a graph G is *strongly regular* with parameters (k, a, b)? Let G be a strongly regular graph with parameters (k, a, b) on n vertices such that $b \geqslant 1$ and $G \neq K_n$. Prove the *rationality condition,* namely that the numbers

$$\frac{1}{2}\left(n - 1 \pm \frac{(n-1)(b-a) - 2k}{\sqrt{(a-b)^2 + 4(k-b)}}\right)$$

are integers.

What are the eigenvalues (with their multiplicities) of the Petersen graph (shown below)?



Show that the set of edges of K_{10} cannot be partitioned into the edges of three copies of the Petersen graph. [*Hint: Suppose that it can be, and that the three Petersen graphs have adjacency matrices A, B and C. You may wish to consider an appropriate common eigenvector of A and B.*]

Paper 1, Section II

33C Integrable Systems

Consider the initial boundary value problem for a function $u = u(x,t)$,

$$u_t = iu_{xx}\,, \qquad 0 < x < \infty\,, \quad t > 0,$$

$$u(x,0) = u_0(x), \qquad u(0,t) = h(t)\,.$$

Show that the equation $u_t = iu_{xx}$ has a formulation as the consistency condition for the following pair of equations for $\psi = \psi(x,t) \in \mathbb{C}$,

$$\begin{aligned} &\psi_t + ik^2\psi = iu_x - ku, \\ &\psi_x - ik\psi = u. \end{aligned}$$

By means of the integrating factor $e^{-ikx+ik^2t}$, or otherwise, deduce that

$$\hat{u}(k,t)e^{ik^2t} - \hat{u}_0(k) = \int_0^t e^{ik^2\tau}\left[ku(0,\tau) - iu_x(0,\tau)\right]d\tau\,,$$

where

$$\hat{u}(k,t) = \int_0^\infty e^{-ikx}u(x,t)dx, \qquad \text{and} \qquad \hat{u}_0(k) = \int_0^\infty e^{-ikx}u_0(x)dx\,.$$

Hence, by considering also $\hat{u}(-k,t)$, find a function $G = G(k,\tau)$ such that

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ik^2t+ikx}\left[\hat{u}_0(k) - \hat{u}_0(-k)\right]dk \\ + \frac{1}{\pi}\int_{-\infty}^{+\infty}\int_0^t e^{-ik^2(t-\tau)+ikx}G(k,\tau)d\tau dk\,.$$

[You may assume that u is smooth and rapidly decreasing so that $\hat{u}(k,t)$ is holomorphic for $\mathrm{Im}\{k\} < 0$, and satisfies $\lim_{|k|\to\infty}\hat{u}(k,t) = 0$.]

Paper 2, Section II

34C Integrable Systems

(a) Explain what it means to say the KdV equation $u_t + u_{xxx} - 6uu_x = 0$, where $u = u(x,t)$, has a *Lax pair formulation* in terms of the linear operators

$$L = -\partial_x^2 + u \quad \text{and} \quad A = 4\partial_x^3 - 3u\partial_x - 3\partial_x u\,.$$

(b) Consider now the case of periodic boundary conditions for the KdV equation, so that at each time t the unknown u is a real-valued function satisfying $u(x+2\pi,t) = u(x,t)$. At each fixed time t, introduce a basis φ_+, φ_- of solutions to the scattering equation $L\phi = k^2\phi$ (for $k^2 > 0$) determined by the initial conditions at $x = 0$,

$$\varphi_\pm(0) = 1, \qquad \partial_x\varphi_\pm(0) = \pm ik\,.$$

(i) Show that there exists a matrix

$$\hat{T} = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix},$$

where $a(t)$ and $b(t)$ are functions of time and $\overline{a}$, $\overline{b}$ denote the complex conjugates, such that

$$\begin{pmatrix} \varphi_+(x+2\pi) \\ \varphi_-(x+2\pi) \end{pmatrix} = \hat{T} \begin{pmatrix} \varphi_+(x) \\ \varphi_-(x) \end{pmatrix}.$$

Show further that $|a|^2 - |b|^2 = 1$.

(ii) Now as t varies let u evolve in time according to the KdV equation. Show that there exists a matrix

$$\Lambda = \begin{pmatrix} \lambda & \mu \\ \overline{\mu} & \overline{\lambda} \end{pmatrix},$$

where $\lambda(t)$ and $\mu(t)$ depend on time, such that

$$\begin{pmatrix} A\varphi_+ + \partial_t\varphi_+ \\ A\varphi_- + \partial_t\varphi_- \end{pmatrix} = \Lambda \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix}.$$

Prove that $\partial_t\hat{T} = [\Lambda, \hat{T}]$.

[*Hint: Consider* $\partial_t(\hat{T}\Psi) = (\partial_t\hat{T})\Psi + \hat{T}\partial_t\Psi$, *with* $\Psi = \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix}$.]

Deduce that $\mathrm{Re}[a] = \frac{1}{2}(a + \overline{a})$ is independent of t.

Paper 3, Section II

32C Integrable Systems

Explain how to find Lie symmetries for an ordinary differential equation $\Delta(x,u,u',\ldots,u^{(N)})=0$ by considering the action of the Nth prolongation

$$\mathrm{pr}^{(N)}V = V_1\partial_x + \phi\,\partial_u + \phi_1\partial_{u'} + \cdots + \phi_N\partial_{u^{(N)}}$$

of the vector field $V = V_1(x,u)\partial_x + \phi(x,u)\partial_u$. Give the inductive formula for the computation of the $\phi_j = \phi_j(x,u,u',\ldots,u^{(j)})$ which determines the prolongation.

For the special case in which $V_1 = f(x)$ depends only on x, compute the third prolongation, and show that

$$\phi_3 = \phi_{xxx} + u'(\alpha\phi_{xxu} - f''') + 3(u')^2\phi_{xuu} + (u')^3\phi_{uuu} + \beta(\phi_{xu} - f'' + u'\phi_{uu})u'' + (\phi_u - 3f')u''',$$

for integers α, β which you should determine.

Find the Lie symmetries of the equation

$$\frac{d^3u}{dx^3} = \frac{1}{u^3}$$

that are generated by a vector field of the form $V = f(x)\partial_x + \phi(x,u)\partial_u$.

Paper 1, Section II

22G Linear Analysis

State and prove the Closest Point Theorem. Deduce that if F is a closed subspace of a Hilbert space H then H is the direct sum of F and $F^{\perp}$.

Let H be a separable Hilbert space. An operator T on H is called a *shift* if there exists an orthonormal (Hilbert) basis $(e_n)_{n=1}^{\infty}$ of H such that $T(e_n) = e_{n+1}$ for all n. Show that T is a shift if and only if T is an isometry with $\cap_{n=1}^{\infty} \operatorname{Im}(T^n) = \{0\}$ and $\dim\,(\operatorname{Im} T)^{\perp} = 1$.

Paper 2, Section II

22G Linear Analysis

State and prove the Baire Category Theorem.

Let X be a Banach space, and let S be a non-empty subset of X that is closed, convex and symmetric (S symmetric means $x \in S$ implies $-x \in S$). Show that if $\cup_{n=1}^{\infty} nS = X$ then S is a neighbourhood of the origin.

Give an example to show that the condition that S is convex cannot be omitted.

Paper 3, Section II

21G Linear Analysis

State and prove the Stone–Weierstrass theorem. [You may assume that the function $x^{1/2}$ is uniformly approximable by polynomials on $[0,1]$.]

Let $C(\mathbb{R})$ denote the space of all bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$, equipped with the uniform norm. Explain briefly why $C(\mathbb{R})$ is a Banach space.

If A is a subalgebra of $C(\mathbb{R})$ that contains the constants and separates the points, must A be dense in $C(\mathbb{R})$? Justify your answer.

Paper 4, Section II

22G Linear Analysis

In this question all spaces are complex.

(a) Let T be an operator on l_2. Prove that the spectrum of T is non-empty. [If you use a power series expression for the resolvent function then you must prove it.]

(b) Let $(e_n)_{n=1}^{\infty}$ be the usual basis of l_2. In each case below, an operator on l_2 is given; determine for each operator whether or not it is compact, and find its spectrum. [Any results about spectra from the course may be used without proof, as long as they are stated clearly.]

(i) $T(e_n) = e_{n+1}$ for all n.

(ii) $S(e_n) = e_{n-1}$ for all $n > 1$, with $S(e_1) = 0$.

(iii) $R(e_n) = \frac{1}{n}e_{n+1}$ for all n.

(iv) $Q(e_n) = \frac{1}{n}e_{n-1}$ for all $n > 1$, with $Q(e_1) = 0$.

Paper 1, Section II

16I Logic and Set Theory

State the Soundness Theorem for Propositional Logic.

Let S be a consistent set of propositions. Explain briefly why the function $v\colon L \to \{0,1\}$ on the set L of all propositions defined by

$$v(t) = \begin{cases} 1 & \text{if } S \vdash t \\ 0 & \text{if } S \nvdash t \end{cases}$$

need not be a model of S. Show that S is contained in a consistent, deductively closed set $T \subset L$ such that the definition of v above with S replaced by T *is* a model of S. [You need not prove that v is a model. The set of primitive propositions here is arbitrary. You may assume Zorn's lemma.]

Show that if every finite subset of an arbitrary set S of propositions has a model, then S has a model.

Let X, Y be infinite sets such that there is an injection from X to Y. For each $x \in X$, let A_x be a non-empty, finite subset of Y. Let P be a set consisting of pairwise distinct primitive propositions $p_{x,y}$ for all $x \in X$ and $y \in Y$. For a valuation v on L, set

$$f_v = \{(x,y) \in X \times Y : v(p_{x,y}) = 1\} .$$

For each of the following statements, either write down a set $S \subset L$ that makes the statement true or prove that no such set S exists.

(i) $\{f_v : v \text{ is a model of } S\}$ is the set of all injective functions from subsets of X to Y.

(ii) $\{f_v : v \text{ is a model of } S\}$ is the set of all injective functions $g\colon X \to Y$ such that $g(x) \in A_x$ for all $x \in X$.

(iii) $\{f_v : v \text{ is a model of } S\}$ is the set of all injective functions from X to Y.

Paper 2, Section II

16I Logic and Set Theory

State and prove Hartogs' Lemma.

Define *ordinal exponentiation.* Show that $\alpha^{\beta+\gamma} = \alpha^{\beta} \cdot \alpha^{\gamma}$ for all ordinals α, β, γ. Given ordinals γ and α, show that there exist unique ordinals β and δ such that $\gamma = \omega^{\alpha} \cdot \beta + \delta$ with $\delta < \omega^{\alpha}$. [You may assume standard properties of ordinal addition and multiplication. Other results used must be proved.]

Let X be a well-ordered set. Say $x \in X$ is a *limit point of* X if $I_x \neq \emptyset$ and I_x has no greatest element, where $I_x = \{y \in X : y < x\}$. Let X' denote the set of limit points of X and define $X^{(\alpha)}$ for all ordinals α by recursion as follows:

$$\begin{aligned} X^{(0)} &= X \\ X^{(\alpha+1)} &= \left(X^{(\alpha)}\right)' \\ X^{(\lambda)} &= \bigcap_{\alpha<\lambda} X^{(\alpha)} \qquad \text{(for non-zero limit ordinal } \lambda) \end{aligned}$$

Show that if $X \neq \emptyset$, then $X' \neq X$. Deduce that $X^{(\alpha)} = \emptyset$ for some α. The least such α is the *index of* X.

Show that if ξ is an ordinal, then

$$\xi' = \{\gamma < \xi : \exists\, \beta > 0 \ \text{ such that } \gamma = \omega \cdot \beta\}.$$

Describe ξ''. Find the index of ω and the index of ω^2.

Paper 3, Section II

16I Logic and Set Theory

(a) State Zorn's Lemma, the Axiom of Choice and the Well-ordering Principle, and prove that they are equivalent.

(b) State Gödel's Completeness Theorem and the Compactness Theorem for First-order Logic.

Let $T_0, T_1, \ldots, T_n$ be first-order theories in some language L that partition the collection of all L-structures in the sense that every L-structure is a model of exactly one T_i. Show that each T_i is finitely axiomatisable.

Paper 4, Section II

16I Logic and Set Theory

(a) In this part of the question, work within ZF.

Define the *rank* of a set. For a non-zero limit ordinal α, what is the rank of the set of all functions from α to α?

Define the *von Neumann hierarchy* and prove that it exhausts the set-theoretic universe, in the sense that every set is an element of some member of the von Neumann hierarchy.

(b) In this part of the question, work within ZFC.

Define the cardinal numbers $\aleph_\alpha$. Is every infinite cardinal equal to some $\aleph_\alpha$? Justify your answer.

Prove that $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$ for every ordinal α. Deduce that

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \aleph_\beta$$

for ordinals $\alpha \leqslant \beta$.

Prove that the following sentence is a theorem of ZFC, where $t \equiv x$ is taken to mean 't and x have the same cardinality.'

$$(\forall x)(\neg(x = \emptyset) \Rightarrow \neg(\exists y)((\forall t)((t \equiv x) \Rightarrow (t \in y))))$$

Paper 1, Section I

6A Mathematical Biology

Consider a model for population growth in which the population $n(t)$ evolves according to

$$\frac{dn}{dt} = \alpha n - \beta n^3,$$

where $\alpha, \beta > 0$.

(a) Find and analyse the stability of the non-negative fixed points.

(b) Sketch the solution starting from a range of (non-negative) population sizes.

(c) Suppose a population either follows the model above, or it follows logistic growth. Explain briefly how these possibilities may be distinguished by observing population dynamics starting from a very small population. Illustrate your answer with a sketch showing the difference between the two models.

Paper 2, Section I

6A Mathematical Biology

The model of a viral infection in a population is given by the system

$$\begin{aligned}
\frac{dX}{dt} &= \mu N - \beta XY - \mu X, \\
\frac{dY}{dt} &= \beta XY - (\mu + \nu)Y, \\
\frac{dZ}{dt} &= \nu Y - \mu Z,
\end{aligned}$$

where μ, β and ν are positive constants and X, Y, and Z are respectively the number of susceptible, infected and immune individuals in a population of size N, independent of t, where $N = X + Y + Z$.

(a) Interpret the biological meaning of each of the parameters μ, β and ν.

(b) Show that there is a critical population size $N_c(\mu, \beta, \nu)$ such that if $N < N_c$ there is no steady state with the infection maintained in the population. Show that in this case the numbers of infected and immune individuals decrease to zero for all possible initial conditions.

(c) Show that for $N > N_c$ there is a steady state $(X, Y, Z) = (X^*, Y^*, Z^*)$ with $0 < X^*, Y^*, Z^* < N$. Show that this steady state is stable.

Paper 3, Section I

6A Mathematical Biology

Consider a birth-death process in a population of size n. The birth rate per individual is λ and the death rate per individual is $\gamma + \beta n$, where λ, γ and β are positive constants.

Let $p_n(t)$ be the probability that the population has size n at time t. Write down the master equation for the system. Show that

$$\frac{d}{dt}\langle n\rangle = (\lambda - \gamma)\,\langle n\rangle - \beta\langle n^2\rangle\,,$$

where $\langle . \rangle$ denotes the mean.

From this result, deduce a bound on the mean in a steady state in the case that $\lambda > \gamma$.

What can be said about the mean at steady state when $\lambda \leqslant \gamma$?

Paper 4, Section I

6A Mathematical Biology

A discrete-time model of alien cell population dynamics considers the coupled dynamics of immature and mature cells. At each time step, a proportion λ of the immature cells mature and a proportion μ of mature cells divides. When a mature cell divides it is replaced by four new immature cells. Finally, a proportion k of mature cells die at each time step.

(a) Explain briefly how this model may be represented by the equations

$$\begin{aligned} a_{t+1} &= (1-\lambda)a_t + 4\mu b_t, \\ b_{t+1} &= \lambda a_t + (1-\mu-k)b_t \end{aligned}$$

where $0 \leqslant \mu, \lambda, k \leqslant 1$ and $\mu + k \leqslant 1$.

(b) What is the expected number of offspring per cell?

(c) By considering the total population of cells, show that population growth is only possible if $3\mu > k$. How does this relate to your answer to part (b)?

(d) At time T, the population is treated with a chemical that completely stops cells from maturing for all $t \geqslant T$, but otherwise has no direct effects. Explain what will happen to the population afterwards. In terms of a_T and b_T, what will be the total number of cells in the long term?

Paper 3, Section II

13A Mathematical Biology

An experiment is run with bacteria in a thin channel. The concentration of bacteria at time t is given by $C(x,t)$, where x is position in the channel. The ends of the channel are at $x = 0$ and $x = L$. The bacteria cannot survive outside the channel. Within the channel, the concentration of bacteria is modelled by

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} + \mu C\,,$$

where D and μ are positive constants.

Initially, the bacteria are at a uniform concentration C_0 for $0 < x < L$.

(a) Suppose that the ends of the channel are open so that bacteria may diffuse out of the ends.

(i) Find the concentration $C(x,t)$ for $t > 0$.

(ii) What is the flux of bacteria out of the channel? Comment on this flux at $t = 0$.

(iii) Show that the population will grow if $\mu > \mu_c$, where $\mu_c(D,L)$ should be given. Give a brief explanation for the dependence of μ_c on D and L.

(iv) Sketch the bacterial concentration as a function of x for a range of times (on the same sketch), paying particular attention to early and late times. Consider separately the cases when $\mu < \mu_c$ and $\mu > \mu_c$ (i.e. two sketches are required).

(b) The experiment is run again with the $x = L$ end of the channel closed ($x = 0$ remains open). Again initially $C(x,t) = C_0$. What is the condition now for population growth in the long term? Comment briefly on how this compares to the condition for growth in part (a).

(c) The experiment is run once more with the channel ends both open, but now the per capita growth rate is a function of time (as the experimental conditions fluctuate each day), given by $\mu(t) = \mu_0 + \mu_1 \cos(t)$. Find the condition for population growth in the long term. Give a brief interpretation of this result.

Paper 4, Section II

14A Mathematical Biology

An activator-inhibitor system is described by the equations

$$\begin{aligned}\frac{\partial u}{\partial t} &= D_1 \frac{\partial^2 u}{\partial x^2} + u - uv + u^2, \\ \frac{\partial v}{\partial t} &= D_2 \frac{\partial^2 v}{\partial x^2} + \alpha u^2 - \beta uv,\end{aligned}$$

where $\alpha, \beta, D_1, D_2 > 0$.

(a) Find conditions on α and β for the spatially homogeneous system to have a stable stationary solution with $u > 0$ and $v > 0$. Sketch this region in the α–β plane (for the quadrant with $\alpha, \beta > 0$).

(b) Consider spatial perturbations of the form $\cos(kx)$ about the solution found in part (a), and set $\lambda = D_1/D_2$. Find conditions for the system to be unstable. For fixed β, sketch in the λ–α plane the region where spatial instability is possible for some $k \in \mathbb{R}$.

(c) Consider a general system of the form

$$\begin{aligned}\frac{\partial u}{\partial t} &= D_1 \frac{\partial^2 u}{\partial x^2} + f(u, v), \\ \frac{\partial v}{\partial t} &= D_2 \frac{\partial^2 v}{\partial x^2} + g(u, v).\end{aligned}$$

A *Turing instability* of this system is when a stationary solution is stable to homogenous perturbations but is unstable to some spatial perturbation. Explain why a Turing instability is not possible when $D_1 = D_2$.

Verify that this is consistent with your answer for part (b) .

Paper 1, Section II

31K Mathematics of Machine Learning

(a) (i) Let $\mathcal{Z}$ be a non-empty set. Given $z_1, \ldots, z_n \in \mathcal{Z}$ and a class $\mathcal{F}$ of functions $f : \mathcal{Z} \to \mathbb{R}$, what is meant by the *empirical Rademacher complexity* $\hat{\mathcal{R}}(\mathcal{F}(z_{1:n}))$? Given i.i.d. random variables $Z_1, \ldots, Z_n$ taking values in $\mathcal{Z}$, what is meant by the *Rademacher complexity* $\mathcal{R}_n(\mathcal{F})$?

(ii) Suppose $\mathcal{H}$ is a class of functions $h : \mathbb{R}^p \to \{0, 1\}$ with $|\mathcal{H}| \geqslant 2$. Define the *shattering coefficient* $s(\mathcal{H}, n)$ and the *VC dimension* $\mathrm{VC}(\mathcal{H})$ of $\mathcal{H}$.

(iii) Let $\mathcal{H} = \{\mathbf{1}_A : A \in \mathcal{A}\}$ where $\mathcal{A} := \left\{ \prod_{j=1}^{p} (-\infty, a_j] : a_1, \ldots, a_p \in \mathbb{R} \right\}$. Show that $\mathrm{VC}(\mathcal{H}) \leqslant p$.

(b) A new painkilling drug is tested on n patients. Let $Y_i^{(0)}$ and $Y_i^{(1)}$ be the pain levels, on a scale from 0 (no pain) to $M > 0$ (maximum pain), of the ith patient, before and after taking the painkiller respectively. Suppose the vector $X_i \in \mathbb{R}^p$ records p additional characteristics of the ith patient, such as their age, weight, height, etc. We treat $\left(Y_i^{(0)}, Y_i^{(1)}, X_i\right) \in [0, M]^2 \times \mathbb{R}^p$ for $i = 1, \ldots, n$ as independent copies of a random triple $\left(Y^{(0)}, Y^{(1)}, X\right)$. Let $\mathcal{A}$ and $\mathcal{H}$ be defined as in part (a) (iii) above. We wish to determine a region $A \in \mathcal{A}$ where if $X \in A$, we expect the drug to be effective. To this end, let h^* and $\hat{h}$ minimise

$$Q(h) := \mathbb{E}\left[\left(Y^{(1)} - Y^{(0)}\right) h(X)\right] \qquad \text{and} \qquad \hat{Q}(h) := \frac{1}{n} \sum_{i=1}^{n} \left(Y_i^{(1)} - Y_i^{(0)}\right) h(X_i)$$

respectively, over $h \in \mathcal{H}$.

(i) Using any results from the course that you need, show that

$$\mathbb{E}Q(\hat{h}) \leqslant Q(h^*) + 2\mathcal{R}_n(\mathcal{F})$$

for an appropriate class of functions $\mathcal{F}$ that you should specify.

(ii) Using any results from the course that you need, show that

$$\mathbb{E}Q(\hat{h}) \leqslant Q(h^*) + 2M\sqrt{\frac{2p\log(n+1)}{n}}.$$

Paper 2, Section II

31K Mathematics of Machine Learning

(a) Carefully describe the construction of the regions $\{\hat{R}_1, \ldots, \hat{R}_J\}$ of a decision tree $x \mapsto \hat{T}(x) = \sum_{j=1}^J \hat{\gamma}_j \mathbf{1}_{\hat{R}_j}(x)$ trained on data D' consisting of input–output pairs $(X_i', Y_i') \in \mathbb{R}^p \times \mathbb{R}$, $i = 1, \ldots, n$. [You need not explain how computations may be performed in a computationally efficient manner.]

(b) In the following, consider the data D' as deterministic (i.e. not random). Let data $D := (X_i, Y_i)_{i=1}^n$ consist of i.i.d. input–output pairs, and let $(X, Y) \in \mathbb{R}^p \times \mathbb{R}$ have the same distribution as (X_1, Y_1) and be independent of D. Set

$$\tilde{T}(x) := \sum_{j=1}^J \tilde{\gamma}_j \mathbf{1}_{\hat{R}_j}(x),$$

$$\text{where} \qquad \tilde{\gamma}_j := \frac{1}{N_j + 1} \sum_{i=1}^n Y_i \mathbf{1}_{\hat{R}_j}(X_i) \qquad \text{and} \qquad N_j := \sum_{i=1}^n \mathbf{1}_{\hat{R}_j}(X_i).$$

Let $\gamma_j := \mathbb{E}[\tilde{\gamma}_j \,|\, X_{1:n}]$. Show that

$$\mathbb{E}\big[(\tilde{\gamma}_j - \gamma_j)^2 \,|\, X_{1:n}\big] = \frac{1}{(1 + N_j)^2} \sum_{i=1}^n \operatorname{Var}(Y_i \,|\, X_i) \mathbf{1}_{\hat{R}_j}(X_i).$$

Now suppose further that $\operatorname{Var}(Y \,|\, X = x)$ is bounded from above by σ^2 for all $x \in \mathbb{R}^p$. Show that

$$\mathbb{E}\Big[\big\{\tilde{T}(X) - \mathbb{E}\big(\tilde{T}(X) \,|\, X, X_{1:n}\big)\big\}^2\Big] \leqslant \frac{\sigma^2 J}{n}.$$

[*Hint: You may use without proof, the fact that if* $N \sim \operatorname{Binomial}(n, q)$, *for success probability* $q \in (0, 1]$, *then* $\mathbb{E}[1/(N+1)] \leqslant 1/(nq)$.]

Paper 4, Section II

30K Mathematics of Machine Learning

(a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. What is meant by the *subdifferential* $\partial f(\alpha)$ of f at $\alpha \in \mathbb{R}^n$?

Prove that α minimises f if and only if $0 \in \partial f(\alpha)$.

(b) Consider a classification setting with input–output pairs $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^p \times \{-1, 1\}$. What is meant by the empirical ϕ-risk $\hat{R}_\phi(h)$ of hypothesis $h : \mathbb{R}^p \to \mathbb{R}$ given convex surrogate loss ϕ?

(c) In the following, we consider hypothesis class $\mathcal{H} := \{h_\beta : \beta \in \mathbb{R}^p\}$, where $h_\beta : x \mapsto x^\top \beta$. Fix $\lambda > 0$ and let $\hat{\beta}$ be the minimiser (assumed to exist) of $q : \mathbb{R}^p \to \mathbb{R}$ given by

$$q(\beta) = \hat{R}_\phi(h_\beta) + \lambda \|\beta\|_2^2.$$

Let $X \in \mathbb{R}^{n \times p}$ be the matrix with ith row x_i for $i = 1, \ldots, n$ and suppose that $XX^\top$ is invertible. Writing $P := X^\top (XX^\top)^{-1} X \in \mathbb{R}^{p \times p}$, show that $q(\beta) \geqslant q(P\beta)$. [*Hint: Consider the decomposition* $\beta = \{P + (I - P)\}\beta$.]

Hence conclude that $\hat{\beta} = X^\top \hat{\alpha}$ where $\hat{\alpha}$ is the minimiser of $r : \mathbb{R}^n \to \mathbb{R}$ given by

$$r(\alpha) = q(X^\top \alpha).$$

(d) Now take ϕ to be the hinge loss. Writing $k_i \in \mathbb{R}^n$ for the ith column of $XX^\top$, show that

$$\frac{1}{n} \sum_{i=1}^{n} k_i t_i = 2\lambda XX^\top \hat{\alpha},$$

where t_i is related to $y_i x_i^\top \hat{\beta}$ in a way you should specify. Hence conclude that whenever $y_i x_i^\top \hat{\beta} > 1$, then $\hat{\alpha}_i = 0$.

Paper 1, Section II

20F Number Fields

Define *algebraic integer.*

Prove that the set of algebraic integers is closed under multiplication. [You may use without proof any characterisation of algebraic integers, provided it is properly stated.]

What is the ring of integers in the number field $\mathbb{Q}(\sqrt{2})$? Prove your claim.

For a polynomial $f \in \mathbb{C}[x]$, we define

$$M(f) = |a_d| \prod_{j=1}^{d} \max\{1, |\alpha_j|\},$$

where d is the degree, a_d is the leading coefficient and $\alpha_1, \ldots, \alpha_d$ are the roots of f in $\mathbb{C}$.

Suppose $f \in \mathbb{Z}[x]$ is irreducible, $M(f) = 2$, and f has a real root $\alpha_1 > 1$.

Prove that α_1 is an algebraic integer and $|N_{\mathbb{Q}(\alpha_1)/\mathbb{Q}}(\alpha_1)| = 2$. [*Hint: Consider the number* $|N_{\mathbb{Q}(\alpha_1)/\mathbb{Q}}(\alpha_1)|/2$*, and show that it is a rational integer.*]

Paper 2, Section II

20F Number Fields

State Dirichlet's unit theorem.

Let $K = \mathbb{Q}(\sqrt{5})$, and determine the units in $\mathcal{O}_K$. [You may use without proof the description of $\mathcal{O}_K$, as long as you state it clearly.]

For $K = \mathbb{Q}(\sqrt{5})$, what are the possible degrees of extensions L/K of number fields such that $1 < |\mathcal{O}_L^\times/\mathcal{O}_K^\times| < \infty$? Give an example for each possible degree.

Paper 4, Section II

20F Number Fields

Let K be a number field.

Define the *norm* $N(I)$ of an ideal I in $\mathcal{O}_K$.

Let $d = [K : \mathbb{Q}]$. Define the *discriminant of a tuple* $(\alpha_1, \ldots, \alpha_d) \in K$. Define the *discriminant of* K.

State a formula for the norm of an ideal in terms of discriminants without proof.

Prove that $N(\alpha\mathcal{O}_K) = |N_{K/\mathbb{Q}}(\alpha)|$ for all $\alpha \in \mathcal{O}_K$.

Now let L/K be an extension of number fields, and let $P \subset \mathcal{O}_K$ and $Q \subset \mathcal{O}_L$ be non-zero prime ideals.

(a) Prove that $Q | P\mathcal{O}_L$ if and only if $P = Q \cap \mathcal{O}_K$.

(b) Suppose that $P = Q \cap \mathcal{O}_K$ and $[O_K : P] = [O_L : Q]$. Prove that if $Q = \alpha\mathcal{O}_L$ for some $\alpha \in \mathcal{O}_L$, then $P = N_{L/K}(\alpha)\mathcal{O}_K$.

Paper 1, Section I

1F Number Theory

Let $N \in \mathbb{N}$ be an odd composite integer, and let $b \in (\mathbb{Z}/N\mathbb{Z})^\times$. Define what it means for N to be a *Fermat pseudoprime* to the base b, and for N to be an *Euler pseudoprime* to the base b.

Now let $N = 105$. Determine the proportion of bases $b \in (\mathbb{Z}/N\mathbb{Z})^\times$ such that N is a Fermat pseudoprime to the base b. Determine the proportion of bases $b \in (\mathbb{Z}/N\mathbb{Z})^\times$ such that N is an Euler pseudoprime to the base b.

Paper 2, Section I

1F Number Theory

Let d be a positive integer.

Define what it means for a positive definite binary quadratic form $f(x, y) = ax^2 + bxy + cy^2$ to be *reduced*. If d is congruent to 0 or 3 mod 4, define the *class number* $h(-d)$.

Show that if d is odd and has k distinct prime factors, then $h(-4d) \geqslant 2^{k-1}$.

Give an example with $d > 1$ to show that the inequality $h(-4d) \geqslant 2^{k-1}$ can be strict.

Paper 3, Section I

1F Number Theory

Let N be an odd positive integer, and let a be an integer.

Define the *Jacobi symbol* $\left(\frac{a}{N}\right)$, and write down a formula for $\left(\frac{2}{N}\right)$.

State the law of quadratic reciprocity for the Jacobi symbol, and use it to compute $\left(\frac{3}{N}\right)$ in terms of the value of N modulo 12.

Show that if d is a positive integer such that $d \equiv 0$ or 3 mod 4, and a, b are positive odd integers such that $a \equiv b \bmod d$, then $\left(\frac{-d}{a}\right) = \left(\frac{-d}{b}\right)$.

Paper 4, Section I

1F Number Theory

Define the *Möbius function* $\mu : \mathbb{N} \to \mathbb{C}$, and state the Möbius inversion formula.

Let $k > 1$ be an integer. We say that $n \in \mathbb{N}$ is k^{th}*-power free* if for all $d \in \mathbb{N}$ with $d > 1$, d^k does not divide n. Show that

$$\sum_{\substack{1 \leqslant d \leqslant n \\ d^k \mid n}} \mu(d) = \begin{cases} 1 & n \text{ is } k^{\text{th}}\text{-power free,} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that for $n \in \mathbb{N}$ we have

$$\mu(n) = \sum_{\substack{1 \leqslant d \leqslant n \\ (d,n)=1}} e^{2\pi i d/n}.$$

Paper 3, Section II

11F Number Theory

Let $k \in \mathbb{N}$.

Let p be an odd prime and $H_k = \{a + p^k\mathbb{Z} \mid a \equiv 1 \bmod p\}$. Show that H_k is a cyclic group under multiplication, and determine its order.

In the following, I denotes the $k \times k$ identity matrix.

Let p be an odd prime and let A be a $k \times k$ matrix with integer entries. Suppose that $A = I + p^m B$ for some $m \in \mathbb{N}$ and some $k \times k$ matrix B with integer entries, and that $A^n = I$ for some $n \in \mathbb{N}$. Show that $A = I$.

Now find the smallest integer $r \geqslant 1$ such that if A is a $k \times k$ matrix with integer entries satisfying $A = I + 2^r B$ for some $k \times k$ matrix B with integer entries, and $A^n = I$ for some $n \in \mathbb{N}$, then in fact $A = I$.

Paper 4, Section II

11F Number Theory

Let $\theta \in \mathbb{R}$ be irrational.

Define the *convergents* $(p_n, q_n)_{n \geqslant 0}$ of the continued fraction expansion $[a_0, a_1, a_2, \ldots]$ of θ. Prove that $p_n/q_n \to \theta$ as $n \to \infty$, and that if $n \geqslant 1$ is odd then

$$\frac{p_{n-1}}{q_{n-1}} < \theta < \frac{p_n}{q_n}.$$

Compute the continued fraction expansion of $\sqrt{7}$.

For each of the following equations, either find a solution in strictly positive integers x, y, or prove that no such solution exists:

(i) $x^2 - 7y^2 = 1$.

(ii) $x^2 - 7y^2 = -1$.

(iii) $x^2 - 7y^2 = 5$.

(iv) $p_x/q_x = 34/13$, where $(p_n, q_n)_{n \geqslant 0}$ is the sequence of convergents of the continued fraction expansion of $\sqrt{7}$.

Paper 1, Section II

41A Numerical Analysis

(a) The Fourier transform of an infinite sequence $(v_m)_{m\in\mathbb{Z}}$ is defined as

$$\widehat{v}(\theta) = \sum_{m\in\mathbb{Z}} v_m e^{-im\theta}, \qquad -\pi \leqslant \theta \leqslant \pi.$$

(i) Prove Parseval's identity:

$$\sum_{m\in\mathbb{Z}} |v_m|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\widehat{v}(\theta)|^2\, d\theta.$$

(ii) Consider the following two-step recurrence in n for u_m^n with $n \in \mathbb{Z}_+$ and $m \in \mathbb{Z}$:

$$u_m^{n+1} = \frac{1}{1+\mu}\Big[(1-\mu)u_m^{n-1} + \mu(u_{m+1}^n + u_{m-1}^n)\Big],$$

with $\mu \geqslant 0$. Use Fourier analysis to determine the range of μ for which the method is stable.

(b) The linear system $d\mathbf{y}/dt = A\mathbf{y}$ is discretised by the scheme

$$\mathbf{y}^{n+1} = (I - kB)^{-1}(I - kC)^{-1}\mathbf{y}^n,$$

with $B + C = A$ and $k = \Delta t$, where $\mathbf{y} \in \mathbb{R}^M$ and A, B and C are $M \times M$ square matrices.

(i) Define the exponential of a matrix. Show that for $k \ll 1$,

$$\exp[kB]\exp[kC] = \exp[k(B+C)] + \tfrac{1}{2}k^2(BC - CB) + O(k^3).$$

(ii) Find the order of the local truncation error of the scheme.

(iii) In the special case when the matrices B and C commute, does the order of the scheme change?

Paper 2, Section II

41A Numerical Analysis

(a) Let $h : \mathbb{R} \to \mathbb{R}$ be a 2-periodic function with Fourier series $h(x) = \sum_{n\in\mathbb{Z}} \widehat{h}_n e^{i\pi n x}$. Let $I(h) = \frac{1}{2}\int_{-1}^{1} h(x)\,dx$. For $N \geqslant 1$, consider the approximation

$$I_N(h) = \frac{1}{2N} \sum_{k=-N+1}^{N} h(k/N).$$

Find the error $|I_N(h) - I(h)|$ in terms of the $\widehat{h}_n$.

Assuming $|\widehat{h}_n| \leqslant M c^{|n|}$ for all $n \in \mathbb{Z}$, where $M > 0$ and $c \in (0,1)$, show that the error decays exponentially fast with N.

(b) Let $w : \mathbb{R} \to \mathbb{R}$ be a 2-periodic function with a finite Fourier expansion

$$w(x) = \sum_{|n|\leqslant d} \widehat{w}_n e^{i\pi n x}.$$

Consider the partial differential equation for $u(x,t)$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \frac{dw}{dx}\frac{\partial u}{\partial x}$$

with initial condition $u(x,0) = u_0(x)$ which is 2-periodic. Seek an approximate solution for $u(x,t)$ for all $t \geqslant 0$ that is 2-periodic in x with an expansion of the form

$$u(x,t) = \sum_{|n|\leqslant D} \widehat{u}_n(t) e^{i\pi n x},$$

where D is the truncation level. Write down a differential equation for the $\widehat{u}_n(t)$ of the form

$$\frac{d\widehat{u}_n(t)}{dt} = \sum_{|m|\leqslant D} B_{nm}\widehat{u}_m(t),$$

for a matrix B that you should specify.

Assume that $w(x) = \cos(\pi x)$. Show that in this case, the eigenvalues of B have non-positive real part. Is B invertible?

Paper 3, Section II

40A Numerical Analysis

Let A be an $n \times n$ real symmetric positive definite matrix and $\mathbf{b} \in \mathbb{R}^n$. Consider the linear system

$$A\mathbf{x} = \mathbf{b}.$$

(a) An iterative method is defined by $\mathbf{x}_{k+1} = H\mathbf{x}_k + \mathbf{v}$ with $n \times n$ matrix H and $\mathbf{v} \in \mathbb{R}^n$.

(i) State, giving a brief justification, necessary and sufficient conditions for $\mathbf{x}_k$ to converge to $\mathbf{x}^* = A^{-1}\mathbf{b}$ as $k \to \infty$.

(ii) Give H and $\mathbf{v}$ in terms of A and $\mathbf{b}$ for the Jacobi method. Prove that the Jacobi method is convergent if A is strictly diagonally dominant. [You may use Gershgorin's circle theorem without proof.]

(b) Define the function $f : \mathbb{R}^n \to \mathbb{R}$ by $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x}$.

(i) Show that the unique global minimizer of f is at $\mathbf{x}^* = A^{-1}\mathbf{b}$. Show also that for any $\mathbf{x}$ we have the identity

$$f(\mathbf{x}) - f(\mathbf{x}^*) = \tfrac{1}{2}\nabla f(\mathbf{x})^T A^{-1} \nabla f(\mathbf{x}),$$

where $\nabla f(\mathbf{x})$ is the gradient of f at $\mathbf{x}$.

(ii) Consider the gradient method with exact line search to find the $\mathbf{x}$ that minimises $f(\mathbf{x})$:

$$\mathbf{x}_{k+1} = \mathbf{x}_k - t_k \nabla f(\mathbf{x}_k) \quad \text{where} \quad t_k = \arg\min_{t \in \mathbb{R}} f(\mathbf{x}_k - t\nabla f(\mathbf{x}_k)).$$

Give an explicit expression for t_k in terms of the residual $\mathbf{r}_k = \mathbf{b} - A\mathbf{x}_k$.

(iii) Deduce that

$$f(\mathbf{x}_{k+1}) - f(\mathbf{x}^*) = \left[1 - \frac{(\mathbf{r}_k^T \mathbf{r}_k)^2}{(\mathbf{r}_k^T A^{-1} \mathbf{r}_k)(\mathbf{r}_k^T A \mathbf{r}_k)}\right] (f(\mathbf{x}_k) - f(\mathbf{x}^*)).$$

Conclude that

$$f(\mathbf{x}_k) - f(\mathbf{x}^*) \leqslant \left[1 - \frac{l}{L}\right]^k (f(\mathbf{x}_0) - f(\mathbf{x}^*)),$$

where l and L are respectively the smallest and largest eigenvalues of A.

What does this inequality tell us in the case when $n = 1$?

Paper 4, Section II

40A Numerical Analysis

(a) Describe the *power method*, and the *inverse iteration method* with shift $s \in \mathbb{R}$. Briefly explain the purpose of these algorithms.

(b) Let A be an $n \times n$ real symmetric matrix. The QR algorithm is as follows. Set $A_0 = A$. For $k = 0, 1, \ldots$, set $A_{k+1} = R_k Q_k$ where R_k and Q_k are determined by the QR factorisation $A_k = Q_k R_k$. Here, Q_k is an $n \times n$ orthogonal matrix and R_k is an $n \times n$ upper triangular matrix with all of its diagonal elements strictly positive.

(i) Briefly explain the purpose of the QR algorithm.

(ii) Show that for all k, A_k is symmetric and has the same eigenvalues as A.

A matrix M is r-*banded* if $M_{ij} = 0$ whenever $|i - j| > r$. Show that if A_k is r-banded then so is A_{k+1}.

(iii) For any $k \geqslant 1$, let $\widetilde{Q}_k = Q_0 \ldots Q_{k-1}$ and $\widetilde{R}_k = R_{k-1} \ldots R_0$. Show that $A^k = \widetilde{Q}_k \widetilde{R}_k$.

(iv) Consider the first and last columns of $\widetilde{Q}_k$. How do these relate to the inverse iteration method and the power method?

Paper 1, Section II

34B Principles of Quantum Mechanics

(a) A two-dimensional Hilbert space is spanned by two normalised vectors $|\phi\rangle$ and $|\psi\rangle$, which are not necessarily orthogonal. Consider the linear operator H defined by

$$H\,|\psi\rangle = g\,|\phi\rangle\ , \qquad\qquad H\,|\phi\rangle = g^*\,|\psi\rangle\ ,$$

where g is a complex constant. Determine the condition on $\langle\psi|\phi\rangle$ under which H is Hermitian. Henceforth assume this condition is satisfied and find the eigenvectors $|\pm\rangle$ of H and the corresponding eigenvalues. Verify that the distinct eigenvectors are orthogonal.

(b) Now assume $g = 1$. The Hamiltonian of a two-dimensional system is given by $H' = H + \Delta(t)$ where H is as in part (a) and

$$\Delta(t)\,|\phi\rangle = \Delta(t)\,|\psi\rangle = V(t)\,(|\phi\rangle + |\psi\rangle)\ ,$$

with $V(t)$ a real and time-dependent function. Working in the $|\pm\rangle$ basis, or otherwise, determine the exact probability to find the system in state $|\phi\rangle$ at time $t > 0$ if it was in state $|\psi\rangle$ at $t = 0$.

Paper 2, Section II

35B Principles of Quantum Mechanics

(a) State the defining properties of a *density operator* ρ in a Hilbert space of finite dimension N, and state the number of real free parameters that determine ρ. Starting from ρ_H in the Heisenberg picture, derive the time-dependent $\rho_S(t)$ in the Schrödinger picture.

(b) State the commutation relations for the spin operators $\mathbf{S}$ with each other and with $\mathbf{S}\cdot\mathbf{S}$. For the pure state $|\psi\rangle$ of a spin-$\frac{1}{2}$ qubit, you are given $\langle S_x\rangle_\psi$, $\langle S_z\rangle_\psi$, and the sign of $\langle S_y\rangle_\psi$. Determine the normalised state $|\psi\rangle$. [*Hint: Recall that the state need only be determined up to an overall phase, so that the normalised state can be parametrised by a single complex number.*]

(c) For a general mixed state of a spin-$\frac{1}{2}$ qubit, you are given $\langle S_x\rangle$, $\langle S_z\rangle$, and $\langle S_y\rangle$. Determine the mixed state. Are the expectation values of any three linearly independent Hermitian operators sufficient to fully specify a general mixed state? Present a proof or a counterexample.

Paper 3, Section II

33B Principles of Quantum Mechanics

(a) Let the components of the angular momentum operator be J_i, for $i = 1, 2, 3$. Define the states $|j, m\rangle$ in terms of the action of certain angular momentum operators and state the possible values of j and m. Let the operators $J_\pm \equiv J_1 \pm iJ_2$ and let $C_\pm(j, m)$ be defined by $J_\pm |j, m\rangle = C_\pm(j, m) |j, m \pm 1\rangle$. Considering particular cases, or otherwise, explicitly derive the four constants $\lambda_1^\pm$ and $\lambda_2^\pm$ appearing in

$$C_\pm(j, m) = \sqrt{j(j+1) + \lambda_1^\pm m(m + \lambda_2^\pm)}\,.$$

Hence, or otherwise, prove that if O is a linear operator that commutes with all components of the angular momentum operator, then

(i) $\langle j, m| O |j', m'\rangle \propto \delta_{jj'}$

(ii) $\langle j, m| O |j', m'\rangle \propto \delta_{mm'}$

(iii) $\langle j, m| O |j, m\rangle$ is independent of m.

(b) Now consider a two qubit system with states $|j_1, m_1; j_2, m_2\rangle$, with $j_1 = j_2 = 1/2$. Let $|j, m\rangle$ be the eigenstates of the total angular momentum of this system. State which values of j and m occur and write down a resolution of the identity in terms of the $|j, m\rangle$ eigenstates. Write down the values of the Clebsch–Gordan coefficients

$$C(j, m; j_1, m_1, j_2, m_2) \equiv \langle j, m|j_1, m_1, j_2, m_2\rangle$$

for all appropriate values of j, m, m_1, m_2.

Suppose that an operator O commutes with all the components of the total angular momentum of this two qubit system. Using the results derived above and the results from part (a), or otherwise, write down the most general expression for

$$\left\langle j_1, m_1; j_2, m_2 \right| O \left| j_1, m_1'; j_2, m_2' \right\rangle$$

for all distinct cases where it does not vanish.

Paper 4, Section II

33B Principles of Quantum Mechanics

(a) For a harmonic oscillator A of unit mass and frequency ω, write down the Hamiltonian in terms of position and momentum operators (X_A, P_A) as well as in terms of creation and annihilation operators $(a_A, a_A^\dagger)$. Briefly describe the Hilbert space of the system.

(b) Now consider the AB system comprising oscillator A and a second harmonic oscillator B with the same frequency and unit mass. The two oscillators are coupled by an interaction Hamiltonian $H_{\text{int}} = \lambda X_A X_B$, where λ is a real parameter.

(i) Compute the corrections to both the ground state of the AB system and its energy to linear order in λ.

(ii) Using the corrected ground state obtained in part (i), compute the density operator of the full system and hence the reduced density operator ρ_A, by tracing over B. Your final answer should be a density operator ρ_A that is normalised up to and including $\mathcal{O}(\lambda^2)$. [You may assume that corrections to the ground state beyond linear order do not affect the normalised density operator up to and including $\mathcal{O}(\lambda^2)$.]

(iii) For any properly normalised density operator ρ, the purity is defined by $\gamma = \text{Tr}\big(\rho^2\big)$. Determine an upper and lower bound on γ for a general state and determine its value for a pure state.

(iv) Compute the purity of ρ_A to order λ^2. For what values of λ does it satisfy the upper and lower bounds? Interpret your findings.

Paper 1, Section II

29L Principles of Statistics

(a) Let $\hat{\theta}$ denote the maximum likelihood estimator based on i.i.d. observations $X_1, \ldots, X_n$ from a parametric statistical model $\{f(\cdot, \theta) : \theta \in \Theta\}$, where $\theta_0 \in \Theta$ is the true parameter. Write down the limiting distribution of $\sqrt{n}(\hat{\theta} - \theta_0)$ under standard regularity conditions. [You should define any quantities involved in the expression of the limiting distribution.]

Now suppose $X_1, \ldots, X_n$ are i.i.d. Uniform$[-\theta, \theta]$ random variables, for a parameter $\theta > 0$.

(b) Derive an expression for the maximum likelihood estimator $\hat{\theta}$ of θ.

(c) Let θ_0 denote the true parameter. By calculating the cumulative distribution function of $\hat{\theta}$ or otherwise, show that $n(\theta_0 - \hat{\theta}) \xrightarrow{d} Z$ for a random variable Z whose distribution you should specify.

(d) Find, with brief justification, the limiting distribution of $n(\hat{\theta}^2 - \theta_0^2)$.

(e) Do we have $\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} W$ for a random variable W with positive variance? Justify your answer.

Paper 2, Section II

29L Principles of Statistics

Let $X_1, \ldots, X_n$ be i.i.d. observations from a statistical model $\{f(\cdot, \theta) : \theta \in \Theta\}$ satisfying the usual regularity conditions, where $\Theta \subseteq \mathbb{R}^p$. Suppose we wish to test

$$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta \neq \theta_0.$$

(a) Define the *Wald statistic* $W_n(\theta)$ and write down a test for H_0 based on $W_n(\theta_0)$ with asymptotic type-I error bounded by a given $\alpha \in (0, 1)$.

(b) Define the *likelihood ratio statistic* Λ_n and write down a test for H_0 based on Λ_n with asymptotic type-I error bounded by a given $\alpha \in (0, 1)$.

(c) Suppose now that $p = 1$, i.e. θ is a scalar parameter, and we are under the null H_0. By considering an appropriate Taylor expansion, show that the two test statistics above are asymptotically equivalent, in the sense that

$$\frac{\Lambda_n}{W_n(\theta_0)} \xrightarrow{P} 1.$$

[You may use, without proof, a uniform law of large numbers, as long as it is clearly stated.]

Paper 3, Section II

28L Principles of Statistics

Consider a binomial model $X \sim \text{Binomial}(n, \theta)$, where $\theta \in \Theta = [0,1]$, and let $\delta_{\text{MLE}}(X) = \frac{X}{n}$ denote the maximum likelihood estimator. In all of the following, we use the weighted quadratic loss

$$L(a, \theta) = \frac{(a-\theta)^2}{\theta(1-\theta)}.$$

(a) Compute the risk $R(\delta_{\text{MLE}}, \theta)$, for each $\theta \in (0,1)$.

(b) Suppose π is a prior for θ. What is meant by the π-*Bayes risk* of an estimator δ of θ? Prove that any estimator of θ which minimises the posterior risk also minimises the π-Bayes risk.

(c) Now suppose π is the uniform prior on $[0,1]$. By differentiating the expression for the posterior risk with respect to $\delta(x)$, prove that δ_{MLE} is the unique π-Bayes rule. [In your calculations, you may interchange differentiation and integration without justification. You may also use the formulas $\int_0^1 \theta^{a-1}(1-\theta)^{b-1} d\theta = \text{Beta}(a,b)$ and $\text{Beta}(a+1,b) = \text{Beta}(a,b) \cdot \frac{a}{a+b}$ without proof.]

(d) Prove that the estimator δ_{MLE} is minimax. What does it mean for an estimator δ of θ to be *admissible*? Is the estimator δ_{MLE} admissible? Justify your answer. [Results from the course should *not* be used without proof.]

Paper 4, Section II

28L Principles of Statistics

Let $X_1, \ldots, X_n$ be i.i.d. draws from a distribution on $\mathbb{R}$, with $\mu = \mathbb{E}(X_1)$ and $\sigma^2 = \mathrm{Var}(X_1) > 0$.

(a) Show that $\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 \xrightarrow{P} \sigma^2$, where $\bar{X}_n$ is the sample mean of the observations.

Let the function $g : \mathbb{R} \to \mathbb{R}$ be differentiable at μ with $g'(\mu) \neq 0$. Consider the plug-in estimator $T_n = T_n(X_1, \ldots, X_n) = g(\bar{X}_n)$ for the parameter $\theta = g(\mu)$.

(b) Using the Delta method, provide a formula for σ_n such that

$$\frac{T_n - \theta}{\sigma_n} \xrightarrow{d} N(0,1).$$

Consider now the jackknife variance estimator

$$v_{\mathrm{JACK}} = \frac{n-1}{n}\sum_{i=1}^{n}\left(T_{n-1,i} - \frac{1}{n}\sum_{j=1}^{n} T_{n-1,j}\right)^2,$$

where $T_{n-1,i} = T_{n-1}(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$, for $i = 1, \ldots, n$, is the leave-one-out estimator. In the following suppose further that g is twice-differentiable with $\sup_{x\in\mathbb{R}} |g''(x)| = 2M < \infty$.

(c) By considering a Taylor expansion of $T_{n-1,i} - T_n$, show that

$$v_{\mathrm{JACK}} = \frac{n-1}{n}(A_n + B_n + 2C_n),$$

where, for some $R_{n,i}$ satisfying $|R_{n,i}| \leqslant M(\bar{X}_{n-1,i} - \bar{X}_n)^2$ that you should specify,

$$A_n = (g'(\bar{X}_n))^2 \sum_{i=1}^{n}(\bar{X}_{n-1,i} - \bar{X}_n)^2, \quad B_n = \sum_{i=1}^{n}\left(R_{n,i} - \frac{1}{n}\sum_{j=1}^{n} R_{n,j}\right)^2$$

and $|C_n| \leqslant \sqrt{A_n B_n}$. Here $\bar{X}_{n-1,i}$ is the sample mean of the observations with X_i excluded.

Further assuming that $\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^4 \xrightarrow{P} c < \infty$, show that $v_{\mathrm{JACK}}/\sigma_n^2 \xrightarrow{P} 1$.

[*Hint: Use the fact (which you need not derive) that* $\bar{X}_{n-1,i} - \bar{X}_n = \frac{1}{n-1}(\bar{X}_n - X_i)$.]

(d) Give, with justification, an asymptotically valid $(1-\alpha)$-level confidence interval for θ based on the jackknife estimator.

Paper 1, Section II

27G Probability and Measure

(a) (i) What does it mean for a measure μ on a measurable space $(\Omega, \mathcal{F})$ to be *σ-finite*? State the uniqueness of extension theorem for a σ-finite measure.

(ii) Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}$. Show that Lebesgue measure λ is translation invariant, i.e., for $x \in \mathbb{R}$ and $B \in \mathcal{B}$,

$$\lambda(B) = \lambda(B + x)$$

where $B + x = \{b + x : b \in \mathcal{B}\}$.

(iii) Show that Lebesgue measure λ is the unique translation invariant σ-finite measure on $\mathcal{B}$ such that $\lambda((0,1]) = 1$.

(b) Let X, $(X_n)_{n\in\mathbb{N}}$ be real-valued random variables with distribution functions F_X, $(F_{X_n})_{n\in\mathbb{N}}$ respectively.

(i) State what it means to say that $X_n \to X$ *in distribution* in terms of their distribution functions.

(ii) Now assume that $X_n \to X$ in distribution. Let $B_{(0,1)}$ and $\lambda|_{(0,1)}$ be the Borel σ-algebra and the Lebesgue measure on $(0,1)$ respectively. On $((0,1), \mathcal{B}_{(0,1)}, \lambda|_{(0,1)})$, define for all $\omega \in (0,1)$,

$$\tilde{X}_n(\omega) = \inf\{x \in \mathbb{R} : \omega \leqslant F_{X_n}(x)\}\,, \quad \tilde{X}(\omega) = \inf\{x \in \mathbb{R} : \omega \leqslant F_X(x)\}\,.$$

Show that $\tilde{X}$ has the same distribution as X and $\tilde{X}_n$ has the same distribution as X_n for all n, and $\tilde{X}_n \to \tilde{X}$ almost surely.

[You may use the fact that for a non-constant, right-continuous, non-decreasing function g, $f(\omega) := \inf\{x \in \mathbb{R} : \omega \leqslant g(x)\}$ is left-continuous non-decreasing and $f(\omega) \leqslant x$ if and only if $\omega \leqslant g(x)$. You may also use the fact that a non-decreasing function has at most a countable set of points of discontinuity.]

Paper 2, Section II

27G Probability and Measure

(a) State the two Borel–Cantelli lemmas.

(b) Let $(X_n)_{n\in\mathbb{N}}$ be independent exponential random variables with rate 1. Let $M_n = \max_{1\leqslant m\leqslant n} X_m$. Show that

(i) $\limsup X_n/\log n = 1$ almost surely;

(ii) $\liminf M_n/\log n \geqslant 1$ almost surely.

[*Hint: You may use without proof the inequality $e^x \geqslant 1 + x$ for all $x \in \mathbb{R}$.*]

(c) Let μ, ν be two measures on a measurable space $(\Omega, \mathcal{F})$ such that $\mu(\Omega) < \infty$. We say $\mu \ll \nu$ if for any $A \in \mathcal{F}$, $\nu(A) = 0$ implies $\mu(A) = 0$. Show that $\mu \ll \nu$ if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any $A \in \mathcal{F}$, $\mu(A) < \varepsilon$ whenever $\nu(A) < \delta$.

Paper 3, Section II

26G Probability and Measure

(a) (i) State Lévy's convergence theorem for characteristic functions.

(ii) Let $(X_n)_{n\in\mathbb{N}}$, X be random variables on $\mathbb{R}^d$. Show that $X_n \to X$ weakly if and only if for all $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$, $\langle u, X_n\rangle \to \langle u, X\rangle$ weakly. [For $a = (a_1, \ldots, a_d), b = (b_1, \ldots, b_d) \in \mathbb{R}^d$, $\langle a, b\rangle := \sum_{i=1}^d a_i b_i$.]

(b) Let X be a real-valued random variable with characteristic function $\phi(t) = \mathbb{E}(e^{itX})$.

(i) Show that for any $u > 0$

$$\frac{1}{u}\int_{-u}^{u}(1-\phi(t))\,dt = 2\mathbb{E}\left(1 - \frac{\sin(uX)}{uX}\right).$$

(ii) Show that

$$\mathbb{P}(|X| > 2/u) \leqslant \frac{1}{u}\int_{-u}^{u}(1-\phi(t))dt\,.$$

(iii) Now let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real-valued random variables with characteristic functions $(\phi_n)_{n\in\mathbb{N}}$. If for all $t \in \mathbb{R}$, $\phi_n(t) \to \phi(t)$ for some function ϕ that is continuous at 0, show that given any $\varepsilon > 0$, there exists $M > 0$ such that $\mathbb{P}(|X_n| > M) \leqslant \varepsilon$ for all n.

[*You may use convergence results for integrals given in the course and Fubini's theorem, without proof.*]

Paper 4, Section II

26G Probability and Measure

(a) Define what it means for a sequence of real-valued random variables $(X_n)_{n\in\mathbb{N}}$ to be *uniformly integrable.* Show that if $(X_n)_{n\in\mathbb{N}}$ is bounded in L^p for some $p > 1$, then $(X_n)_{n\in\mathbb{N}}$ is uniformly integrable. Give, with justification, a counterexample to show that a sequence of random variables bounded in L^1 need not be uniformly integrable.

(b) Let $(X_n)_{n\in\mathbb{N}}$ be an identically distributed sequence of real-valued random variables in L^1. Let $S_n = X_1 + \cdots + X_n$. Show that the sequence $(S_n/n)_{n\in\mathbb{N}}$ is uniformly integrable. Assuming that $S_n/n \to \mathbb{E}(X_1)$ in probability, prove that $S_n/n \to \mathbb{E}(X_1)$ in L^1.

(c) Now assume that $(X_n)_{n\in\mathbb{N}}$ is an identically distributed sequence of random variables in L^2. Let $M_n = \max_{k\leqslant n} |X_k|$.

(i) Show that $n^{-1/2}M_n \to 0$ in probability as $n \to \infty$.

(ii) Show that $n^{-1/2}\mathbb{E}(M_n) \to 0$ as $n \to \infty$.

Paper 1, Section I

10E Quantum Information and Computation

Let U_f denote a quantum oracle which acts on $(n+1)$ qubits as follows: $\forall\, x \in \{0,1\}^n$ and $y \in \{0,1\}$,

$$U_f \,|x\rangle\,|y\rangle = |x\rangle\,|y \oplus f(x)\rangle\,,$$

where the addition is taken modulo 2 and

$$f(x) = a.x \oplus b := \sum_{i=1}^{n} a_i x_i \oplus b \quad \text{for some } a \in \{0,1\}^n \text{ and } b \in \{0,1\}.$$

(a) (i) Write down an expression for the state $|\Phi_1\rangle$ obtained when the state $|0\rangle^{\otimes n}\,|1\rangle$ of $(n+1)$ qubits is acted on by $(U_f H^{\otimes(n+1)})$, where H denotes the Hadamard gate.

(ii) Let $|\Phi_2\rangle := H^{\otimes(n+1)}\,|\Phi_1\rangle$. Write an expression for this state.

(iii) Let $|\Phi_3\rangle := U_f\,|\Phi_2\rangle$. Write an expression for this state.

(b) (i) $|\Phi_3\rangle$ is a state of $n+1$ qubits. What is the probability of obtaining the n-bit string a by doing a measurement of the first n qubits in the computational basis?

(ii) What is the state of the last (i.e., the $(n+1)^{\text{th}}$) qubit after the above measurement?

(iii) Find the probability that a measurement on this qubit yields the value of b when a contains an odd number of 1s and when a contains an even number of 1s.

Paper 2, Section I

10E Quantum Information and Computation

Consider a function $f : \mathbb{Z}_{12} \to \mathbb{Z}_9$ defined by

$$f(x) = 4^x \bmod 9.$$

(a) The function f is periodic. Find its period r.

(b) Suppose we are given the following quantum state of 2 registers:

$$|f\rangle := \frac{1}{\sqrt{12}} \sum_{x \in \mathbb{Z}_{12}} |x\rangle \, |f(x)\rangle \, ,$$

and a measurement of the second register yields a value y.

What are the possible values of y and what are the corresponding probabilities?

(c) If $y = 4$, find the resulting state $|\alpha\rangle$ of the first register after the above measurement.

(d) Let QFT_{12} denote the quantum Fourier transform modulo 12. How does it act on a state $|x\rangle$ for $x \in \mathbb{Z}_{12}$?

(e) Suppose a measurement of the state $\mathrm{QFT}_{12} \, |\alpha\rangle$ yields a value c. What are the possible values of c and what are the corresponding probabilities?

Paper 3, Section I

10E Quantum Information and Computation

Recall the Schmidt decomposition theorem for bipartite states: For any bipartite state $|\psi_{AB}\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} X_{ij}|i\rangle|j\rangle$ of an $n-$dimensional system $A \simeq \mathbb{C}^n$ and an $m-$dimensional system $B \simeq \mathbb{C}^m$, there exist (non-unique) orthonormal bases $\{|\alpha_i\rangle\}_{i=1}^{n} \subset \mathbb{C}^n$, $\{|\beta_j\rangle\}_{j=1}^{m} \subset \mathbb{C}^m$, and a (unique) set of non-negative numbers $\lambda_1, \lambda_2, \ldots, \lambda_d$ (for $d = \min\{n, m\}$) such that

$$|\psi_{AB}\rangle = \sum_{i=1}^{d} \lambda_i |\alpha_i\rangle|\beta_i\rangle.$$

The number of non-zero λ_i is called the *Schmidt rank* of $|\psi_{AB}\rangle$ and is equal to $\operatorname{rank}(X)$.

(a) Using the Schmidt decomposition theorem, show that a bipartite state $|\psi_{AB}\rangle$ is entangled if and only if its Schmidt rank is at least 2.

(b) Consider the following tripartite states of 3 qubits:

$$\begin{aligned}
|\chi^{(1)}_{ABC}\rangle &= |0_A\rangle|0_B\rangle|0_C\rangle\,,\\
|\chi^{(2)}_{ABC}\rangle &= |0_A\rangle|\phi^{+}_{BC}\rangle\,,\\
|\chi^{(3)}_{ABC}\rangle &= \frac{1}{\sqrt{2}}(|0_A\rangle|0_B\rangle|0_C\rangle + |1_A\rangle|1_B\rangle|1_C\rangle)\,.
\end{aligned}$$

Here, $|\phi^{+}_{BC}\rangle = \frac{1}{\sqrt{2}}(|0_B\rangle|0_C\rangle + |1_B\rangle|1_C\rangle)$ is a Bell state of two qubits. Show that

(i) $|\chi^{(1)}_{ABC}\rangle$ is product across all three bipartitions,

(ii) $|\chi^{(2)}_{ABC}\rangle$ is product across the $A - BC$ bipartition, and is entangled across the $B - AC$ and the $C - AB$ bipartitions,

(iii) $|\chi^{(3)}_{ABC}\rangle$ is entangled across all three bipartitions.

(c) Suppose that Alice (A), Bob (B), and Charlie (C) share either the state $|\chi^{(3)}_{ABC}\rangle$ or $|\chi^{(4)}_{ABC}\rangle = \frac{1}{\sqrt{3}}(|0_A\rangle|0_B\rangle|1_C\rangle + |0_A\rangle|1_B\rangle|0_C\rangle + |1_A\rangle|0_B\rangle|0_C\rangle)$. Charlie performs a measurement in the computational basis and obtains the result 0. Show that the resulting state of Alice and Bob will be

(i) product in the case they share $|\chi^{(3)}_{ABC}\rangle$,

(ii) entangled in the case they share $|\chi^{(4)}_{ABC}\rangle$.

Paper 4, Section I

10E Quantum Information and Computation

(a) Verify that for any 2×2 matrix, A, the following relation holds

$$(I \otimes A) \left|\Phi^+\right\rangle = (A^T \otimes I) \left|\Phi^+\right\rangle,$$

where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, I denotes the 2×2 identity matrix and T denotes transposition taken with respect to the standard basis $\{|0\rangle, |1\rangle\}$.

(b) Let $\theta \in [0, 2\pi)$,

$$U_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

$|\psi_1\rangle = U_\theta |0\rangle$ and $|\psi_2\rangle = U_\theta |1\rangle$.

(i) Show that $ZX|\psi_2\rangle = |\psi_1\rangle$, where X and Z are the usual one-qubit gates.

(ii) Show that the Bell state $|\Phi^+\rangle$ can be written as $\frac{1}{\sqrt{2}}(|\psi_1\rangle|\psi_1\rangle + |\psi_2\rangle|\psi_2\rangle)$. [*Hint: Use part (a).*]

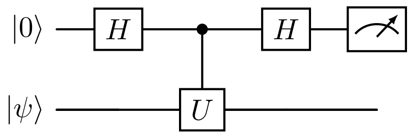
(iii) Suppose Alice and Bob initially share the Bell state $|\Phi^+\rangle$, with the first qubit being with Alice and the second qubit being with Bob. Alice then applies U_θ^{-1} to her qubit and then measures it in the computational basis. What are Alice's possible outcomes and what are the corresponding states of Bob's qubit after Alice's measurement?

(iv) Suppose Alice knows the value of θ but Bob does not. Give a protocol by which Alice can transmit $|\psi_1\rangle$ to Bob by sending just one classical bit to him, given that they share the Bell state $|\Phi^+\rangle$. (This is in contrast to general teleportation of an unknown qubit, which uses one Bell state and two classical bits of communication.)

Paper 2, Section II

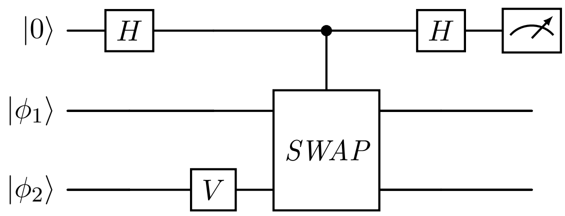
15E Quantum Information and Computation

(a) Consider a circuit which uses a controlled unitary gate with unitary operator U:



Here H represents a Hadamard gate, $|\psi\rangle$ denotes a single qubit state which is an eigenstate of U, $U|\psi\rangle = e^{2\pi i\theta}\,|\psi\rangle$, and the measurement is done in the computational basis. Express the phase θ in terms of the probability of the measurement giving zero.

(b) Consider the following circuit which uses a controlled $SWAP$ gate. A $SWAP$ gate acts as: $SWAP|i\rangle\,|j\rangle \mapsto |j\rangle\,|i\rangle \;\forall\; |i\rangle\,, |j\rangle \in \mathbb{C}^2$.



Here V is a unitary operator, $|\phi_1\rangle\,, |\phi_2\rangle$ are single qubit states, and the measurement is done in the computational basis. Find the probability of the measurement giving zero.

(c) Consider the operator U and the state $|\psi\rangle$ introduced in part (a). It is given that $\theta = j/2^m$ for some $j \in \{0, 1, 2, \ldots, 2^m - 1\}$ and some $m \in \mathbb{N}$. Define the controlled unitary operator $\Theta_m(U)$ which acts on a state $|k\rangle\,|\psi\rangle$ of $m+1$ qubits as:

$$\Theta_m(U)\,|k\rangle\,|\psi\rangle = |k\rangle\,U^k\,|\psi\rangle\,,$$

where $U^k\,|\psi\rangle$ is the state obtained by k successive applications of the operator U on the state $|\psi\rangle$ and $k \in \{0, 1, 2, \ldots, 2^m - 1\}$.

(i) Find an expression for the following state of $(m+1)$ qubits:

$$|\Phi\rangle := \Theta_m(U)\left(H^{\otimes m} \otimes I\right)|0\rangle^{\otimes m}\,|\psi\rangle\ .$$

[You should write out the result of applying the operators to $|0\rangle^{\otimes m}\,|\psi\rangle$.]

(ii) Write an expression for the corresponding state $|\phi_j\rangle$ of the first m qubits. Show that $\{|\phi_j\rangle\}_{j=0}^{2^m-1}$ is an orthonormal basis.

(iii) Let F be an operator acting on m qubits as $F\,|j\rangle = |\phi_j\rangle$. Justify why F is a unitary operator.

(iv) State the 2 sequential operations that you can do on $|\phi_j\rangle$ to find the value of j. What is the probability p_j of finding the value of j (and hence θ)?

[QUESTION CONTINUES ON THE NEXT PAGE]

(d) Recall the Breidbart basis used in the intercept and resend attack by Eve in the BB84 protocol. Suppose you do a measurement in this basis to discriminate between two equiprobable states $|\psi_1\rangle = |0\rangle$ and $|\psi_2\rangle = |-\rangle$. What is your probability of success? Justify why this is the best measurement that you can do to distinguish between the states (clearly stating any relevant result from the course).

Paper 3, Section II

15E Quantum Information and Computation

Suppose $\mathcal{H}_n \simeq (\mathbb{C}^2)^{\otimes n}$ can be partitioned into 2 mutually orthonormal subspaces $\mathcal{S}_g$ and $\mathcal{S}_b$. Let P_g and P_b denote the orthogonal projections onto $\mathcal{S}_g$ and $\mathcal{S}_b$, respectively. We call $\mathcal{S}_g$ the *good* subspace and $\mathcal{S}_b$ the *bad* subspace. All kets in this question are normalised.

(a) Show that any $|\varphi\rangle \in \mathcal{H}_n$ can be uniquely decomposed as follows:

$$|\varphi\rangle = a\,|\psi_1\rangle + b\,|\psi_2\rangle\,,$$

where $|\psi_1\rangle \in \mathcal{S}_g$ and $|\psi_2\rangle \in \mathcal{S}_b$. Express a, b, $|\psi_1\rangle$ and $|\psi_2\rangle$ in terms of $|\varphi\rangle$, P_g and P_b.

For any $|\alpha\rangle \in \mathcal{H}_n$, consider a unitary operator $R_{|\alpha\rangle}$ which acts as follows:

$$\begin{aligned} &R_{|\alpha\rangle}\,|\alpha\rangle = -\,|\alpha\rangle\ , \\ &R_{|\alpha\rangle}\,|\beta\rangle = |\beta\rangle \quad \forall\ |\beta\rangle \text{ such that } \langle\beta|\alpha\rangle = 0\,. \end{aligned}$$

In particular define $R_0 := R_{|0_n\rangle}$, where $|0_n\rangle := |0\rangle^{\otimes n}$. Further, let A denote a unitary operator which acts on $|0_n\rangle$ as follows:

$$|\Omega\rangle := A\,|0_n\rangle := \sqrt{p}\,|\psi_g\rangle + \sqrt{1-p}\,|\psi_b\rangle\,,$$

where $|\psi_g\rangle \in \mathcal{S}_g$, $|\psi_b\rangle \in \mathcal{S}_b$, and $p \in (0,1)$. Let $R_g := R_{|\psi_g\rangle}$.

Suppose you are given quantum circuits which implement R_0, R_g and A. Below, we consider a protocol that increases the amplitude $\sqrt{p}$ of the *good* state $|\psi_g\rangle$.

(b) Show that $AR_0A^{-1}\,|\Omega\rangle = R_{|\Omega\rangle}\,|\Omega\rangle$.

(c) Consider the 2-dimensional real Euclidean plane $\mathcal{P}$ spanned by $|\psi_g\rangle$ and $|\psi_b\rangle$, and let θ denote the angle between $|\Omega\rangle$ and the horizontal axis $|\psi_b\rangle$. Express θ in terms of p.

(d) How can you geometrically interpret the actions of R_g and $(-R_{|\Omega\rangle})$ on any vector $|\phi\rangle$ in $\mathcal{P}$?

(e) Justify that R_g and $(-R_{|\Omega\rangle})$ leave the plane $\mathcal{P}$ invariant.

(f) Start with the state $|\Omega\rangle$ and apply the operator $Q := -R_{|\Omega\rangle}R_g$ to it. What is the action of a single use of Q on the state $|\Omega\rangle$ and k successive uses of Q on the state $|\Omega\rangle$? [*Hint: Draw clear diagrams.*] What is the final state $|\Omega_k\rangle$ obtained after k successive applications of Q on $|\Omega\rangle$?

(g) What is the probability that a measurement of $|\Omega_k\rangle$ will yield the *good* state $|\psi_g\rangle$? Show that $k = \mathcal{O}(1/\sqrt{p})$ iterations suffice to make the amplitude of the *good* state close to 1.

Paper 1, Section II

19H Representation Theory

What is a *complex representation* (ρ, V) of a group G? What does it mean to say that a representation (ρ, V) of G is *faithful*? Show that every finite group has a faithful representation over the complex numbers.

Let G be a finite group, (ρ, V) be a complex representation of G, and $g \in G$. Writing $S(g)$ for the set of eigenvalues of $\rho(g)$, show that if g is conjugate to g^k in G then

$$\lambda \in S(g) \implies \lambda^k \in S(g).$$

Deduce that if p is a prime number and (ρ, V) is a faithful complex representation of S_p, then $\dim V \geqslant p-1$.

Consider the group G of all invertible functions $\sigma \colon \mathbb{N} \to \mathbb{N}$, under composition. Does G have a faithful complex representation?

Paper 2, Section II

19H Representation Theory

Let G be a finite group. What is the *character* χ_V of a complex representation (ρ, V) of G?

Suppose that (ρ, V) and (σ, W) are complex representations of G. Show that the vector space $\mathrm{Hom}_{\mathbb{C}}(V, W)$ of $\mathbb{C}$-linear maps $\alpha \colon V \to W$ can be made into a representation of $G \times G$ via

$$((g,h)\cdot\alpha)(v) = \sigma(h)\left(\alpha(\rho(g^{-1})v)\right) \text{ for } (g,h) \in G \times G, \alpha \in \mathrm{Hom}_{\mathbb{C}}(V,W) \text{ and } v \in V.$$

Show that the character $\chi_{\mathrm{Hom}_{\mathbb{C}}(V,W)}$ satisfies

$$\chi_{\mathrm{Hom}_{\mathbb{C}}(V,W)}(g,h) = \overline{\chi_V(g)}\chi_W(h).$$

Consider the permutation representation $\mathbb{C}G$ of $G \times G$ arising from the action of $G \times G$ on G via

$$(g,h)\cdot x = gxh^{-1} \text{ for } (g,h) \in G \times G, x \in G$$

What is $\chi_{\mathbb{C}G}$?

Suppose that $V_1, \ldots, V_r$ are all the simple representations of G (up to isomorphism). Show there is an isomorphism

$$\mathbb{C}G \cong \bigoplus_{i=1}^{r} \mathrm{Hom}_{\mathbb{C}}(V_i, V_i)$$

of representations of $G \times G$.

Paper 3, Section II

19H Representation Theory

Let p be a prime number and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the field with p elements. Consider the group of unitriangular 3×3-matrices with coefficients in $\mathbb{F}_p$,

$$G := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{F}_p \right\}.$$

(a) Describe the conjugacy classes of G.

(b) Show that G has precisely p^2 complex representations of degree 1 and describe them explicitly.

(c) Find an abelian subgroup A of G of order p^2 and construct $p-1$ irreducible complex representations of G of degree p by induction from A.

(d) Determine the character table of G.

Paper 4, Section II

19H Representation Theory

What is the topological group S^1? Assuming any necessary facts about continuous homomorphisms with domain $(\mathbb{R}, +)$, show that every irreducible complex representation of S^1 is of the form

$$z \mapsto z^n : S^1 \to GL_1(\mathbb{C})$$

for some $n \in \mathbb{Z}$.

Let $\rho_V \colon SU(2) \to GL(V)$ be a complex representation of the topological group $SU(2)$ and let χ_V be its character.

(a) Show that χ_V is determined by its restriction to a subgroup T of $SU(2)$ isomorphic to S^1 and deduce that χ_V may be written in the form $\sum_{n \in \mathbb{Z}} a_n z^n$ for some non-negative integers a_n such that $\sum_{n \in \mathbb{Z}} a_n < \infty$. Show moreover that $a_n = a_{-n}$ for all $n \in \mathbb{Z}$.

(b) Let V_n be the $(n+1)$-dimensional irreducible representation of $SU(2)$. Write down χ_{V_n} in the form given in part (a). Decompose $V_4 \otimes V_4$, $S^2 V_4$ and $\Lambda^2 V_4$ as a direct sum of irreducible representations up to isomorphism.

Paper 1, Section II

24H Riemann Surfaces

Given two suitable topological spaces Y and X, define a *covering map* $\pi : Y \to X$. What does it mean to say that X is *simply connected*? Write down the simply connected Riemann surfaces (up to analytic isomorphism).

What is a *lattice* of $\mathbb{C}$? Prove that for any lattice L there exists a non-constant analytic function $f : \mathbb{C} \to \mathbb{C}_\infty$ where $f(z + l) = f(z)$ for all $z \in \mathbb{C}$ and $l \in L$.

Assuming now that the quotient space $\mathbb{C}/L$ is a Riemann surface where the natural projection $q : \mathbb{C} \to \mathbb{C}/L$ is analytic, deduce the existence of a unique analytic function $\overline{f} : \mathbb{C}/L \to \mathbb{C}_\infty$ such that $f = \overline{f}q$ for your function f above.

Show that neither f nor $\overline{f}$ are covering maps. Does there exist a covering map from $\mathbb{C}/L$ to $\mathbb{C}_\infty$? Justify your answer, stating clearly any results which you use.

Paper 2, Section II

24H Riemann Surfaces

For a non-constant analytic map $f : R \to S$ between compact Riemann surfaces and a point $z \in R$, let $m_f(z)$ denote the multiplicity of f at z and $\deg(f)$ the degree of f.

State the valency theorem. For the Riemann surface $\mathbb{C}_\infty$ and a non-constant analytic function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$, which you may assume is of the form $f(z) = p(z)/q(z)$ for non-zero polynomials p, q, explain how to find $\deg(f)$. Which f are the analytic isomorphisms of $\mathbb{C}_\infty$?

If $h : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is the Möbius transformation that swaps ∞ with 1 and swaps 0 with -1, write down a formula for h, as well as a quadratic equation satisfied by the fixed points of h.

Now consider the rotational symmetry group G of a regular octahedron P. You may assume that G is realised as a group of Möbius transformations isomorphic to S_4 with the six vertices of P corresponding to the points $0, \infty, \pm 1, \pm i \in \mathbb{C}_\infty$. Write down the possible sizes of the orbits under this action of G on $\mathbb{C}_\infty$.

Consider the function $F : \mathbb{C}_\infty \to \mathbb{C}_\infty$ given by the formula

$$F(z) = \frac{(z^4+1)^2(z^4+6z^2+1)^2(z^4-6z^2+1)^2}{z^4(z^4-1)^4}.$$

Which points in $\mathbb{C}_\infty$ are mapped to ∞ by F and with what multiplicities? What is $\deg(F)$?

You may now assume that F is constant on orbits, namely that if z_1 and z_2 are in the same orbit of this action of G on $\mathbb{C}_\infty$ then $F(z_1) = F(z_2)$. By using the valency theorem, or otherwise, show that F distinguishes orbits, namely if $F(z) = F(w)$ for $z, w \in \mathbb{C}_\infty$ then z and w are in the same orbit.

Paper 3, Section II

23H Riemann Surfaces

State and prove the open mapping theorem for Riemann surfaces R and S. [You may assume the identity theorem.] Suppose that $f : R \to S$ is analytic and non-constant. Deduce that if R is compact then f is surjective.

Suppose that $f : \mathbb{D} \to \mathbb{D}$ is non-constant and analytic with $f(0) = 0$. Show that $g(z) = f(z)/z$ is analytic on $\mathbb{D}$, and that for the open disc D_r of radius $r < 1$ and centre 0 we have $|g(z)| \leqslant 1/r$ for $z \in D_r$.

Describe all of the analytic isomorphisms $h : \mathbb{D} \to \mathbb{D}$ such that $h(0) = 0$.

Suppose that $F : \mathbb{C} \to \mathbb{C}$ is an analytic isomorphism with $F(0) = 0$. Show that $|F(z)| \to \infty$ as $|z| \to \infty$ and that F extends to an analytic isomorphism $\mathbb{C}_\infty \to \mathbb{C}_\infty$.

Given an analytic isomorphism $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$, does f extend to an analytic isomorphism $\mathbb{C} \to \mathbb{C}$? Does every analytic isomorphism $\mathbb{D} \to \mathbb{D}$ extend to an analytic isomorphism $\mathbb{C} \to \mathbb{C}$? Justify your answers.

Paper 1, Section I

5L Statistical Modelling

(a) What are the three main components of a *generalised linear model* for observations $(Y_1, x_1), \ldots, (Y_n, x_n)$?

(b) Assuming the model holds with the canonical link function, give expressions for the mean and variance of the Y_i as functions of the covariates x_i.

(c) Define the *Poisson generalised linear model* with the canonical link function.

Paper 2, Section I

5L Statistical Modelling

Suppose we observe the proportion $Y \sim n^{-1}\text{Binomial}(n, p)$ where $n \in \mathbb{N}$ is known and $p \in (0, 1)$ is unknown. Compute the score function and the Fisher information for this statistical model.

State the Newton–Raphson and Fisher scoring algorithms for computing the maximum likelihood estimator.

How many steps do these algorithms take to converge to the maximum likelihood estimator when initialised at $p^{(0)} = Y$? How many steps does Fisher scoring take when initialised at some $p^{(0)} \neq Y$?

Paper 3, Section I

5L Statistical Modelling

Explain mathematically why the two results returned in the R output below are as they are.

```
> n <- 50
> Y <- rnorm(n)
> p <- 2
> Z1 <- matrix(rnorm(n*p), nrow=n)
> lm1 <- lm(Y ~ Z1)
> sum(lm1$residuals)
[1] 0
>
> p <- 49
> Z2 <- matrix(rnorm(n*p), nrow=n)
> lm2 <- lm(Y ~ Z2)
> summary(lm2)$r.squared
[1] 1
```

Paper 4, Section I

5L Statistical Modelling

Let the concentrations of insecticides $Y_1, \ldots, Y_n$ relative to covariates $x_1, \ldots, x_n \in \mathbb{R}^p$ follow the model

$$Y_i = \mu + x_i^T \beta + \varepsilon_i,$$

where $\mu \in \mathbb{R}$ and $\beta \in \mathbb{R}^p$ are unknown and the ε_i are i.i.d. with distribution $N(0, \sigma^2)$ for known $\sigma^2 > 0$. Suppose that it is not possible to measure Y_i directly, but we only have access to a random variable Z_i, satisfying

$$Z_i = \begin{cases} 1 & \text{if } Y_i > \tau, \\ 0 & \text{otherwise,} \end{cases}$$

where $\tau \in \mathbb{R}$ is some fixed known tolerance.

(a) Find a binary regression model with a suitable link function and some linear predictor for the data $(Z_1, x_1), \ldots, (Z_n, x_n)$. How can the parameters μ and β be estimated from these data?

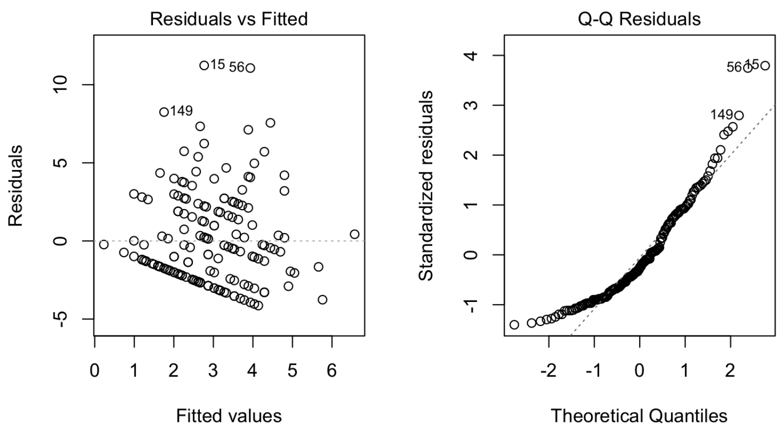
(b) Given a new data point $x^* \in \mathbb{R}^p$, find an asymptotic $1 - \alpha$ confidence interval for the expected mean response at x^* in the binary regression model.

Paper 1, Section II

13L Statistical Modelling

During spawning season, female horseshoe crabs lay clusters of eggs which are fertilised externally by a number of nearby male crabs. We are given a dataset with information on $n = 173$ female crabs. It contains, among other variables, the weight and width of each crab, as well as the number y of male crabs in the vicinity. Consider the following (shortened) R output from an analysis of this dataset.

```
> head(Crabs[c("y", "weight", "width")])

    y weight width
  1 8   3.05  28.3
  2 0   1.55  22.5

> crabs.lm <- lm(y ~ weight + width, data=Crabs)
> summary(crabs.lm)

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
  (Intercept)  -4.5721     4.2132  -1.085   0.2794
  weight        1.6817     0.9015   1.865   0.0639
  width         0.1446     0.2218   0.652   0.5153

> anova(lm(y ~ 1, data=Crabs), crabs.lm)

Model 1: y ~ 1
Model 2: y ~ weight + width
  Res.Df    RSS Df Sum of Sq      F    Pr(>F)
1    172 1704.9
2    170 1477.7  2     227.2 13.069 5.252e-06 ***

> cor(Crabs$weight, Crabs$width)
[1] 0.8769373
> plot(crabs.lm, add.smooth=FALSE, which=c(1,2))
```



[QUESTION CONTINUES ON THE NEXT PAGE]

(a) Write down the statistical model fitted by `crabs.lm`.

(b) State mathematical formulas for what the columns displayed in the output of `crabs.lm` describe.

(c) State the null and alternative hypotheses for the hypothesis test performed using the `anova` command. What can we conclude from the output?

Explain this in relation to the final column of the `summary` command output by referring to the output of the `cor` command.

(d) What would you expect to see in the two diagnostic plots if the model `crabs.lm` were to fit the data well? Which model assumptions appear to be violated according to these plots?

(e) Suggest two modifications that may help to improve the quality of the fit.

Paper 4, Section II

13L Statistical Modelling

A researcher has collected a dataset on 53 patients to study the occurrence of a certain disease. The dataset contains for each patient, a variable (`dis`) indicating if they are suffering from the disease (`dis` = 1) or not (`dis` = 0), a continuous variable (`pollution`) describing the level of air pollution near their home, as well as a variable (`risk`) recording which of two risk categories the patient falls under. Below is given the (shortened) `R` output of the analysis performed by the researcher.

```
> head(disease)
    dis  pollution risk
  1   0   58.44204    2
  2   0   45.29248    1

> binom.lm1 <- glm(dis ~ pollution + risk, family=binomial, data=disease)
> summary(binom.lm1)

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept) -11.83686    3.62821  -3.262  0.00110  **
pollution     0.18243    0.05679   3.212  0.00132  **
risk2         1.53893    0.74665  -2.061  0.03929  *

  (Dispersion parameter for binomial family taken to be 1)

      Null deviance: 73.5  on ?  degrees of freedom
  Residual deviance: 55.3  on ?  degrees of freedom
  AIC: 61.3
```

(a) Write down the generalised linear model being fitted. Why is there no coefficient for `risk1`?

(b) Give an interpretation for the coefficient of `pollution`.

(c) What are the missing degrees of freedom in the output?

(d) How should the `R` code above be changed to fit the model that corresponds to the 'Null deviance' in the output? What is the AIC value of that model?

(e) State the null and alternative hypotheses for the test performed in the code below and describe the form of the test.

What can the researcher conclude from the following ouput?

```
> pchisq(73.5-55.3, df = 2, lower.tail = FALSE)
[1] 0.0001142191
> binom.lm2 <- glm(dis ~ pollution * risk, family=binomial, data=disease)
> pchisq(binom.lm1$deviance-binom.lm2$deviance, 1, lower.tail=FALSE)
[1] 0.8705801
```

Paper 1, Section II

36B Statistical Physics

(a) What systems are described by a *canonical ensemble*? If the energy of the ith microstate is E_i, with $i = 0, 1, 2, \ldots$, write down an expression for the partition function Z in terms of temperature T and the Boltzmann constant k_B.

(b) Calculate the partition function Z_1 for a single classical ultra-relativistic spinless particle moving in three-dimensional space in a potential $U(\mathbf{x})$. [Ultra-relativistic means that the energy-momentum relation is $E = pc$, where c is the speed of light.]

(c) A system of a large number N of identical, non-interacting particles of the type described in part (b) is in equilibrium at temperature T in a potential

$$U(\mathbf{x}) = \frac{(x^2 + y^2 + z^2)^n}{V^{2n/3}} \,,$$

where n is a positive integer and $V > 0$ is an external parameter analogous to volume.

(i) Calculate the partition function and hence show that the Helmholtz free energy is

$$F = -Nk_BT\Big[\ln V + A\ln(k_BT) + \ln I_n + B\Big] \,,$$

where

$$I_n = \int_0^\infty u^2\, e^{-u^{2n}}\, du \,,$$

and you should determine A and B.

(ii) Considering the conjugate pressure to V, $p = -\left(\frac{\partial F}{\partial V}\right)_{T,N}$, derive the equation of state.

(iii) Compute the average energy E, the variance of energy $(\Delta E)^2$ and the heat capacity C_V for the system. Comment on the behaviour of $(\Delta E)/E$ in the thermodynamic limit.

(iv) Obtain the local particle number density as a function of $\mathbf{x}$ and hence determine the most likely $|\mathbf{x}|$ to find a particle.

Paper 2, Section II

37B Statistical Physics

(a) Explain what is meant by an *intensive quantity* and what is meant by an *extensive quantity*. Give two examples of each.

(b) A real-valued homogeneous function f of degree k satisfies

$$f(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^k f(x_1, x_2, \ldots, x_n)\,,$$

for any real λ. Show that

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = kf\,.$$

(c) Explain why the energy $E(S,V,N)$ is a homogeneous function of degree 1, where S is the entropy, V is the volume and N is the number of particles. Hence, using the first law of thermodynamics, find an expression for E in terms of S, V, N, μ, p and T, where μ is the chemical potential, p is the pressure and T is the temperature. Show that $d\mu = (Vdp - SdT)/N$.

(d) Consider a chemical reaction at constant T and p where each molecule of chemical A can change into two molecules of chemical B and one molecule of chemical C, and vice-versa, i.e. $A \leftrightarrow 2B + C$. By minimising the Gibbs free energy G, derive a relation between the chemical potentials of the three chemicals at equilibrium, where the chemical potential of chemical i is $\mu_i = \frac{\partial G}{\partial N_i}$.

Paper 3, Section II

35B Statistical Physics

(a) What systems are described by a *grand canonical ensemble*? Use the grand partition function to derive the formula for the Fermi-Dirac distribution for the mean occupation numbers n_r of discrete single-particle states r with energies $E_r \geqslant 0$ in a gas of non-interacting identical Fermions in terms of $\beta = 1/(k_B T)$ and the chemical potential μ.

(b) Show that the density of states $g(E)$ for a gas of non-interacting non-relativistic identical Fermions with spin degeneracy g_s in a large d-dimensional volume V, where $d \geqslant 2$, is

$$g(E) = g_s B V E^{(d-2)/2} \, ,$$

where B is a constant that you should determine. Compute the Fermi energy E_F, *i.e.* the chemical potential at zero temperature and with N particles. Find an expression at zero temperature for pV in terms of N and E_F, where p is the pressure. [You may denote the surface area of a unit $(d-1)$-dimensional sphere by S_{d-1}.]

(c) Consider a semi-infinite block of metal in 3-dimensional space occupying $z \leqslant 0$ with infinite extent in the x and y directions, and empty space for $z > 0$. An electron in the metal with momentum (p_x, p_y, p_z) will escape from the metal if p_z is large enough, $\frac{p_z^2}{2m} \geqslant E_F + V_0$, where m is the mass of the electron and V_0 is the potential barrier.

The flux density, J_z, of electrons escaping from the metal is the number of electrons escaping per unit area per unit time. Model the electrons as an ideal non-relativistic gas of Fermions, and assume that $k_B T \ll V_0$ and $k_B T \ll E_F$. You may assume that $\mu = E_F$ in this limit. Thereby derive an expression for the flux density of electrons escaping from the metal.

[*Hint:* $\displaystyle\int_{-\infty}^{\infty} e^{-a\,x^2} dx = \sqrt{\pi/a} \quad \textit{for} \quad a > 0.$]

Paper 4, Section II

35B Statistical Physics

(a) What is meant by a *first-order phase transition* and what is meant by a *second-order phase transition*?

(b) Derive the Clausius-Clapeyron equation for dp/dT, with p the pressure, along the first-order phase transition curve between a liquid and a gas in terms of the temperature T, the latent heat L, and the volume per unit particle in the two phases v_{liquid} and v_{gas}. [You may assume that the chemical potentials of the two phases are equal on the phase transition curve.]

(c) Assuming that L is constant, $v_{\text{gas}} \gg v_{\text{liquid}}$, and that the gas obeys the ideal gas law, derive an equation for the phase transition curve $p(T)$. Determine the change in volume of the gas with T along the phase transition curve (i.e. dV_{gas}/dT along the curve).

(d) Consider the Dieterici equation of state

$$p = \frac{k_B T}{v-b} \exp\left(-\frac{a}{k_B T v}\right) .$$

Here v is the volume per unit particle.

(i) Provide a brief physical interpretation for the constants a and b.

(ii) Calculate the second virial coefficient.

(iii) Find the critical temperature T_c for the Dieterici equation of state in terms of a, b and k_B.

Paper 1, Section II

30L Stochastic Financial Models

(a) In the context of a one-period market model, define the terms *arbitrage* and *risk-neutral measure*. State the one-period fundamental theorem of asset pricing.

(b) Given an $n \times d$ matrix P, let

$$\mathcal{A} = \{\varphi \in \mathbb{R}^d : (P\varphi)_i \geqslant 0 \text{ for all } i \text{ and } (P\varphi)_i > 0 \text{ for some } i\}$$

and

$$\mathcal{Q} = \left\{ q \in \mathbb{R}^n : P^\top q = 0, \ \sum_{i=1}^{n} q_i = 1, \quad q_i > 0 \text{ for all } i \right\}.$$

Prove that $\mathcal{Q} = \emptyset$ if and only if $\mathcal{A} \neq \emptyset$.

Consider a one-period market model with d risky assets, where S_t^i is the price of asset i at time $t \in \{0, 1\}$ and r is the interest rate. Assume that there exists at least one risk-neutral measure for the model, and that the random variables $\{S_1^i - (1+r)S_0^i : 1 \leqslant i \leqslant d\}$ are linearly independent.

(c) Let Y be a random variable such that $\mathbb{E}^{\mathbb{Q}}(Y) > 0$ for all risk-neutral measures $\mathbb{Q}$. Show that there exists a vector $\theta \in \mathbb{R}^d$ such that

$$Y \geqslant \theta^\top [S_1 - (1+r)S_0] \ \text{ almost surely,}$$

and we have strict inequality with positive probability.

(d) Let Z be a random variable such that $\mathbb{E}^{\mathbb{Q}}(Z) \geqslant 0$ for all risk-neutral measures $\mathbb{Q}$. Show that there exists a vector $\phi \in \mathbb{R}^d$ such that

$$Z \geqslant \phi^\top [S_1 - (1+r)S_0] \ \text{ almost surely.}$$

Paper 2, Section II

30L Stochastic Financial Models

Let $(\mathcal{F}_n)_n$ be a filtration such that $\mathcal{F}_0$ is trivial. Let $(M_n)_n$ be a martingale and let T be a stopping time with respect to the filtration.

(a) Show that the stopped process $(M_{n\wedge T})_n$ is a martingale. [Results on the martingale transform may *not* be assumed without proof.]

(b) Assuming $(M_{n\wedge T})_n$ is bounded and T is finite, show that $\mathbb{E}(M_T) = M_0$. [Versions of the optional stopping theorem may *not* be assumed without proof.]

For the rest of the problem, let $X_0 = 0$ and $X_n = \xi_1 + \cdots + \xi_n$ for $n \geqslant 1$, where $(\xi_n)_n$ are IID and generate the filtration. Suppose $\mathbb{P}(\xi_n = +1) = \mathbb{P}(\xi_n = -1) = 1/2$ for all n.

(c) Given a real number $w > 1$, show that there exists a real number $z > 0$ such that the process

$$M_n = \left(Aw^{X_n} + Bw^{-X_n}\right) z^n$$

is a martingale for all constants A and B.

(d) Fix positive integers a, b and define

$$T = \min\{n \geqslant 0 : X_n \in \{-a, b\}\}.$$

For any $0 < z < 1$, compute $\mathbb{E}(z^T)$. [You may use the fact that T is a finite stopping time without proof.]

Paper 3, Section II

29L Stochastic Financial Models

Consider a discrete-time market model with interest rate $r > -1$ and one stock with time-n price S_n. Suppose $S_0 > 0$ is given and that $S_n = S_{n-1}\xi_n$ for all $n \geqslant 1$, where the sequence $(\xi_n)_{n\geqslant 1}$ is IID, and

$$\mathbb{P}(\xi_1 = 1+b) = p = 1 - \mathbb{P}(\xi_1 = 1+a)$$

for constants $-1 < a < b$ and $0 < p < 1$. Assume that there exists a risk-neutral measure $\mathbb{Q}$ for the model.

(a) Show that $a < r < b$, and that the sequence $(\xi_n)_{n\geqslant 1}$ is IID under $\mathbb{Q}$. Find $\mathbb{Q}(\xi_1 = 1+b)$.

Fix a maturity date $N > 0$ and a payout function g. For all $s > 0$ define

$$\begin{aligned} V(N,s) &= g(s), \\ V(n-1,s) &= \frac{q}{1+r}V(n,s(1+b)) + \frac{1-q}{1+r}V(n,s(1+a)) \text{ for all } 1 \leqslant n \leqslant N, \end{aligned}$$

where $q = \mathbb{Q}(\xi_1 = 1+b)$.

(b) Consider the case where $g(s) = \log s$. Find, with justification, a formula for $V(n,s)$ of the form $V(n,s) = A_n \log s + B_n$, for families of constants $(A_n)_{0\leqslant n\leqslant N}$ and $(B_n)_{0\leqslant n\leqslant N}$ which should be specified.

(c) Prove that an investor with initial capital $X_0 = V(0,S_0)$ can replicate the payout of the vanilla European claim with time-N payout $g(S_N)$. How many shares of the stock should the investor hold during the time interval $(n-1,n]$?

Now assume $a \leqslant 0$, and fix a barrier $B > S_0$. For all $s > 0$ define

$$\begin{aligned} U(N,s) &= g(s), \\ U(n-1,s) &= \frac{q}{1+r}U(n,s(1+b))\mathbf{1}_{\{s(1+b)<B\}} + \frac{1-q}{1+r}U(n,s(1+a)) \text{ for all } 1 \leqslant n \leqslant N. \end{aligned}$$

(d) In terms of the function U, derive an explicit formula for the number of shares of the stock an investor should hold during the time interval $(n-1,n]$ in order to replicate the payout of the up-and-out European claim with time-N payout $g(S_N)\mathbf{1}_{\{\max_{0\leqslant n\leqslant N} S_n<B\}}$.

Paper 4, Section II

29L Stochastic Financial Models

Let $(W_t)_{t \geqslant 0}$ be a Brownian motion.

(a) Using the definition of Brownian motion, show that $(W_t)_{t \geqslant 0}$ is a Gaussian process with $\mathbb{E}(W_t) = 0$ and $\mathbb{E}(W_s W_t) = s$ for all $0 \leqslant s \leqslant t$.

(b) Let

$$I_t = \int_0^t W_s \, ds \text{ for all } t \geqslant 0.$$

Show that $\mathbb{E}(I_t) = 0$ and $\mathbb{E}(I_s I_t) = \frac{1}{6} s^2 (3t - s)$ for all $0 \leqslant s \leqslant t$.

Consider a continuous time market with interest rate r and a stock whose time-t price is

$$S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t},$$

for a given volatility parameter $\sigma > 0$.

(c) Show that the discounted stock price $(e^{-rt} S_t)_{t \geqslant 0}$ is a martingale with respect to the filtration generated by the Brownian motion.

(d) Show that the time-0 Black–Scholes price of a European call with strike K and maturity date T is $S_0 \; F(\sigma^2 T, K e^{-rT}/S_0)$, where

$$F(v, m) = \mathbb{E}[(e^{-\frac{1}{2}v + \sqrt{v} Z} - m)^+] \text{ for all } v, m \geqslant 0$$

and Z is a standard normal random variable.

(e) Show that the time-0 Black–Scholes price of a European claim with time-T payout $\left(\exp\left(\frac{1}{T} \int_0^T \log S_t \, dt \right) - K \right)^+$ is

$$S_0 e^{-(\frac{1}{2}r + \alpha\sigma^2)T} F(\tfrac{1}{3}\sigma^2 T, K e^{-(\frac{1}{2}r - \alpha\sigma^2)T}/S_0),$$

for a constant α to be determined. [You may assume that the process $(I_t)_{t \geqslant 0}$ defined in part (b) is Gaussian.]

Paper 1, Section I

2G Topics in Analysis

(a) Define the nth *Chebychev polynomial* T_n. Show that it is indeed a polynomial and that $-1 \leqslant T_n(x) \leqslant 1$ for all $x \in [-1,1]$.

(b) Show that, if $n \geqslant 1$, the leading coefficient of T_n is 2^{n-1}.

(c) Show that $T_n(x) = T_n(-x)$ if n is even, and $T_n(x) = -T_n(-x)$ if n is odd, explaining why these results hold for all $x \in \mathbb{R}$.

(d) By looking at the roots of $T_n^{(k)}(x)$, or otherwise, show that, if $0 \leqslant r \leqslant n-1$, then $T_n^{(r)}(x)$ is increasing for $x \geqslant 1$.

(e) Compute $T_n'(1)$ and show that $T_n(x) \geqslant n(x-1)+1$ for all $x \geqslant 1$.

Paper 2, Section I

2G Topics in Analysis

State Baire's category theorem. Define an *isolated point* for a metric space.

Which of the following statements are true and which are false? Give a proof or a counterexample. (By a metric space we mean a non-empty metric space.)

(i) A countable metric space must have isolated points.

(ii) A complete metric space cannot have isolated points.

(iii) An uncountable metric space without isolated points must be complete.

(iv) A complete metric space must be uncountable.

(v) A countable complete metric space must have isolated points.

(vi) All the points in a countable complete metric space are isolated.

(vii) A countable complete metric space containing at least two points must contain at least two isolated points.

(viii) A countable complete metric space containing infinitely many points must contain infinitely many isolated points.

Paper 3, Section I

2G Topics in Analysis

In this question, a contour Γ will be given by a piecewise smooth map $\gamma : [0,1] \to \mathbb{C}$ that is injective on $[0,1)$ and satisfies $\gamma(0) = \gamma(1)$. You may assume the Cauchy integral formula for a square contour.

If Γ is a contour and $f : \Gamma \to \mathbb{C}$ is continuous, show that

$$F(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(w)}{w - z}\, dw$$

defines a continuous function on $\mathbb{C} \setminus \Gamma$. Give an example to show that, even if f is analytic, F cannot always be extended to a continuous function on $\mathbb{C}$.

Suppose Ω is an open subset of $\mathbb{C}$ and K is a compact non-empty subset of Ω. If $f : \Omega \to \mathbb{C}$ is analytic, show that we can find a finite collection of contours $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ lying in $\Omega \setminus K$ such that

$$f(z) = \sum_{j=1}^{n} \frac{1}{2\pi i} \oint_{\Gamma_j} \frac{f(w)}{w - z}\, dw.$$

Paper 4, Section I

2G Topics in Analysis

(a) Show that the collection of algebraic numbers is countable.

(b) Suppose that $a_j > 0$ and $\sum_{j=1}^{\infty} a_j$ converges. Show that we can find $\theta_j \in \{0, 1\}$ so that

$$\sum_{j=1}^{\infty} \theta_j a_j$$

is transcendental.

(c) Suppose $\theta_j \in \{0, 1\}$ and $\theta_j = 1$ for infinitely many values of j. Show that

$$\sum_{j=1}^{\infty} \frac{\theta_j}{j!}$$

is irrational.

Paper 2, Section II

11G Topics in Analysis

Suppose that a_j, b_j are real and strictly positive for all $j \geqslant 0$, and we have $p_0 = a_0$, $p_{-1} = 1$, $q_0 = 1$, $q_{-1} = 0$ with

$$p_j = a_j p_{j-1} + b_{j-1} p_{j-2}$$
$$q_j = a_j q_{j-1} + b_{j-1} q_{j-2}$$

for $j \geqslant 1$. Show that

$$\frac{p_n}{q_n} = a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{\ddots \; \overline{} \atop a_{n-1} + \cfrac{b_{n-1}}{a_n}}}}}$$

and that

$$\begin{pmatrix} p_n & b_n p_{n-1} \\ q_n & b_n q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & b_0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & b_n \\ 1 & 0 \end{pmatrix}.$$

We now specialise to the case when $b_j = 1$ and the a_j are strictly positive integers for all j. Show that $p_n q_{n-1} - q_n p_{n-1} = (-1)^{n+1}$ for all $n \geqslant 1$.

Show that p_n/q_n tends to a limit x and

$$\left| \frac{p_n}{q_n} - x \right| + \left| \frac{p_{n+1}}{q_{n+1}} - x \right| = \frac{1}{q_n q_{n+1}}$$

for each $n \geqslant 0$.

Now specialise still further to the case when $a_j = 1$ for all j. Show that $p_n = F_{n+2}$, $q_n = F_{n+1}$ for all $n \geqslant 1$ where $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Solve this difference equation to obtain an expression for F_n in terms of powers of $\phi = (1+\sqrt{5})/2$. Hence show that, in this case, the limit x discussed in the previous paragraph is ϕ. Show also that

$$F_n F_{n+1} \left| \frac{F_{n+1}}{F_n} - \phi \right| \to \frac{\phi}{\sqrt{5}} \quad \text{and} \quad F_n F_{n+1} \left| \frac{F_{n+2}}{F_{n+1}} - \phi \right| \to \frac{1}{\phi\sqrt{5}}$$

as $n \to \infty$.

Paper 4, Section II

12G Topics in Analysis

(a) Suppose we have n points A_1, A_2, $\ldots$, A_n, in order, along a straight line with each point coloured red or green. If A_1 is coloured red and A_n green, show that there must be an odd number of segments A_jA_{j+1} with A_j and A_{j+1} of different colours.

(b) Consider an equilateral triangle DEF divided up into a grid of small equilateral triangles by 3 sets of n equidistant lines parallel to each side. We refer to the lines joining neighbouring points of the grid as segments.

(i) Suppose that each vertex of the grid is coloured red, green or blue, that every vertex on DE is coloured red or green, every vertex on EF is coloured green or blue and every vertex on FD is coloured blue or red. Show that there is an odd number of triangles in the grid with vertices of different colours.

(ii) Suppose instead that each vertex of the grid is coloured red, green or blue, and that the three vertices D, E and F have different colours. Must there be a triangle in the grid with vertices of different colours? Give reasons.

(c) Let us write

$$\triangle = \{\lambda_1\mathbf{d} + \lambda_2\mathbf{e} + \lambda_3\mathbf{f} \,:\, \lambda_1 + \lambda_2 + \lambda_3 = 1, \lambda_u \geqslant 0\},$$

where $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ are the position vectors of D, E and F, and write I for the closed line segment DE, J for the closed line segment EF, K for the closed line segment FD.

Which of the following statements are true and which are false? Give a proof or a counterexample.

(i) There does not exist a continuous function $f : \triangle \to \partial\triangle$ with

$$f(\mathbf{x}) \in I \text{ for all } \mathbf{x} \in I,\ \ f(\mathbf{x}) \in J \cup K \text{ for all } \mathbf{x} \in J \cup K.$$

(ii) There does not exist a continuous function $g : \triangle \to \partial\triangle$ with

$$g(\mathbf{x}) \in I \text{ for all } \mathbf{x} \in I,\ \ g(\mathbf{x}) \in J \text{ for all } \mathbf{x} \in J,\ \ g(\mathbf{x}) \in K \text{ for all } \mathbf{x} \in K.$$

Paper 1, Section II

40D Waves

(a) Starting from the linearized mass and momentum conservation equations governing sufficiently small and smooth perturbations of a compressible homentropic inviscid fluid at rest with constant reference density ρ_0, pressure p_0 and sound speed c_0, show that the pressure perturbation $\tilde{p}$ satisfies a wave equation. How is $\tilde{p}$ related to the velocity potential ϕ?

(b) Consider a semi-infinite straight duct of uniform cross-section, aligned along the $x-$axis for $-L \leqslant x < \infty$. There is a piston at the end of the duct which performs oscillations $\epsilon e^{i\omega t}$ about its equilibrium position at $x = -L$. The duct is filled with compressible fluid of density ρ_- and sound speed c_- in the region $-L < x < 0$ and with compressible fluid of density ρ_+ and sound speed c_+ in the region $0 < x < \infty$. The piston's oscillations are sufficiently small so that you may assume $0 < \epsilon \ll L$ and $|\epsilon\omega| \ll \min(c_-, c_+)$.

(i) Show that the complex amplitude of the velocity potential in $x > 0$ is given by

$$\epsilon c_- \frac{i\frac{\rho_+}{\rho_-}\sin\lambda - \frac{c_-}{c_+}\cos\lambda}{\left(\frac{\rho_+}{\rho_-}\right)^2 \sin^2\lambda + \left(\frac{c_-}{c_+}\right)^2 \cos^2\lambda}, \qquad \text{where} \qquad \lambda = \frac{\omega L}{c_-}.$$

(ii) Consider the two sets of frequencies of oscillation such that $\lambda = n\pi$ and $\lambda = (n + \frac{1}{2})\pi$ for integer n. Calculate the time-averaged acoustic energy flux in $x > 0$ for each set, and briefly comment on the behaviour in the case where $\rho_+ \ll \rho_-$ and $c_+ \approx c_-$.

Paper 2, Section II

40D Waves

The linearized Cauchy momentum equation governing small and smooth displacements $\boldsymbol{u}(\boldsymbol{x}, t)$ in a uniform, linear isotropic elastic solid of density ρ is

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = (\lambda + \mu) \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + \mu \nabla^2 \boldsymbol{u},$$

where the constants λ and μ are the Lamé moduli.

(a) Show that this equation supports two distinct classes of wave-like motion: P-waves for the dilatation $\vartheta = \boldsymbol{\nabla} \cdot \boldsymbol{u}$ with phase speed c_p; and S-waves for the rotation $\boldsymbol{\Omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$ with phase speed c_s. You should express c_p and c_s explicitly in terms of the Lamé moduli.

(b) Now consider a region of this solid with a horizontal plane boundary at $z = 0$ in which plane waves propagate with wave vector $\boldsymbol{k} = \kappa(\sin\theta, 0, \cos\theta)$, i.e. θ is the angle the wave vector makes with the vertical $z-$direction and κ is the magnitude of the wave vector. Explain briefly why such a domain can in general support:

(i) harmonic P-waves: $\boldsymbol{u} = \boldsymbol{A} \exp[i\kappa(x \sin\theta + z \cos\theta) - i\omega t]$;

(ii) harmonic SV-waves: $\boldsymbol{u} = \boldsymbol{B}_V \exp[i\kappa(x \sin\theta + z \cos\theta) - i\omega t]$;

(iii) and harmonic SH-waves: $\boldsymbol{u} = \boldsymbol{B}_H \exp[i\kappa(x \sin\theta + z \cos\theta) - i\omega t]$.

You should define explicitly the orientations of the complex vector amplitudes $\boldsymbol{A}$, $\boldsymbol{B}_V$ and $\boldsymbol{B}_H$.

(c) Now consider a region of the solid between a rigid plane boundary at $z = 0$ and a free surface at $z = h > 0$.

(i) Show that this region can support propagating SH-waves with wave vector in the $x-$direction (i.e. with $\theta = \pi/2$), calculating explicitly the dispersion relation.

(ii) Deduce that there is a cut-off frequency ω_n for each mode in the vertical, given by

$$\omega_n = \frac{(2n+1)\pi}{2h} c_s\,,$$

for non-negative integers $n = 0, 1, 2 \ldots$.

(iii) Express the phase velocity c and the group velocity c_g of each mode in terms of c_s, ω_n and κ.

(iv) Deduce that, for any given wave number $\kappa > 0$ and mode with $n \geqslant 0$, $c = m c_g$ with $m > 1$, where you should express m in terms of n, κ and h.

(v) Calculate m explicitly for the specific wave with horizontal wavelength h and $n = 1$.

[*Hint: You may find it useful to recall that* $\nabla^2 \boldsymbol{q} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{q}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{q})$.]

Paper 3, Section II

39D Waves

Let $\boldsymbol{u} = (u, 0, w)$ denote the components of a solenoidal two-dimensional velocity field with $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$. Consider small-amplitude vertical-velocity perturbations $w(x, z, t)$ about a state of rest in a two-dimensional stratified fluid of sufficiently slowly varying background density $\rho_0(z)$ such that

$$\left| \frac{1}{\rho_0} \frac{\mathrm{d}\rho_0}{\mathrm{d}z} \frac{\partial w}{\partial z} \right| \ll \left| \frac{\partial^2 w}{\partial z^2} \right| .$$

(a) Show that $w(x, z, t)$ satisfies the equation

$$\nabla^2 \left(\frac{\partial^2 w}{\partial t^2} \right) + N^2(z) \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{where} \quad N^2(z) = -\frac{g}{\rho_0} \frac{\mathrm{d}\rho_0}{\mathrm{d}z} ,$$

and N is the buyoancy frequency.

(b) Consider a semi-infinite region of fluid with constant buoyancy frequency N, flowing with uniform velocity $U > 0$ in the $x-$direction and bounded below by a sinusoidal range of hills at $h = h_0 \sin kx$, where $k > 0$.

(i) Determine the frequency of the motion induced in a frame of reference in which the background fluid is stationary.

(ii) You may assume that the vertical component of the fluid velocity at $z = 0$ is given by the vertical component of the velocity of fluid particles that follow the undulations of the lower boundary, i.e. $w = U \partial h / \partial x$ at $z = 0$. Hence establish that for $z > 0$, the vertical velocity satisfies

$$w = w_0 \exp[i(kx + mz - \omega t)],$$

where $w_0 = Ukh_0$, and m satisfies a dispersion relation which you should express in terms of N, U and k.

(iii) The time- and horizontally-averaged vertical wave energy flux at $z = 0$ is given by $\langle \overline{I_z} \rangle_0 = \langle \overline{\tilde{p} w} \rangle_0$, where $\tilde{p}$ is the pressure perturbation, the angle brackets denote an average over the wave period and the overline denotes an average over the hill wavelength. Calculate $\langle \overline{I_z} \rangle_0$ for sufficiently large wavelengths $k \leqslant k_c = N/U$ making clear the reasoning for selecting the sign of all terms.

(iv) Show that $\langle \overline{I_z} \rangle_0 = 0$ for sufficiently short wavelengths $k > k_c$.

(v) Give a brief physical interpretation of the cut-off wavenumber $k_c = N/U$.

Paper 4, Section II

39D Waves

Consider one-dimensional homentropic flow in an ideal gas with ratio of specific heats γ and equilibrium sound speed c_0.

(a) Starting from the conservation equations for mass and momentum, show that the Riemann invariants $R_\pm$, defined as

$$R_\pm = u \pm 2\frac{(c-c_0)}{(\gamma-1)},$$

are constant on characteristics $C_\pm$ given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u \pm c,$$

where u is the fluid velocity, and $c = \sqrt{\mathrm{d}p/\mathrm{d}\rho}$ is the associated sound speed.

(b) Consider a semi-infinite tube filled with such an ideal gas in the region $x > X(t)$ to the right of a piston at $x = X(t)$. At time $t = 0$, the piston and the gas are at rest, $X = 0$, and the gas is uniform with $c = c_0$. For $t > 0$, the piston accelerates smoothly in the positive $x-$direction.

(i) Show that, prior to the formation of a shock, the motion of the gas can be written in terms of a parameter τ by

$$u(x,t) = \dot{X}(\tau) \quad \text{on} \quad x = X(\tau) + \left[c_0 + \frac{1}{2}(\gamma+1)\dot{X}(\tau)\right](t-\tau),$$

in a region which you should specify carefully.

(ii) Suppose $X(t)$ is given by

$$X(t) = \frac{c_0 t^3}{T^2},$$

where T is a positive constant. Show that a shock first forms in the gas when $t/T = f(\gamma)$, for some function $f(\gamma)$ which you should determine.

END OF PAPER