UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA 2024

List of Courses

Paper 1, Section I

3F Analysis I

Let (a_n) and (b_n) be two sequences of positive real numbers such that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge.

(a) Show that the series $\sum_{n=1}^{\infty} a_n^2$ converges.

(b) Show that the series $\sum_{n=1}^{\infty} \sqrt{a_n b_n}$ converges.

(c) Show that the series $\sum_{n=1}^{\infty} \sqrt{a_n}\, n^{-p}$ converges if $p > 1/2$. Give an example to show that this series need not converge for $p = 1/2$.

[*You may use any results from the course provided you state them clearly.*]

Paper 1, Section I

4D Analysis I

(a) Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients. Show that there exists $R \in [0, \infty]$, the 'radius of convergence' of the series, such that the series is convergent when $|z| < R$ and divergent when $|z| > R$.

(b) Now suppose that the a_n are real and positive, with $a_n \to \infty$. Can it happen that the two power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} a_n^2 z^n$ have the same finite non-zero radius of convergence? What about the two power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} a_n^{a_n} z^n$?

Paper 1, Section II

9F Analysis I

(i) State and prove the ratio test about the convergence of series.

(ii) Let (a_n) be a sequence of positive real numbers. If $a_{n+1}/a_n \to L$ for some $L \in \mathbb{R}$, show that $a_n^{1/n} \to L$.

(iii) Show that $\sum_{n=1}^{\infty} n!/n^n$ converges.

(iv) Compute $\lim_{n\to\infty} n/(n!)^{1/n}$.

Paper 1, Section II

10E Analysis

Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ and a is a real number.

(a) What does it mean to say that f is *differentiable* at a? If f is differentiable at a what is the value of $f'(a)$, the derivative of f at a?

(b) Let $f \cdot g\colon \mathbb{R} \to \mathbb{R}$ be the pointwise product of f and g:

$$(f \cdot g)(x) = f(x)g(x) \text{ for all } x \in \mathbb{R}.$$

(i) Suppose that f and g are both differentiable at a. Show that $f \cdot g$ is differentiable at a. What is the value of $(f \cdot g)'(a)$ in terms of $f(a), f'(a), g(a)$ and $g'(a)$?

(ii) Suppose that f and $f \cdot g$ are both differentiable at a with $(f \cdot g)'(a) \neq 0$. Must g be differentiable at a? Justify your answer.

(c) (i) Suppose now that f is differentiable at a and g is differentiable at $f(a)$. Show that the composite $g \circ f$ is differentiable at a. What is the relationship between $(g \circ f)'(a)$, $g'(f(a))$ and $f'(a)$?

(ii) Suppose that g is differentiable at $f(a)$ and $g \circ f$ is differentiable at a with $(g \circ f)'(a) \neq 0$. Must f be differentiable at a? Justify your answer.

(iii) What does it mean to say that f is *twice differentiable* at a? Suppose now that f is twice differentiable at a and g is twice differentiable at $f(a)$. Show that $g \circ f$ is twice differentiable at a.

Paper 1, Section II

11E Analysis

State and prove the intermediate value theorem.

(a) Suppose that $f\colon \mathbb{C} \to \mathbb{R}$ is a continuous function that takes both positive and negative values. Show that there exists $z \in \mathbb{C}$ such that $f(z) = 0$.

(b) Let n be a positive integer and let a be a positive real number. Suppose that $g\colon [0, na] \to \mathbb{R}$ is a continuous function such that

$$g(na) = g(0) + na.$$

Show that there exists $x \in [0, (n-1)a]$ such that $g(x + a) = g(x) + a$.

Paper 1, Section II

12D Analysis I

(a) (i) Let $f : [0,1] \to \mathbb{R}$ be a bounded function. Define the *upper* and *lower integrals* of f, and explain what it means for f to be *Riemann integrable.*

(ii) Show that every continuous function is Riemann integrable.

(b) Let $f : [0,1] \to \mathbb{R}$ be a non-negative function that is unbounded. We say that f is *improperly integrable* with integral I (where I is a real number) if, for each positive real r, the function $f_r(x) = \min(f(x), r)$ is Riemann integrable, with $\int f_r(x)dx \to I$ as $r \to \infty$.

(i) If f is continuous at all points of $(0,1)$, must f be improperly integrable?

(ii) If there exists a countable set $A \subset [0,1]$ such that f is bounded on the set $[0,1] \setminus A$, must f be improperly integrable?

(iii) Given $x \in [0,1]$, we write x in decimal as $0.a_1a_2a_3\ldots$ (choosing say the terminating-in-9s form in case of ambiguity), and we set $f(x) = k$ if a_k is the first digit that is 5 and $f(x) = 0$ if no digit is 5. Prove that f is improperly integrable. What is the integral of f?

Paper 2, Section I

1A Differential Equations

(a) Find all solutions of the differential equation for $y(x)$

$$xyy'' - xy'^2 = yy'.$$

[Hint: you may find the substitution $z(x) = y'(x)/y(x)$ helpful.]

(b) For $n \neq 0, 1$, show that the substitution $z = y^{1-n}$ transforms the differential equation for $y(x)$

$$y' + P(x)y = Q(x)y^n$$

into a linear differential equation that you should state explicitly.

Hence, or otherwise, solve the differential equation for $x(t) > 0$

$$\frac{1}{\sqrt{x}}\dot{x} = 2te^{-t^3} - 6t^2\sqrt{x}$$

subject to the condition $x(1) = 4$.

Paper 2, Section I

2A Differential Equations

A real-valued function $f(x)$ is differentiable on some interval $(-a, a)$. For any $x, y \in (-a, a)$ such that $x + y \in (-a, a)$, the equality

$$f(x+y) = \frac{f(x) + f(y)}{1 - f(x)f(y)}$$

holds.

(i) Show that $f(0) = 0$.

(ii) By considering the definition of the derivative as a limit, or otherwise, show that there exists a number C such that $f'(x) = C(1 + f^2(x))$ everywhere on the interval $(-a, a)$.

(iii) Hence find the most general form of $f(x)$. Also, find $f(x)$ that satisfies $f'(0) = 2$.

Paper 2, Section II

5A Differential Equations

Let $\varphi_1(x)$ and $\varphi_2(x)$ be non-trivial solutions of the equations

$$\varphi_1'' + q_1(x)\varphi_1 = 0$$

and

$$\varphi_2'' + q_2(x)\varphi_2 = 0,$$

where $q_1(x)$ and $q_2(x)$ are continuous functions such that $q_1(x) \leqslant q_2(x)$ for all x.

(i) Let x_1 and x_2 with $x_1 < x_2$ be consecutive zeroes of φ_1. By considering

$$\int_{x_1}^{x_2} (q_1(x) - q_2(x))\varphi_1(x)\varphi_2(x)\, dx$$

or otherwise, show that if both $\varphi_1(x)$ and $\varphi_2(x)$ are strictly positive on (x_1, x_2) then $q_1(x) \equiv q_2(x)$ on (x_1, x_2).

(ii) Hence prove that between any two consecutive zeroes x_1 and x_2 of $\varphi_1(x)$, there exists at least one zero of $\varphi_2(x)$, unless $q_1(x) \equiv q_2(x)$ on (x_1, x_2).

(iii) Hence show that any solution of the equation

$$y'' + (2 + \cos 3x)y = 0$$

has at least one zero on the interval $[-1, \pi - 1]$.

(iv) Show that each non-trivial solution of the equation

$$\sqrt{1 + x^3}\, y'' + y = 0$$

has at most one zero on the interval $[2, 6]$.

Paper 2, Section II

6A Differential Equations

Define the *generating function* $G(x)$ for a difference equation $F(u_n, u_{n-1}, \ldots, u_0) = 0$ as

$$G(x) = u_0 + u_1 x + u_2 x^2 + \cdots .$$

(a) Consider the difference equation $u_n + u_{n-1} - 6u_{n-2} = n$ for $n \geqslant 2$. Find the solution of this equation, given $u_0 = 0, u_1 = 2$.

Show that

$$(1 + x - 6x^2)G(x) = x + \frac{x}{(1-x)^2}$$

and use this expression to find the power series expansion of $G(x)$. Verify that this expansion is consistent with the u_n determined directly above.

[Hint: it may be helpful to note that $1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}$.]

(b) Find the generating function $G(x)$ for the difference equation

$$u_n - 2u_{n-1} = \left\lfloor \frac{n}{2} \right\rfloor, \quad n \geqslant 1, \quad u_0 = 1,$$

where $\lfloor \frac{n}{2} \rfloor$ is the greatest integer less than or equal to $\frac{n}{2}$. Hence solve this equation.

Paper 2, Section II

7A Differential Equations

Consider the following system of equations involving two functions, $x(t)$ and $y(t)$:

$$\begin{aligned} \dot{x} &= y + kx(x^2 + y^2), \\ \dot{y} &= -x + ky(x^2 + y^2), \end{aligned}$$

where k is a constant.

(i) Show that there exists a function $F(x(t), y(t))$ (which you should state explicitly in terms of $x(t)$ and $y(t)$) such that

$$\frac{dF}{dt} = 2kF^2.$$

Solve this equation assuming that $F = 1$ at $t = 0$.

(ii) Find the equilibrium point of this system and show that the linearised system has a centre at this point. Taking into account the nonlinear terms, deduce for which values of k this equilibrium point is stable, and why. Do the trajectories rotate clockwise or anticlockwise as t increases, and why?

(iii) By changing variables to polar coordinates, via $x(t) = r(t)\cos\theta(t)$ and $y(t) = r(t)\sin\theta(t)$, find $f(r)$ and $g(\theta)$ such that

$$\begin{aligned} \dot{r} &= f(r), \\ \dot{\theta} &= g(\theta). \end{aligned}$$

Integrate these equations to find $r(t)$ and $\theta(t)$ if $r(0) = 1$ and $\theta(0) = 0$.

(iv) Now the system is modified to:

$$\begin{aligned} \dot{x} &= y + x - 2x(x^2 + y^2), \\ \dot{y} &= -x + y - 2y(x^2 + y^2). \end{aligned}$$

At $t = 0$, the system is at a point on the circle $x^2 + y^2 = 1$. Determine $x^2 + y^2$ as a function of t. Find $\lim_{t\to\infty}(x^2 + y^2)$.

Paper 2, Section II

8A Differential Equations

The dynamics of a gas is described by a partial differential equation for the complex function $\psi(x,t) = \sqrt{\rho(x,t)}\exp[iS(x,t)]$ as

$$-2i\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} + (1 - |\psi|^2)\psi. \qquad (*)$$

Let $v(x,t) = \partial S/\partial x$.

(i) Determine the real-valued equations describing the gas dynamics in terms of ρ and v as

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= \frac{\partial A}{\partial x},\\ \frac{\partial v}{\partial t} &= \frac{\partial B}{\partial x} + \frac{\partial C}{\partial x},\end{aligned}$$

where you should specify the functions A that depends only on ρ and v, B that depends only on ρ and its derivatives in x, and C that depends only on v.

(ii) Write down the ordinary differential equation for $a(x) = \sqrt{\rho(x,t)}$ in the case of a stationary gas ($S =$ constant). What is the constant solution $a(x) = d > 0$ of this equation?

(iii) There are solutions of $(*)$ of the form $\psi(x,t) = \psi_0(\xi)$ where $\xi = x - Ut$ with U a constant satisfying $0 \leqslant U < \frac{1}{\sqrt{2}}$. Determine the ordinary differential equation that $\psi_0(\xi)$ satisfies if it is known that $\mathrm{Im}(\psi_0(\xi)) = \sqrt{2}U$ for all ξ and $|\psi_0(\xi)| \to d$ as $\xi \to \pm\infty$.

(iv) Plot $|\psi_0(\xi)|^2$ for the solutions when $U = 0$ and when $U = 1/2$ as a function of ξ, and discuss how these solutions evolve in time.

Paper 4, Section I

3C Dynamics and Relativity

(a) State the *parallel axis theorem* for a rigid body. Now consider rigid bodies A and B and let C denote the rigid body obtained by connecting A to B with a massless rod. For $T = A, B, C$, denote by M_T the mass, by $\mathbf{x}_T$ the centre of mass and by $I_T(\mathbf{x})$ the moment of inertia about an axis passing through $\mathbf{x}$ in a given fixed direction $\hat{\mathbf{d}}$. Derive an expression for M_B in terms of $I_A(\mathbf{x}_A)$, $I_B(\mathbf{x}_B)$, $I_C(\mathbf{x}_C)$, $\mathbf{x}_A$, $\mathbf{x}_B$, $\mathbf{x}_C$, and M_C.

(b) For a two-dimensional rigid body lying in the (x, y) plane, prove the *perpendicular axis theorem*, namely the relation $I_z = I_x + I_y$, where I_x, I_y and I_z are the moments of inertia about the x, y and z axes. Using this relation or otherwise, compute I_x, I_y and I_z for a uniform-density ellipse of mass M and semi-major and semi-minor axes a and b.

Paper 4, Section I

4C Dynamics and Relativity

Consider three distinct events A, B and C in 1+1 spacetime dimensions.

(a) For each of the following statements provide an explicit algebraic proof that they always hold or a counterexample:

(i) If B is in the future lightcone of A and C is in the future lightcone of B then C is in the future lightcone of A.

(ii) If B is in the future lightcone of A and C is in the future lightcone of A then C is in the future lightcone of B.

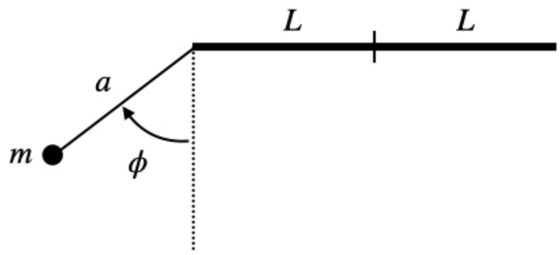
(iii) If B is in the future lightcone of A and C is in the past lightcone of A then B is in the future lightcone of C.

(iv) If A and B are lightlike separated and A and C are lightlike separated then B and C are lightlike separated.

(b) Assume that B is in the future lightcone of A, that C is in the future lightcone of B and that $|x_C - x_A| < |x_C - x_B|$ in some inertial frame S. Prove that there exists an inertial frame S' in which $|x'_C - x'_A| > |x'_C - x'_B|$.

Paper 4, Section II

9C Dynamics and Relativity



(a) Consider a rigid rod of length $2L$ spinning on the horizontal plane about its centre at a constant angular velocity $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is an upward unit vector. A second rod of length a, which is light, connects one end of the first rod to a particle of mass m. The rods lie in the same vertical plane and the second rod makes an angle ϕ with the downward vertical direction $-\hat{\mathbf{z}}$, as in the figure. Compute the constant angular velocity ω for which ϕ takes a constant value ϕ_1. Check that your result is dimensionally correct, and briefly discuss the dependence of the result on m and L.

(b) The second rod is now substituted by a spring of rest length a, which at displacement from rest x has potential energy $V = (k/2)x^2$. Assuming again that ϕ takes a constant value ϕ_2, solve for x and hence compute the constant angular velocity ω in terms of ϕ_2 and the parameters of the problem. Contrast this result with the one obtained in part (a).

Paper 4, Section II

10C Dynamics and Relativity

(i) Consider a particle of mass m moving in a potential V. State which standard quantity is conserved for a central potential $V = V(r)$. Hence, starting from the radial equation of motion in polar coordinates,

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{dV}{dr}\,,$$

derive the orbit equation for the variable $u = 1/r$.

(ii) For the potential $V = -km/r$, where k is a positive constant, solve the orbit equation and show that the integration constants can be chosen such that the orbit satisfies the equation of an ellipse in Cartesian coordinates.

(iii) Evaluate the orbit equation for the case of a potential $V = -km/(r - r_0)$, where k and r_0 are positive constants and $r > r_0$. Discuss the existence of circular orbits and their stability.

(iv) Expand the above orbit equation up to first order in small r_0. Show that this expanded equation has a solution of the form

$$u(\theta) = X\left[1 + A\cos(\omega\theta)\right]\,,$$

for any $0 < A < 1$, where ω and X are parameters you should determine. Hence compute the angle between two successive periapses.

Paper 4, Section II

11C Dynamics and Relativity

(a) For a collection of particles of masses m_i and positions $\mathbf{x}_i$, define the *centre of mass* $\mathbf{R}$, the *total momentum* $\mathbf{P}$ and the *total angular momentum* $\mathbf{L}$ about the origin. Prove that

$$\mathbf{L}_{\mathrm{CoM}} = \mathbf{L} - \mathbf{R} \times \mathbf{P}\,,$$

where $\mathbf{L}_{\mathrm{CoM}}$ is the total angular momentum about the centre of mass.

(b) (i) At some initial time $t = 0$, particles with a total mass of m form a circle of radius R in the (x, y) plane at $z = 0$ and have a velocity v tangential to the circle. Compute the moment of inertia I of the system.

(ii) At later times, the particles will form a circle of radius $R(t)$. Compute $R(t)$ and the total angular momentum with respect to the origin for all times, and hence deduce the angle $\theta(t)$ by which the circle has rotated at time t.

(c) (i) A disc of radius R and negligible mass is filled uniformly with a mass m_0 of water. Compute the moment of inertia I of the disc.

(ii) As the disc spins, with angular velocity $\omega(t)$, from each point of the boundary of the disc water is sprayed out in the tangential direction to the boundary at a relative velocity u. By mimicking the derivation of the rocket equation, or otherwise, derive a first order differential equation in time for the angular velocity $\omega(t)$ of the disc that depends on R, u and the mass of water $m(t)$ that remains inside the disc at time t. [You may assume that the remaining water $m(t)$ is always uniformly distributed inside the disc.]

(d) Assume now that water is leaking out at zero relative velocity, $u = 0$, and at a constant rate $\dot{m}(t) = -\mu$. Compute the time it takes for the disc to stop spinning from an initial angular velocity ω_0.

Paper 4, Section II

12C Dynamics and Relativity

(a) Write down the relativistic four-momenta of a massive and a massless particle. For the decay of particle 1 into particles 2 and 3 (all massive), where particle i has mass m_i and energy E_i, use relativistic invariants to compute all energies in the frame where $E_1 = m_1 c^2$. Derive the condition on the masses for the decay to be kinematically allowed.

(b) Now consider a particle Q of mass m and $N+1$ different types of particle, $R_0, \ldots, R_N$, of masses $(nM + m_0)$, for $n = 0, 1, \ldots, N$ respectively, where $m_0 < m$ and $M = \lambda m$ with $\lambda > 1$. For each $n > 0$, particle R_n decays into particle Q and particle R_{n-1}, as long as this is kinematically allowed. If we start with one R_N particle, what is the end product after all allowed decays have happened, and how much rest mass has been converted into energy?

(c) Consider the relativistic scattering of particles A and B into C and D. Write down the relativistic kinematic invariants that can be written as bilinear functions of the 4-momenta P_a^μ for $a = A, B, C, D$. Express each kinematic invariant in terms of the masses m_a and of the following variables

$$s = (P_A + P_B)^2 , \qquad t = (P_A - P_C)^2 , \qquad u = (P_A - P_D)^2 .$$

Show that the sum $s + t + u$ depends only on a certain combination of the masses.

Paper 3, Section I

1D Groups

Prove that every Möbius map sends circles and straight lines to circles and straight lines. [You may use any statement from the course about generating sets for the group of Möbius maps, provided that you state it precisely.]

Is the subgroup of the Möbius maps consisting of those that send circles to circles a normal subgroup?

Paper 3, Section I

2D Groups

(i) Define the groups $SO(n)$. Prove that every element of $SO(2)$ is a rotation about the origin, and that every element of $SO(3)$ is a rotation about some axis. [You may assume simple facts about orthogonal maps and rotations and reflections, but if you wish to quote any statement of the form 'these elements generate this group' then you must prove it.]

(ii) Explain why $SO(2)$ has a subgroup isomorphic to $\mathbb{Z}$. Does it have a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$? Justify your answer.

Paper 3, Section II

5D Groups

(i) Define the *sign* of a permutation $\sigma \in S_n$, explaining why it is well-defined. Show also that it gives a homomorphism from S_n to $\{\pm 1\}$.

(ii) Prove that A_5 is simple. [You may assume facts about conjugacy classes in A_5, provided that you state them precisely.]

(iii) Show that there is no surjective homomorphism from A_5 to $\{\pm 1\}$.

(iv) Is there a surjective homomorphism from $A_5 \times A_5$ to $\{\pm 1\}$? Justify your answer.

Paper 3, Section II

6D Groups

(i) A group G is called *n-dicyclic*, where $n > 1$, if it is generated by elements a and b such that a has order $2n$ and $b^2 = a^n$ and $bab^{-1} = a^{-1}$. Prove that such a group must have order $4n$. Show that, for each n, there exists an n-dicyclic group. [Hint: find it as a subgroup of $GL_2(\mathbb{C})$, choosing a as a suitable diagonal matrix.]

(ii) Find 5 pairwise non-isomorphic groups of order 12, explaining carefully why your groups are non-isomorphic.

Paper 3, Section II

7D Groups

(i) State and prove Cayley's theorem.

(ii) State and prove Cauchy's theorem.

(iii) For each $n > 1$ that is a prime power, exhibit a group G of order n such that G is not a subgroup of S_{n-1}.

(iv) Let G be a finite group of order n, where $n > 1$ is *not* a prime power. Show that G is a subgroup of S_{n-1}. [Hint: for two suitable subgroups H and K of G, consider the standard actions of G on the left cosets of H and of K.]

Paper 3, Section II

8D Groups

(a) State and prove the orbit-stabiliser theorem for a finite group.

(b) (i) Let G be the group of all isometries of a cube in $\mathbb{R}^3$. By considering the action of G on the vertices of the cube, show that $|G| = 48$. To which standard group is the stabiliser of a vertex isomorphic?

(ii) Now consider the action of G on the edges of the cube. How large is the stabiliser of an edge? To which standard group is it isomorphic?

(iii) Now consider the action of G on the main diagonals of the cube. How large is the stabiliser of a main diagonal? To which standard group is it isomorphic? Is this action faithful?

(iv) Are every two elements of G of order 3 conjugate? Are every two elements of G of order 2 conjugate? Justify your answers.

[*Throughout this question you may assume standard properties of isometries, rotations, reflections, etc.*]

Paper 4, Section I

1E Numbers and Sets

State and prove Wilson's theorem. State Fermat's little theorem.

Calculate the residue of $28! \cdot 7^{29}$ modulo 31.

Paper 4, Section I

2E Numbers and Sets

What is a *relation* on a set S? What is an *equivalence relation*?

For each of the following relations determine whether or not $\sim$ defines an equivalence relation on $\mathbb{R}$:

(i) $x \sim y$ if and only if $xy > 0$;

(ii) $x \sim y$ if and only if $xy \geqslant 0$;

(iii) $x \sim y$ if and only if $x - y \in \mathbb{Z}$;

(iv) $x \sim y$ if and only if $x - y \geqslant 0$.

For any cases (i)-(iv) where $\sim$ is an equivalence relation, describe the set of equivalence classes.

Paper 4, Section II

5E Numbers and Sets

State and prove the Fermat–Euler theorem.

For a $(k+1)$-digit positive integer written in base 10,

$$n = a_k a_{k-1} \cdots a_0$$

with each $a_i \in \{0, 1, 2, \ldots, 9\}$ and $a_k \neq 0$, a *cyclic permutation of* n is a number of the form

$$c_j(n) = a_{k-j} a_{k-j-1} \cdots a_0 a_k \cdots a_{k-j+1} \text{ with } j = 0, \ldots, k.$$

For example

$$c_0(120) = 120, c_1(120) = 201 \text{ and } c_2(120) = 012 = 12.$$

For each natural number $d > 1$, a number n is *d-cyclic-divisible* if every cyclic permutation of n is a multiple of d.

(i) Show that every multiple of 3 is 3-cyclic-divisible.

(ii) Show that if $d = 2$ or 5, n is d-cyclic-divisible if and only if every digit of n is a multiple of d.

(iii) Show that if n is 7-cyclic-divisible then either all its digits are equal to 7 or its number of digits is a multiple of 6.

(iv) Suppose that $p > 7$ is a prime. Show that $p - 1$ has a factor l such that, if the number of digits of n is a multiple of l, then n is p-cyclic-divisible if and only if n is a multiple of p.

Paper 4, Section II

6E Numbers and Sets

The *Fibonacci numbers* are defined for all non-negative integers by

$$F_0 = 0,\ \ F_1 = 1,\ \ F_{n+1} = F_n + F_{n-1} \text{ for all } n \geqslant 1.$$

Prove the following properties of the Fibonacci numbers by induction:

(i) $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ for all $n \geqslant 1$;

(ii) $F_{n+l}F_{n+m} - F_nF_{n+m+l} = (-1)^n F_m F_l$ for all $n, m, l \geqslant 0$.

Deduce that if $j \geqslant k \geqslant 0$ then

$$(F_{j+k} - F_{j-k})(F_{j+k} + F_{j-k}) = F_{2k}F_{2j}$$

and

$$F_{j+k+1}^2 + F_{j-k}^2 = F_{2k+1}F_{2j+1}.$$

Paper 4, Section II

7E Numbers and Sets

Given a non-empty bounded subset S of $\mathbb{R}$, what is its *supremum* $\sup S$?

What does it mean to say that a sequence (x_n) of real numbers *converges*?

(a) (i) Show that an increasing sequence of real numbers converges if and only if it is bounded.

(ii) Does the sequence $\left(12 - \frac{1000n^2+4}{1.1^n}\right)$ converge? Carefully justify your answer from first principles.

(b) Suppose that S and T are non-empty bounded subsets of $\mathbb{R}$.

(i) If $S + T = \{s + t : s \in S, t \in T\}$, then must $\sup(S + T) = \sup S + \sup T$?

(ii) If $ST = \{st : s \in S, t \in T\}$, then must $\sup ST = (\sup S)(\sup T)$?

(iii) If all the elements of S are positive and $S^T = \{s^t : s \in S, t \in T\}$, then must $\sup\left(S^T\right) = (\sup S)^{(\sup T)}$?

Paper 4, Section II

8E Numbers and Sets

(a) What does it mean to say a set is *countable*?

Show that a countable union of countable sets is countable. Deduce that $\mathbb{Q}$ is countable. Show that if A and B are countable then $A \times B$ is countable.

Show that $\mathbb{R}$ is not countable.

(b) A *line* in $\mathbb{R}^2$ is a subset of the form

$$\{(x, y) \in \mathbb{R}^2 : ax + by + c = 0\}$$

for some $a, b, c \in \mathbb{R}$ with $(a, b) \neq (0, 0)$.

(i) Must a collection of lines whose intersection is non-empty be countable? Justify your answer carefully.

(ii) Must a collection of lines each of which contains at least two distinct elements of $\mathbb{Q}^2$ be countable? Justify your answer carefully.

Paper 2, Section I

3F Probability

(a) State Markov's inequality. Prove that for any random variable X and any $t > 0$,

$$\mathbb{P}(X \geqslant x) \leqslant e^{-tx} M_X(t)\,,$$

where $M_X(t) = \mathbb{E}(e^{tX})$ is the moment generating function of X.

(b) Let $X_1, X_2, \ldots, X_n$ be i.i.d. Poisson random variables with mean 1. Let $S = X_1 + \cdots + X_n$.

(i) Compute the moment generating function of S. Find the distribution of S.

(ii) Prove that

$$\mathbb{P}(S \geqslant 2n) \leqslant (e/4)^n\,.$$

[*You may use the fact that the moment generating function $M_X(t)$ of a Poisson random variable X with mean λ is $e^{\lambda(e^t-1)}$.*]

Paper 2, Section I

4F Probability

Let (X_1, X_2) have a bivariate normal distribution with $\mathbb{E}(X_i) = \mu_i, \mathrm{var}(X_i) = \sigma_i^2$ for $i = 1, 2$ and $\mathrm{corr}(X_1, X_2) = \rho$.

(a) Write down the joint probability density function of (X_1, X_2).

(b) Find the conditional probability density function of $X_1 | X_2$.

(c) If $\sigma_1 = \sigma_2 = \sigma$, show that $X_1 + X_2$ and $X_1 - X_2$ are independent random variables. Find their distributions.

Paper 2, Section II

9F Probability

Let $S_1, S_2, \ldots$ be independent exponential random variables with means $\mathbb{E}(S_i) = 1/q_i$ for $i = 1, 2, \ldots$. Let $T = \min\{S_1, S_2, \ldots, S_n\}$ and let K be the value of i for which $S_i = T$.

(i) Find $\mathbb{P}(K = k, T \geqslant t)$ for $k \in \{1, 2, \ldots, n\}, t \geqslant 0$.

(ii) Find the distributions of the random variables K and T. Show that K and T are independent.

Now assume that $q_i = 1$ for all $i = 1, 2, \ldots$.

(iii) Show that for all $n \geqslant 1$, the probability density function of $X_n = \sum_{i=1}^{n} S_i$ is given by

$$f(x) = \frac{x^{n-1}}{(n-1)!} e^{-x}, \quad x > 0 .$$

(iv) Let N be a geometric random variable independent of the sequence $S_1, S_2, \ldots$, with $\mathbb{P}(N = n) = p(1-p)^{n-1}$ for $n = 1, 2, \ldots$. Define

$$Y = \sum_{i=1}^{N} S_i .$$

Find $\mathbb{E}(e^{\theta Y})$ for $\theta < p$. Hence or otherwise, find the distribution of Y.

[*You may use the fact that the moment generating function $M_X(t)$ of an exponential random variable X with mean $1/\lambda$ is $\lambda/(\lambda - t)$ for $t < \lambda$.*]

Paper 2, Section II

10F Probability

A fair n-sided die is rolled repeatedly so that each roll is independent. We say a *match* occurs if the face i appears on the i-th roll.

(i) Find the probability p_n that at least one match occurs in the first n rolls. What is the value of $\lim_{n\to\infty} p_n$?

Now let T_n be the minimum number of rolls required until all the n faces have appeared at least once.

(ii) Show that T_n is the sum of n independent geometric random variables.

(iii) Find the expectation $\mathbb{E}(T_n)$.

(iv) Find the variance $\operatorname{var}(T_n)$. Show that $\operatorname{var}(T_n) \leqslant Cn^2$ where $C = \sum_{i=1}^{\infty} i^{-2}$.

[You may use the fact that the variance of a geometric random variable of parameter p is $(1-p)/p^2$.]

(v) Show that for any $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{T_n}{n\log n} - 1\right| > \varepsilon\right) = 0\,.$$

[You may use the fact that $\sum_{i=1}^{n} i^{-1}/\log n \to 1$ as $n \to \infty$. You may use standard inequalities from lectures if you state them clearly.]

Paper 2, Section II

11F Probability

Let $(S_n : n \geqslant 0)$ be a simple random walk on $\mathbb{Z}$ with $S_0 = 0$ and $\mathbb{P}(S_n - S_{n-1} = 1) = p$ and $\mathbb{P}(S_n - S_{n-1} = -1) = q = 1 - p$ for all $n \geqslant 1$.

(i) Find the distribution of S_n.

(ii) Find b_n, c_n so that

$$\mathbb{P}\left(\frac{S_n - b_n}{c_n} \leqslant x\right) \to \Phi(x)$$

as $n \to \infty$, where Φ is the standard normal distribution function.

[You may quote standard results from lectures.]

From now on, assume that $p = q = 1/2$.

(iii) Let T be the random number of steps taken by the random walk until it first hits $-a$ or b for some $a, b \in \mathbb{N}$. Find $\mathbb{E}(T)$.

(iv) Let V_n be the number of visits to the origin until time n, that is, $V_n = |\{0 \leqslant i \leqslant n : S_i = 0\}|$. Using Stirling's formula or otherwise, prove that there exists some $c > 0$ such that

$$\mathbb{E}(V_{2n}) \geqslant c\sqrt{n}$$

for all n.

Paper 2, Section II

12F Probability

A *graph* on a set V is a set of some unordered pairs of (distinct) elements of V: we call these the *edges* of the graph and the elements of V are called the *vertices*. The *degree* of a vertex is the number of edges that contain it.

We form a random graph with n vertices $v_1, v_2, \ldots, v_n$ by including the edge $v_i v_j$ with probability p for all $i \neq j$ independently.

(i) Find the distribution of the degree of the vertex v_i.

We call a vertex *isolated* if its degree is 0. Let N be the number of isolated vertices.

(ii) Find the expectation $\mathbb{E}(N)$.

(iii) Let $p = c \log n / n$. Show that if $c > 1$, then $\mathbb{P}(N = 0) \to 1$ as $n \to \infty$.

(iv) Show that if p is such that $\operatorname{var}(N)/\mathbb{E}(N)^2 \to 0$ as $n \to \infty$, then $\mathbb{P}(N = 0) \to 0$ as $n \to \infty$.

(v) Find $\mathbb{E}(N^2)$. Now let $p = c \log n / n$ with $c < 1$. Show that $\mathbb{P}(N = 0) \to 0$ as $n \to \infty$.

[*You may want to use the inequalities* $e^{-x} \geqslant 1 - x$ *for all* x*; and for any* $\alpha > 1$, $e^{-\alpha x} \leqslant 1 - x$ *for all* $x \geqslant 0$ *small enough (depending on* α*). You may use standard inequalities from lectures if you state them clearly.*]

Paper 3, Section I

3B Vector Calculus

Let $\mathbf{F}$ and $\mathbf{G}$ be smooth vector fields in $\mathbb{R}^3$.

(i) Define the divergence $\nabla \cdot \mathbf{F}$ and the curl $\nabla \times \mathbf{F}$ in the standard Cartesian basis $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$.

(ii) Prove the following identities:

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}),$$
$$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}.$$

[You may use standard properties of the antisymmetric tensor ε_{ijk}.]

(iii) Prove that the identity $\mathbf{F} \cdot (\nabla \times \mathbf{G}) = \nabla \cdot (\mathbf{G} \times \mathbf{F})$ is true or find a counterexample.

(iv) Compute $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ in the case that $\mathbf{F} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$.

Paper 3, Section I

4B Vector Calculus

Define the *curvature* κ of a curve $\boldsymbol{\gamma}$ in $\mathbb{R}^3$ and describe its geometric significance. Determine the curvature for the curve

$$\boldsymbol{\gamma}(t) = (\cos(2t), \sqrt{5}\, t, \sin(2t)), \quad t \in [0, \pi]$$

as a function of its arclength (starting from the initial point $\boldsymbol{\gamma}(0)$). What is the integral of the curvature over the curve? Without performing any further computations, write down the integral of the curvature for the curve

$$\widetilde{\boldsymbol{\gamma}}(t) = (\cos(2t), 0, \sin(2t)), \quad t \in [0, \pi]$$

and explain why your result is different from the result for $\boldsymbol{\gamma}$.

Paper 3, Section II

9B Vector Calculus

(a) Consider a bounded volume V in $\mathbb{R}^3$ with smooth surface $\partial V = S$. Show that if $f(x,y,z)$ and $g(x,y,z)$ are smooth functions then there exists at most one smooth function $\phi(x,y,z)$ that satisfies $\nabla^2\phi = f$ in V and the Dirichlet boundary condition $\phi = g$ on S. If we change the boundary condition to a Neumann boundary condition $\hat{\mathbf{n}} \cdot \nabla\phi = g$ on S, can there be more than one solution? (Here $\hat{\mathbf{n}}$ denotes the outward unit normal to the surface S.)

(b) Now suppose that V, f and g all have spherical symmetry. Argue briefly why any solution to the Dirichlet problem must be spherically symmetric.

(c) Compute the unique solution to the Dirichlet problem in each of the following cases:

(i) V is the volume $r \leqslant 1$, $f(r) = (r-1)e^r$ and $g(r) = 3$.

(ii) V is the unbounded volume $r \geqslant 1$, $f(r) = -1/r^3$, $g(r) = 1$ and $\phi(r) \to 0$ as $r \to \infty$.

(iii) V is the volume $r \leqslant 1$, $f(r) = 1/(r^2+1)$ and $g(r) = -1$.

Paper 3, Section II

10B Vector Calculus

State the divergence theorem.

Let S be the surface formed by the part of the paraboloid $z = 1 - x^2 - y^2$ lying above the xy-plane and let $\mathbf{F}(x,y,z) = (x^2, -y^2, z^4)$. Calculate the flux of $\mathbf{F}$ across S (in the upward direction). Do this in two ways:

(i) By direct calculation of a surface integral.

(ii) By computing the flux over a simpler surface and applying the divergence theorem.

[*Hint: Use cylindrical polar coordinates for the surface integral.*]

Paper 3, Section II

11B Vector Calculus

(i) Prove that if ϕ is a smooth scalar field on $\mathbb{R}^3$ then

$$\nabla \times \nabla \phi = 0.$$

(ii) Prove that if $\mathbf{A}$ is a smooth vector field on $\mathbb{R}^3$ then

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0.$$

(iii) State Stokes' theorem. State what it means for a vector field defined on some region to be conservative. Prove that if a vector field $\mathbf{F}$ on $\mathbb{R}^3$ can be written as the gradient of a function then it is conservative.

(iv) Consider the following vector fields on $\mathbb{R}^3 \backslash \{x^2 + y^2 = 0\}$:

$$\mathbf{B}_1(x, y, z) = \left(2xe^{x^2} \cos(yz), -ze^{x^2} \sin(yz), -ye^{x^2} \sin(yz)\right)$$
$$\mathbf{B}_2(x, y, z) = \left(-y/(x^2 + y^2), x/(x^2 + y^2), 0\right)$$
$$\mathbf{B}_3(x, y, z) = \left(2x/(x^2 + y^2 + z^2), 2y/(x^2 + y^2 + z^2), 2z/(x^2 + y^2 + z^2)\right)$$

For each of these vector fields, compute the line integral around the curve C, where C is the closed curve $s \mapsto (\cos(s), \sin(s), 0)$, $s \in [0, 2\pi]$. Which of these vector fields are conservative and which can be written as gradients of functions? For those that can be written as gradients of functions, give a suitable potential function.

Paper 3, Section II

12B Vector Calculus

For this question, all tensors are in $\mathbb{R}^3$.

(i) Define a *rank n tensor.*

(ii) Define what it means for a tensor to be *totally antisymmetric.* For each integer $n \geqslant 2$, find all the totally antisymmetric rank n tensors.

(iii) Define what it means for a tensor to be *isotropic* and state the general form of isotropic rank 4 tensors.

(iv) Prove that

$$\varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}$$

and find $\varepsilon_{ijk}\varepsilon_{ijk}$.

(v) Find an isotropic rank 4 tensor (not identically zero) that can be written as a contraction of two antisymmetric tensors. Write down a (not identically zero) isotropic rank 5 tensor. Show that the most general isotropic rank 5 tensor must have at least ten independent components.

Paper 1, Section I

1A Vectors and Matrices

Consider a function of the complex variable z with $z \neq -1$

$$f(z) = \frac{az+b}{z+1},$$

where the coefficients a and b are complex numbers such that $a \neq b$.

(a) Find a and b such that the equation $f(z) = z$

(i) has a unique solution $z = i$;

(ii) has exactly two solutions $3i$ and $1+i$.

(b) Sketch the locus of all $z \neq 0$ satisfying

$$\arg\left(\frac{z-2}{z}\right) = \frac{\pi}{4}$$

and find its Cartesian equation.

Paper 1, Section I

2C Vectors and Matrices

(a) Consider a 3×3 matrix A with elements $A_{ij} = i + aj + b$, with a and b two non-zero real constants and $i, j = 1, 2, 3$.

(i) Compute the determinant of A.

(ii) Compute the kernel of A and hence its nullity.

(iii) Compute the image of A and hence its rank.

(iv) State the rank-nullity theorem and verify it for A.

(b) Now consider an $n \times m$ matrix B with $n, m > 1$ and elements $B_{ij} = i^2 + aj + b$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, where a and b are two non-zero real constants. Show that the nullity of B is $m - 2$.

Paper 1, Section II

5A Vectors and Matrices

(a) Use the summation convention and basic properties of the Levi-Civita symbol and the Kronecker delta to prove that

(i) $\epsilon_{ijk}\epsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}$.

(ii) $|\mathbf{x}|^2|\mathbf{y}|^2 \geqslant |\mathbf{x}\cdot\mathbf{y}|^2$, for any vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.

(iii) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$, for any vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ in $\mathbb{R}^3$.

(b) Let $\mathbf{y},\mathbf{z}$ be two fixed linearly independent unit vectors in $\mathbb{R}^3$. Define a scalar function S of a unit vector $\mathbf{x}$ as

$$S = |\mathbf{x}\times\mathbf{y}|^2 + |\mathbf{x}\times\mathbf{z}|^2 + \mathbf{x}\cdot\mathbf{y} + \mathbf{z}\cdot\mathbf{x}.$$

Find $\mathbf{F}(\mathbf{y},\mathbf{z})$ such that every $\mathbf{x}$ that maximises S satisfies $\mathbf{x}\times(\mathbf{y}\times\mathbf{z}) = \mathbf{F}(\mathbf{y},\mathbf{z})$.

Another scalar function S' of a unit vector $\mathbf{x}$ is defined by

$$S' = |R(\theta_1)\mathbf{x}\times R(\theta_1)\mathbf{y}|^2 + |\mathbf{x}\times\mathbf{z}'|^2 + R(\theta_2)\mathbf{x}\cdot R(\theta_2)\mathbf{y} + \mathbf{z}'\cdot\mathbf{x},$$

where $\mathbf{z}' = 2\mathbf{z}$, $R(\theta)$ is the matrix of rotation around the $z-$axis

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and θ_1, θ_2 are two angles. Find $\mathbf{G}(\mathbf{y},\mathbf{z})$ such that every $\mathbf{x}$ that maximises S' satisfies $\mathbf{x}\times(\mathbf{y}\times\mathbf{z}) = \mathbf{G}(\mathbf{y},\mathbf{z})$.

Paper 1, Section II

6C Vectors and Matrices

(a) Find an orthogonal linear transformation that maps the triangle formed by

$$P=(0,0,0)\,, \qquad Q=(0,1,0)\,, \qquad R=(\sqrt{3}/2,1/2,0)\,,$$

to the triangle formed by

$$P'=(0,0,0)\,, \qquad Q'=(-1/2,0,\sqrt{3}/2)\,, \qquad R'=(1/2,0,\sqrt{3}/2)\,.$$

(b) Consider the linear transformation M of $\mathbb{R}^4$ representing a reflection in the hyperplane $\sum_{m=1}^{4} a_m x_m = 0$, where a_m are four real constants (not all zero). Write down the matrix of M. Hence, or otherwise, compute its determinant.

(c) Consider the linear transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $A : \mathbf{b} \mapsto \mathbf{a} \times \mathbf{b}$ for some non-zero vector $\mathbf{a}$. Write A as a 3×3 matrix and compute its trace and its determinant.

(d) A *tridiagonal* matrix is a square matrix A such that $A_{ij} = 0$ for $|i-j| \geqslant 2$. For (a_i), (b_i) and (c_i) three infinite real sequences and n any positive integer, the $n \times n$ tridiagonal matrix $A^{(n)}$ is defined by $A^{(n)}_{ii} = a_i$ for $i = 1, \ldots, n$ and $A^{(n)}_{i(i+1)} = b_i$ and $A^{(n)}_{(i+1)i} = c_i$ for $i = 1, \ldots, n-1$.

(i) Prove that the determinants $d_n = \det(A^{(n)})$ satisfy the recurrence relation

$$d_n = X_n d_{n-1} + Y_n d_{n-2}\,,$$

for each $n \geqslant 3$, where X_n and Y_n are parameters you should determine in terms of (a_i), (b_i), (c_i) and n.

(ii) In the case $a_i = -2$, $b_i = c_i = 1$ for all i, compute d_n.

Paper 1, Section II

7B Vectors and Matrices

(a) Define the trace, $\mathrm{Tr}(A)$, of a complex $n \times n$ matrix A. If B is another complex $n \times n$ matrix, show that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Is it always the case that $\mathrm{Tr}(ABC) = \mathrm{Tr}(ACB)$ for 2×2 matrices?

(b) Define the characteristic polynomial of a complex $n \times n$ matrix A, and define the eigenvalues and eigenvectors of A.

(c) Let A, B be real $n \times n$ matrices such that $\det(A + iB) \neq 0$. Show that there exists $\lambda \in \mathbb{R}$ such that $A + \lambda B$ is invertible. [Hint: Consider a suitable polynomial.]

(d) Prove that if C, D are real $n \times n$ matrices related by a complex similarity transformation S (i.e., an invertible complex $n \times n$ matrix S with $C = SDS^{-1}$), then they are related by a real similarity transformation (i.e., an invertible real $n \times n$ matrix T with $C = TDT^{-1}$).

(e) Show that if a complex $n \times n$ matrix A has n distinct eigenvalues then it is diagonalisable over $\mathbb{C}$. Deduce that if a real $n \times n$ matrix A has n distinct real eigenvalues then it is diagonalisable over $\mathbb{R}$.

Paper 1, Section II

8B Vectors and Matrices

(a) Prove that any complex $n \times n$ matrix A has at least one eigenvalue and eigenvector.

(b) (i) Define the *adjoint* (or *Hermitian conjugate*) of a square matrix.

(ii) Define what it means for a complex $n \times n$ matrix U to be *unitary* and prove that the eigenvalues of any such matrix have modulus 1.

A complex $n \times n$ matrix A is *normal* if it commutes with its adjoint.

(iii) Prove that if A is diagonalisable with a unitary change of basis matrix then it is normal.

(c) Let A be an $n \times n$ normal matrix and let λ be an eigenvalue of A. Show that there exists a unitary change of basis matrix U such that

$$U^{\dagger}AU = \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}, \quad \text{where } B \text{ is an } (n-1) \times (n-1) \text{ normal matrix.}$$

Hence prove by induction that any normal matrix A is diagonalisable with a unitary change of basis matrix.

(d) A complex $n \times n$ matrix B is *upper triangular* if $B_{ij} = 0$ for $i > j$.

(i) Prove that given any complex $n \times n$ matrix A, there exists a unitary matrix U and an upper triangular matrix B such that $U^{\dagger}AU = B$.

(ii) Find such a unitary matrix U for the matrix

$$A = \begin{pmatrix} 4 & 5 & 0 \\ 0 & 6 & 0 \\ 2 & 3 & 1 \end{pmatrix}.$$

END OF PAPER