UNIVERSITY OF CAMBRIDGE

MAT2

MATHEMATICAL TRIPOS Part II

Friday, 09 June, 2023 9:00am to 12:00pm

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most six questions** from Section I and from any number of questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Number Theory

Compute the continued fraction expansion of $\sqrt{11}$. Show that for all $n \geqslant 0$ the convergents p_n/q_n satisfy

$$p_{n+1} + q_{n+1}\sqrt{11} = \begin{cases} \alpha(p_n + q_n\sqrt{11}) & \text{if } n \text{ is odd,} \\ \beta(p_n + q_n\sqrt{11}) & \text{if } n \text{ is even,} \end{cases}$$

for real numbers α and β which you should determine.

2F Topics In Analysis

We say that a function $f : X \to X$ has a fixed point if there exists an $x \in X$ with $f(x) = x$.

(i) Use the intermediate value theorem to show that, if $f : [0,1] \to [0,1]$ is continuous, it has a fixed point. Show also that, if $0, 1 \in f([0,1])$, then f is surjective.

(ii) Suppose that A and B are homeomorphic subsets of $\mathbb{R}^2$. Show that, if every continuous function $g : A \to A$ has a fixed point, then so does every continuous function $f : B \to B$.

(iii) State Brouwer's fixed point theorem for the closed unit disc $\bar{D}$.

(iv) Show that the closed unit disc is not homeomorphic to the annulus

$$A = \{(x, y) \in \mathbb{R}^2 : 1 \leqslant x^2 + y^2 \leqslant 2\}.$$

(v) Suppose that B is a subset of $\mathbb{R}^2$ containing at least two points. If every continuous function $g : B \to B$ has a fixed point, does it follow that B is homeomorphic to the closed unit disc? Give reasons.

3I Coding and Cryptography

(a) If $C \subseteq \mathbb{F}_2^n$ is a linear code, define the *dual code* $C^{\perp}$ and explain why it is also linear. If C is cyclic, show directly that $C^{\perp}$ is cyclic. Explain briefly how the generator polynomials of C and $C^{\perp}$ are related.

(b) Factorise $X^7 - 1$ over the field $\mathbb{F}_2$ and hence list all the binary cyclic codes of length 7. Identify versions of Hamming's original code and its dual in your list. What are the other cyclic codes of length 7? You should relate them to codes defined explicitly in the course.

4I Automata & Formal Languages

(i) Define what it means for a grammar to be *regular*.

(ii) Let $G = (\Sigma, V, P, S)$ be a regular grammar and $\Omega = \Sigma \cup V$. Prove that if $\alpha \in \Omega^*$ and $S \xrightarrow{G} \alpha$, then there are $w \in \mathbb{W}$ and $A \in V$ such that $\alpha = wA$ or $\alpha = w$.

(iii) Let $G = (\Sigma, V, P, S)$ be a regular grammar, $A, B \in V$, and $w, v \in \mathbb{W}$. Prove that if $wA \xrightarrow{G} vB$, then there is some word $u \in \mathbb{W}$ such that $A \xrightarrow{G} uB$.

If $G = (\Sigma, V, P, S)$ is a regular grammar and A is a variable, we call A *accessible in* G if there is a word $w_1 \in \Sigma^*$ such that $S \xrightarrow{G} w_1 A$; we call A *looping in* G if there is a word $w_2 \in \Sigma^*$ such that $A \xrightarrow{G} w_2 A$; we call A *terminable in* G if there is a word $w_3 \in \Sigma^*$ such that $A \xrightarrow{G} w_3$.

(iv) Let G be a regular grammar. Prove that if $\mathcal{L}(G)$ is infinite then there is a variable that is accessible, looping, and terminable in G.

5J Statistical Modelling

Below is a simplified 1993 dataset of US cars. The columns list make, model, price (in $1000), miles per gallon, number of passengers, length and width in inches, and weight (in pounds). The data are displayed in R as follows (abbreviated):

```
> cars
         make   model price mpg psngr length width weight
1       Acura Integra  15.9  31     5    177    68   2705
2       Acura  Legend  33.9  25     5    195    71   3560
3        Audi      90  29.1  26     5    180    67   3375
          ...               ...                       ...
91 Volkswagen Corrado  23.3  25     4    159    66   2810
92      Volvo     240  22.7  28     5    190    67   2985
93      Volvo     850  26.7  28     5    184    69   3245
```

It is reasonable to assume that prices for different makes are independent. How would you instruct R to model the logarithm of the price as a linear combination of an error term and

(i) an intercept;

(ii) an intercept and all other quantitative properties of the cars;

(iii) an intercept, all other quantitative properties of the cars, and the make of the cars?

Suppose the fitted models are assigned to objects fit1, fit2, and fit3, respectively. Suppose R provides the following analysis of variance table for these models:

```
> anova(fit1, fit2, fit3)

[...]

  Res.Df    RSS Df Sum of Sq       F    Pr(>F)
1     92 8584.0
2     87 3349.1  5    5234.9 69.7334 < 2.2e-16 ***
3     56  840.8 31    2508.3  5.3891 2.541e-08 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

What are your conclusions about the statistical models in fit1, fit2 and fit3 based on this table? Explain how you can determine the number of unique car manufacturers in this dataset from this table.

6C Mathematical Biology

The concentration $C(x,t)$ of a morphogen obeys the differential equation

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} + f(C),$$

in the domain $0 \leqslant x \leqslant L$, with boundary conditions $C(0,t) = 0$ and $\partial C(L,t)/\partial x = 0$, with D a positive constant and $f(C)$ a nonlinear function of C with $f(0) = 0$ and $f'(0) > 0$. Linearising the dynamics around $C = 0$, and representing $C(x,t)$ as a suitable Fourier expansion, find the condition on L such that the system is linearly stable. Express your answer in terms of D and $f'(0)$.

7E Further Complex Methods

The hypergeometric function $F(a,b;c;z)$ is the solution of the hypergeometric equation, i.e. the Fuchsian equation determined by the Papperitz symbol

$$P\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{array}\right\}$$

with $F(a,b;c;z)$ analytic at $z=0$ and satisfying $F(a,b;c;0)=1$.

Explain carefully the meaning of each of the elements appearing in the Papperitz symbol, including any aspects that are required for it to correspond to the hypergeometric equation.

Show that

$$F(a, c-b; c; \frac{z}{z-1}) = (1-z)^a F(a,b;c;z),$$

stating clearly any general results for transforming Fuchsian differential equations or manipulating Papperitz symbols that you use.

8D Classical Dynamics

What is meant by an *adiabatic invariant* of a mechanical system?

A particle of mass m and energy E moves between two fixed, parallel walls that are a distance L apart. The particle travels freely in a direction perpendicular to the walls except when it collides elastically with a wall at which point its velocity changes instantaneously. Compute the action $I = \oint p\,dq$ and verify that $T = dI/dE$ is the period of oscillation.

Suppose that the distance between the walls is varied very slowly so that $L(t)$ depends on time. How does the energy of the particle depend on time? Give a brief physical explanation for why the particle's energy changes.

9B Cosmology

What is the *flatness problem*? By using the Friedmann and continuity equations, show that a period of accelerated expansion of the scale factor $a(t)$ in the early stages of the universe can solve the flatness problem if $\rho + 3P < 0$, where ρ is the energy density and P is the pressure. *[Hint: it may be useful to compute* $\mathrm{d}(\rho a^2)/\mathrm{d}t$*.]*

In the very early universe one can neglect the spatial curvature and the cosmological constant. Suppose that in addition there is a homogenous scalar field ϕ with potential

$$V(\phi) = m^2\phi^2 \, ,$$

and the Friedmann equation is

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) \, ,$$

where $H = \dot{a}/a$ is the Hubble parameter. The field ϕ obeys the evolution equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0 \, .$$

During inflation, ϕ evolves slowly after starting from a large initial value ϕ_i at $t = 0$. State what is meant by the *slow-roll approximation.* Show that in this approximation

$$\begin{aligned} \phi(t) &= \phi_i - \frac{2}{\sqrt{3}} m t \\ a(t) &= a_i \exp\left[\frac{m\phi_i}{\sqrt{3}} t - \frac{1}{3} m^2 t^2\right] = a_i \exp\left[\frac{\phi_i^2 - \phi(t)^2}{4}\right] \, , \end{aligned}$$

where a_i is the initial value of a.

10D Quantum Information and Computation
[In this question you do not need to draw any circuits and you can assume that Alice can perform a measurement on two qubits in the Bell basis.]

(a) Suppose that Alice and Bob share the quantum state

$$|\psi^+_{AB}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$

and can communicate classically. Alice wants to send an arbitrary qubit state to Bob. State the steps that Alice and Bob need to execute to achieve this goal.

(b) Suppose Alice, Bob and Charlie share the following state of three qubits:

$$|\Psi_{ABC}\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right),$$

where the qubits A, B and C are with Alice, Bob and Charlie, respectively. Moreover, Alice has the qubit state $|\alpha\rangle = a\,|0\rangle + b\,|1\rangle$, with $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$. She now performs the Bell measurement on the two qubits in her possession. Depending on the measurement outcome, she asks Bob and Charlie to perform the necessary correction operations on their individual qubits, as is done in the standard teleportation protocol. Show that the final *joint* state of Bob and Charlie at the end of this protocol is either the state $|\varphi_1\rangle := a\,|00\rangle + b\,|11\rangle$ or the state $|\varphi_2\rangle := a\,|00\rangle - b\,|11\rangle$. Show that these states are entangled if and only if $a \neq 0$ and $b \neq 0$.

SECTION II

11G Number Theory

(a) Define the *Legendre symbol* $\left(\frac{a}{p}\right)$. State and prove Euler's criterion. Deduce a formula for $\left(\frac{-1}{p}\right)$.

(b) Let A be a 2×2 matrix with integer entries. Explain why if p is a prime number then

$$(I+A)^p \equiv I + A^p \pmod{p}.$$

Taking $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $p = 4k \pm 1$, show that $(-4)^k \equiv 1$ or $2 \pmod{p}$. Deduce a formula for $\left(\frac{2}{p}\right)$.

(c) Let p be an odd prime number and

$$T = \sum_{a=1}^{p-1} a\left(\frac{a}{p}\right).$$

(i) Show that if $p \equiv 1 \pmod 4$ then $T = 0$.

(ii) Show that if $p > 3$ then $T \equiv 0 \pmod p$.

(d) Show that if $p \equiv 7 \pmod 8$ then the sum of the quadratic residues modulo p in the interval $(0, p/2)$ is $(p^2-1)/16$.

12F Topics In Analysis

Let $C([0,1])$ denote the space of continuous real functions on $[0,1]$ equipped with the uniform norm $\|\cdot\|_\infty$.

(a) Consider $\mathbb{R}^{n+1}$ with the standard Euclidean norm $\|\cdot\|$, and let T be the map $T:\mathbb{R}^{n+1}\to C([0,1])$ given by $T(\mathbf{a})=\sum_{r=0}^{n} a_r t^r$. Let S be the map $S:\mathbb{R}^{n+1}\to\mathbb{R}$ given by $S(\mathbf{a})=\|T\mathbf{a}\|_\infty$. Show that there exists a $\delta>0$ such that

$$|S(\mathbf{a})|\geqslant\delta \text{ whenever } \|\mathbf{a}\|=1.$$

Conclude that $\|T(\mathbf{a})\|_\infty\to\infty$ as $\|\mathbf{a}\|\to\infty$.

(b) If $f\in C([0,1])$ and $n\geqslant 0$, show that there exists a (not necessarily unique) 'best fit' polynomial P of degree at most n such that

$$\|P-f\|_\infty\leqslant\|Q-f\|_\infty \text{ whenever } Q \text{ is a polynomial of degree at most } n.$$

(c) State Chebychev's equiripple criterion and show that it is a sufficient condition for a polynomial to be best fit.

(d) Let $g\in C([0,1])$, $M=\|g\|_\infty$ and suppose that

$$0=u_0<v_0<u_1<v_1<\ldots<v_{m-1}<u_m<v_m=1$$

are such that

$$\begin{array}{lll} M\geqslant g(t)>-M & \text{for } t\in[u_{2j},v_{2j}], & (2j\leqslant m) \\ -M\leqslant g(t)<M & \text{for } t\in[u_{2j+1},v_{2j+1}], & (2j+1\leqslant m) \\ -M<g(t)<M & \text{for } t\in[v_{j-1},u_j], & (j\leqslant m). \end{array}$$

Let $w_j=(v_{j-1}+u_j)/2$ and set $Q(t)=(-1)^{m-1}\prod_{j=1}^{m-1}(t-w_j)$. Show that, if $\eta>0$ is sufficiently small, we have

$$\|\eta Q-g\|_\infty<M.$$

Deduce that Chebychev's criterion is also a necessary condition for a polynomial to be best fit.

13J Statistical Modelling

The data frame `worldcup22` contains information about the matches played in a sports competition, including for each team in the match the starting formations (indicated by letters A-L), the expected goals (xg) and the actual goals. In the questions below we will assume that the match results are independent.

```
> worldcup22
     team1   team2 team1_xg team2_xg team1_form team2_form team1_goal team2_goal
1    Qatar Ecuador      0.3      1.2          I          H          0          2
2  England IR Iran      2.1      1.4          E          J          6          2
       ...                       ...                                        ...
63 Croatia Morocco      0.7      1.2          E          F          2          1
64   Japan  France      3.3      2.2          F          E          3          3
> fit1 <- glm(team1_goal ~ team1_form + team2_form, worldcup22,
              family = poisson)
```

(i) Let Y denote the response vector and X denote the design matrix for `fit1`. Write down the likelihood function that is maximized by the command above. [Recall that if Y follows a Poisson distribution with mean μ, then $\mathbb{P}(Y = k) = \mu^k e^{-\mu}/k!$, $k = 0, 1, \ldots$.]

(ii) Comment on the following abbreviated summary of `fit1`. Is there enough information to conclude that the formation of `team1` does not affect its goals? If not, what is the name of the statistical procedure you can use to test this hypothesis?

```
> summary(fit1)
               Estimate Std. Error z value Pr(>|z|)
(Intercept)       1.890      0.581     3.3    0.001 **
team1_formB      -0.672      0.595    -1.1    0.259
team1_formC     -17.865   2446.075     0.0    0.994
team1_formD       0.595      1.293     0.5    0.645
team1_formE      -0.361      0.441    -0.8    0.413
team1_formF      -0.098      0.414    -0.2    0.812
team1_formG      -1.120      1.089    -1.0    0.304
team1_formH      -0.332      0.490    -0.7    0.498
team1_formI      -1.855      1.104    -1.7    0.093 .
team1_formJ       0.285      0.830     0.3    0.731
team2_formK     -18.831   3467.859     0.0    0.996
team2_formB      -1.199      0.565    -2.1    0.034 *
team2_formC      -1.792      1.080    -1.7    0.097 .
team2_formL      -0.905      0.558    -1.6    0.105
team2_formE      -1.482      0.478    -3.1    0.002 **
team2_formF      -1.464      0.504    -2.9    0.004 **
team2_formH      -0.728      0.494    -1.5    0.140
team2_formI      -0.980      0.588    -1.7    0.095 .
team2_formJ      -0.143      0.612    -0.2    0.816
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

[QUESTION CONTINUES ON THE NEXT PAGE]

(iii) Expected goals (xg) is a new metric in sports analytics that computes the number of goals a team should have scored based on the quality of the chances created. State the following two hypotheses mathematically: (a) H_1: `team1_goal` has mean `team1_xg`; (b) H_2: `team1_goal` follows a Poisson distribution with mean `team1_xg`. Then name the result in probability theory that suggests `team1_goal` should approximately follow a Poisson distribution.

(iv) An analyst fitted the following model to test H_1. Does the model fit suggest evidence against H_1? Give one reason why we should be skeptical about the standard errors in the table.

```
> fit2 <- lm(team1_goal ~ team1_xg - 1, worldcup22)
> summary(fit2)
Coefficients:
        Estimate Std. Error t value Pr(>|t|)
team1_xg  1.15790    0.08643    13.4   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

(v) The analyst then fitted the following model and computed the 95% confidence interval for the coefficients. Explain why the observation that the confidence interval for `log(team1_xg)` contains 1 does not directly imply that H_2 cannot be rejected at the 5% significance level.

```
> fit3 <- glm(team1_goal ~ log(team1_xg), worldcup22, family = poisson)
> confint(fit3)
                  2.5 %    97.5 %
(Intercept)  -0.1387542 0.3836497
log(team1_xg)  0.6691166 1.3395731
```

14C Mathematical Biology

Consider a population subject to the following birth-death process. When the number of individuals in the population is n, the probability of an increase from n to $n+1$ per unit time is $\gamma+\beta n$ and the probability of a decrease from n to $n-1$ is $\alpha n(n-1)$, where α, β, and γ are constants.

Draw a transition diagram and show that the master equation for $P(n,t)$, the probability that at time t the population has n members, is

$$\frac{\partial P}{\partial t} = \alpha n(n+1)P(n+1,t) - [\alpha n(n-1)+\gamma+\beta n]\,P(n,t) + [\gamma+\beta(n-1)]\,P(n-1,t). \quad (1)$$

Show that $\langle n\rangle$, the mean number of individuals in the population, satisfies

$$\frac{d\langle n\rangle}{dt} = -\alpha\langle n^2\rangle + (\alpha+\beta)\langle n\rangle + \gamma.$$

Deduce that, in a steady state,

$$\langle n\rangle = \frac{\alpha+\beta}{2\alpha} \pm \sqrt{\frac{(\alpha+\beta)^2}{4\alpha^2} + \frac{\gamma}{\alpha} - (\Delta n)^2},$$

where Δn is the standard deviation of n. Given the form of the expression above, when is the choice of the minus sign not admissible?

Show that, under conditions to be specified, the master equation (1) may be approximated by a Fokker-Planck equation of the form

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial n}\left[g(n)P(n,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial n^2}\left[h(n)P(n,t)\right].$$

Find the functions $g(n)$ and $h(n)$.

In the case $\alpha \ll \gamma$ and $\alpha \ll \beta$, find the leading-order approximation to n_* such that $g(n_*)=0$. Defining the new variable $x = n - n_*$, explain how an approximate form of $P(x)$ may be obtained in the neighbourhood of $x=0$ in the steady-state limit, showing clearly the dependence of $P(x)$ on the properties of the functions $g(n)$ and $h(n)$ at $n=n_*$. Deduce leading order estimates for $\langle n\rangle$ and $(\Delta n)^2$ in terms of α, β and γ.

Compare your results to those obtained from the master equation above and give justification of why the conditions for applicability of the Fokker-Planck equation hold in this case.

15D Classical Dynamics

What does it mean for a phase space coordinate transformation to be *canonical*? Consider a coordinate transformation $(q,p) \mapsto (Q,P)$ on the phase space of a system with one degree of freedom. Show that if this transformation is defined in terms of a generating function $F(q,P)$ via

$$Q = \left.\frac{\partial F}{\partial P}\right|_q \qquad \text{and} \qquad p = \left.\frac{\partial F}{\partial q}\right|_P$$

then it is canonical.

Find the phase space coordinate transformation associated to the generating function

$$F(q,P) = \int_0^q \sqrt{2P - u^2}\, du\,.$$

Obtain Hamilton's equations for Q and P in the case $H(q,p) = \frac{1}{2}(p^2+q^2)$. Hence find $Q(t)$ and $P(t)$ and check that these agree with the usual solution for a simple harmonic oscillator.

A particle of energy E has Hamiltonian $H(q,p) = \frac{1}{2}(p^2+q^2) + \epsilon q^4$, where $2q^2\epsilon \ll 1$ for all q in the range $-\sqrt{2E} \leqslant q \leqslant \sqrt{2E}$. By choosing an appropriately modified generating function $F_\epsilon(q,P)$, show that

$$\frac{q(t)}{p(t)} = \tan(t-t_0) - \epsilon\, I(q_0(t),E)\,(1+\tan^2(t-t_0)) + \epsilon q_0^2(t)\tan^3(t-t_0) + \mathcal{O}(\epsilon^2),$$

where $q_0(t) = \sqrt{2E}\sin(t-t_0)$ and $I(x,y)$ is defined by

$$I(x,y) = \int_0^x \frac{u^4}{(2y-u^2)^{3/2}}\, du\,.$$

16H Logic and Set Theory

In this question we work in ZF, not ZFC. As usual, for cardinals κ and λ we write $\kappa \leqslant \lambda$ if there is an injection from K to L, where K and L are sets of cardinalities κ and λ, respectively.

(a) Show that the assertion that $\leqslant$ is a total ordering on cardinals (in other words, that for any κ and λ we have $\kappa \leqslant \lambda$ or $\lambda \leqslant \kappa$) is equivalent to the Axiom of Choice.

(b) Show that the Axiom of Choice implies that, for any infinite cardinal κ, we have $\kappa^2 = \kappa$.

(c) Suppose that κ and λ are non-zero cardinals such that $\kappa\lambda \leqslant \kappa + \lambda$. Prove that there must exist either an injection or a surjection from K to L.

(d) Show that the assertion that for any infinite cardinal κ we have $\kappa^2 = \kappa$ is equivalent to the Axiom of Choice. [*Hint: for a given set X, you may wish to consider the disjoint union of X with $\gamma(X)$.*]

[*You may assume Hartogs' Lemma, and you may use the equivalence of the Axiom of Choice with any of its equivalents from the course.*]

17H Graph Theory

(a) State Menger's theorem relating the size of x-y separators in a graph to the number of independent x-y paths.

(b) State Hall's theorem and, assuming Menger's theorem, prove Hall's theorem.

(c) Let $k \geqslant 1$ and let $[0,1]^3 = A_1 \cup \cdots \cup A_k$ and $[0,1]^3 = B_1 \cup \cdots \cup B_k$ be two partitions of the unit cube into sets of equal volume. Show there is a permutation σ of $[k]$ so that $A_i \cap B_{\sigma(i)} \neq \emptyset$, for all $i \in [k]$.

(d) Let G be a $2k$-connected graph that contains a K_{2k} and let $x_1, \ldots, x_k, y_1, \ldots, y_k$ be distinct vertices in G. Show that there exist paths $P_1, \ldots, P_k$ for which

$$V(P_i) \cap V(P_j) = \emptyset,$$

for all $i \neq j$ where, for each i, P_i is an x_i-y_i path.

[*In parts (c) and (d) you may assume results from the course provided they are stated clearly.*]

18I Galois Theory

(a) Define the *discriminant* of a monic polynomial.

Let K be a field with $\mathrm{char}(K) \neq 2$, and let $f \in K[T]$ be a monic, separable polynomial of degree n. Show that the Galois group of f is contained in A_n if and only if the discriminant of f is a square in K.

Compute the Galois group of $T^3 - 2T + 2$ over $\mathbb{Q}$ and over $\mathbb{Q}(\sqrt{-19})$.

[*The discriminant of* $T^3 + aT + b$ *is* $-4a^3 - 27b^2$.]

(b) Let K be a field of characteristic 2, and $f = T^3 + aT + b \in K[T]$. Let L be a splitting field for f over K.

(i) Show that f is separable if and only if $b \neq 0$.

(ii) Assuming that f is separable, show that $g = T^2 + bT + a^3 + b^2$ splits into distinct linear factors in $L[T]$. By considering the action of the Galois group G of f on the roots of g, or otherwise, show that G is contained in A_3 if and only if g splits into linear factors in $K[T]$.

19H Representation Theory

State Schur's lemma.

What is a *complex representation* of a topological group G? What does it mean to say a complex representation (ρ, V) of G is *unitary*?

Explain why every complex representation of S^1 is unitary. Deduce that every complex representation of S^1 is a direct sum of 1-dimensional representations.

Let G be the group of 3×3 upper unitriangular real matrices

$$G := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}$$

under matrix multiplication. Let Z be the centre of G and Z_0 the cyclic subgroup of Z given by

$$Z_0 = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| z \in \mathbb{Z} \right\} \leqslant Z = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| z \in \mathbb{R} \right\}.$$

By considering elements of the form $g^{-1}h^{-1}gh$ with $g, h \in G$, show that every 1-dimensional representation of G has kernel containing Z.

Show that any complex representation (ρ, V) of G/Z_0 decomposes as a direct sum of subrepresentations $(\rho_i, V_i)_{i=1,\ldots,d}$ with the property that

$$\operatorname{Res}^{G/Z_0}_{Z/Z_0} \rho_i = \theta_i \operatorname{id}_{V_i}$$

for some distinct 1-dimensional representations $\theta_1, \ldots, \theta_d$ of Z/Z_0. By considering $\det \rho_i$, or otherwise, deduce that $d = 1$ and that θ_1 is the trivial representation. Hence show that G/Z_0 does not have a faithful representation.

20H Number Fields

(a) State Minkowski's lemma.

Let $K = \mathbb{Q}(\sqrt{d})$, with $d \in \mathbb{Q}$, $d > 0$, d not a square. Prove that there are infinitely many $\alpha \in \mathcal{O}_K$ with $N(\alpha) < |D_K|^{1/2}$, where D_K is the discriminant of K.

(b) Determine the units in the ring of integers $\mathcal{O}_K$ in the cases (i) $K = \mathbb{Q}(\sqrt{10})$ and (ii) $K = \mathbb{Q}(\sqrt{-3})$. You must prove that your answers are correct.

(c) Let $K = \mathbb{Q}(\zeta)$, where $\zeta^5 = 1$. Determine $\mathcal{O}_K^*$ as an abelian group. [You do not have to describe explicit generators.]

Find explicit elements of $\mathcal{O}_K^*$ which generate a subgroup H of finite index (that is, for which $\mathcal{O}_K^*/H$ is finite).

21G Algebraic Topology

Suppose that (C, d) and (C', d') are chain complexes, and that $f, g : C \to C'$ are chain maps. Show that f induces a map $f_* : H_*(C) \to H_*(C')$. Define what it means for f and g to be *chain homotopic*. Show that if f and g are chain homotopic, they induce the same map on homology.

Define a chain complex $(M(f), d_f)$ as follows: $M(f)_i = C_{i-1} \oplus C'_i$ and the map $(d_f)_i : M(f)_i \to M(f)_{i-1}$ is given by the matrix

$$\begin{pmatrix} d_{i-1} & 0 \\ (-1)^i f_{i-1} & d'_i \end{pmatrix}.$$

Verify that $(M(f), d_f)$ is a chain complex. Show that there is a long exact sequence

$$\ldots \to H_i(C) \xrightarrow{(-1)^{i+1} f_*} H_i(C') \to H_i(M(f)) \to H_{i-1}(C) \xrightarrow{(-1)^i f_*} H_{i-1}(C') \to \ldots$$

If f is chain homotopic to g, show that $(M(f), d_f)$ and $(M(g), d_g)$ are isomorphic as chain complexes.

22F Linear Analysis

Below, H denotes a Hilbert space over $\mathbb{C}$.

(a) Consider a sequence (x_n) in H with the property that there exists an $x \in H$ such that for any $y \in H$, $\langle x_n, y \rangle$ converges to $\langle x, y \rangle$ in $\mathbb{C}$. Prove that the sequence (x_n) is bounded. [The uniform boundedness principle may be used without proof, provided it is properly stated.]

(b) With (x_n) and x as above, prove that there exists another sequence $(\tilde{x}_k)$ in H such that $\|\tilde{x}_k - x\|_H \to 0$ and such that each $\tilde{x}_k$ is a convex combination of terms in (x_n).

(c) Deduce that if $C \subset H$ is closed and convex, and (x_n) is a sequence in C as in part (a), i.e. with the property that there exists $x \in H$ such that for any $y \in H$, $\langle x_n, y \rangle \to \langle x, y \rangle$, then in fact $x \in C$.

(d) Is the statement in part (c) still true when C is closed but not necessarily convex? [You must either provide a proof if true or a detailed counterexample if untrue.]

23F Analysis of Functions

(a) Prove that the embedding $H^1(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ is not compact.

(b) Construct a bounded linear functional on $L^\infty(\mathbb{R}^n)$ that cannot be expressed as $f \in L^\infty(\mathbb{R}^n) \mapsto \int f(x)g(x)\,dx$ for any $g \in L^1(\mathbb{R}^n)$. [You may use theorems from the course if you state them carefully.]

(c) Prove that $H^n(\mathbb{R}^n)$ embeds continuously into $C^{0,\alpha}(\mathbb{R}^n)$, for some $\alpha \in (0,1)$.

(d) Let θ be the Heaviside function, defined by $\theta(x) = 1_{x \geqslant 0}$, $x \in \mathbb{R}$. Find the Hardy–Littlewood maximal function $M\theta$.

24G Algebraic Geometry

What does it mean for two irreducible varieties to be *birational*? Prove that birational varieties have the same dimension.

Let K be a finitely generated field extension of $\mathbb{C}$. Prove that there exists a projective variety X over $\mathbb{C}$ whose function field is K.

Let X be the affine plane curve $\mathbb{V}(f) \subset \mathbb{A}^2$, where

$$f(x,y) = y^2 - x(x-1)^2.$$

For what values of d is X birational to a smooth projective plane curve of degree d?

Construct an affine variety X of dimension 2 that is birational to $\mathbb{A}^2$, and whose set of singular points is an irreducible subvariety of dimension 1.

25G Differential Geometry

(a) Given a compact orientable surface with smooth boundary, define the *area element dA*, *Euler characteristic* χ, and *geodesic curvature* k_g of the boundary, explaining briefly why the first two are well defined. State the Gauss–Bonnet theorem for the surface. [You need not consider the case of corners.]

(b) Let S be a compact orientable surface without boundary, and let $\gamma : I \to S$ be a smooth closed curve on S, parametrised by arc length, which separates S into two surfaces with boundary, S_1 and S_2, such that S is the union $S = S_1 \cup S_2$ where $\partial S_1 = \partial S_2 = S_1 \cap S_2 = \gamma(I)$. Suppose there exists an isometry $\phi : S_1 \to S_2$, and moreover, for each $x, y \in \gamma(I)$, an isometry $\phi_{x,y} : S \to S$ such that $\phi_{x,y}(\gamma(I)) = \gamma(I)$ and such that $\phi_{x,y}(x) = y$. Show that γ is a geodesic.

(c) In the above problem, suppose we drop the assumption of the existence of the isometry ϕ. Is γ still necessarily a geodesic?

(d) Alternatively, suppose we drop the assumption of the existence of the isometries $\phi_{x,y}$. Is γ still necessarily a geodesic?

26K Probability and Measure

(a) Let $(Y_n : n \in \mathbb{N})$ be an infinite sequence of i.i.d. random variables such that $\mathbb{E}|Y_1| = \infty$. Show that $\limsup_{n\to\infty} |Y_1 + \cdots + Y_n|/n = \infty$ almost surely.

(b) Show that one can find $(Y_n : n \in \mathbb{N})$ as in part (a) but such that $(Y_1 + \cdots + Y_n)/n$ converges weakly to some random variable Z.

[*You may use theorems from lectures provided you state them clearly.*]

27J Applied Probability

(a) Let $d \geqslant 1$ and let $\lambda : \mathbb{R}^d \mapsto \mathbb{R}$ be a non-negative measurable function such that $\int_A \lambda(x)dx < \infty$ for all bounded Borel sets A. Define a *non-homogeneous spatial Poisson process* on $\mathbb{R}^d$ with *intensity function* λ.

(b) Assume that the positions $(x, y, z) \in \mathbb{R}^3$ of stars in space are distributed according to a Poisson process on $\mathbb{R}^3$ with constant intensity λ. Show that their distances from the origin $g(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ form another (non-homogeneous) Poisson process on $\mathbb{R}_+$. Find its intensity function. Find the density function for the distribution of the distance of the closest star from the origin.

(c) An art gallery has ten rooms, and visitors are required to view them all in sequence. Visitors arrive at the instants of a non-homogeneous Poisson process on $\mathbb{R}_+$ with intensity function $\lambda(x)$. The ith visitor spends time $Z_{i,j}$ in the jth room, where the random variables $Z_{i,j}$ for $i \geqslant 1, 1 \leqslant j \leqslant 10$ are i.i.d. random variables, independent of the arrival process. Let $t \geqslant 0$ and let $V_j(t)$ be the number of visitors in room j at time t. Show for fixed t that $V_j(t)$ for $1 \leqslant j \leqslant 10$ are independent random variables. Find their distributions.

[*You may quote any result from the lectures that you need, without proof, provided it is clearly stated.*]

28K Principles of Statistics

Suppose $X \sim f$ takes values in $\mathcal{X}$, and h is a reference density on $\mathcal{X}$ from which it is possible to generate i.i.d. samples.

(a) State the steps of the *importance sampling algorithm* and explain why it can be used to approximate $\mathbb{E}[g(X)]$, where g is a function defined on $\mathcal{X}$.

Now consider the following algorithm:

1. Generate $Y_1, \ldots, Y_m$ i.i.d. from h. Let
$$q_i = \frac{f(Y_i)/h(Y_i)}{\sum_{j=1}^{m} f(Y_j)/h(Y_j)}, \qquad \forall 1 \leqslant i \leqslant m.$$

2. Let X^* be a random variable generated from the discrete distribution on $\{Y_1, \ldots, Y_m\}$, such that $\mathbb{P}(X^* = Y_k) = q_k$ for all $1 \leqslant k \leqslant m$.

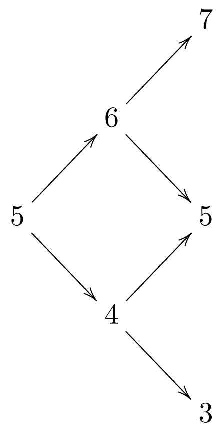
(b) Show that X^* converges in distribution to f, i.e., for all $t \in \mathbb{R}$, we have
$$\mathbb{P}(X^* \leqslant t) \overset{a.s.}{\to} F(t),$$
as $m \to \infty$, where F is the cumulative distribution function corresponding to f.

(c) Suppose F is continuous. Prove that the convergence in part (b) is uniform:
$$\sup_{t \in \mathbb{R}} |\mathbb{P}(X^* \leqslant t) - F(t)| \overset{a.s.}{\to} 0,$$
as $m \to \infty$.

[*You may quote any result from the lectures that you need, without proof.*]

29K Stochastic Financial Models

Consider the following two-period market model. There is a single risky stock with prices $(S_n)_{n\in\{0,1,2\}}$ given by



where in each period the price is equally likely to go up as to go down.

(a) Suppose that the interest rate $r = 1/6$. Find an arbitrage $(\varphi_n)_{n\in\{1,2\}}$.

For the rest of the problem, suppose $r = 1/8$.

(b) Find the time-0 no-arbitrage price of a European put option maturing at $T = 2$ with strike $K = 5$. How many shares of the stock should be held in the first period to replicate the payout of the put?

(c) Find the time-0 no-arbitrage price of a European call option maturing at $T = 2$ with strike $K = 5$.

(d) Now find the time-0 no-arbitrage price of an American put option maturing at $T = 2$ with strike $K = 5$. What is an optimal exercise policy?

30J Mathematics of Machine Learning

(a) Let $(X_1, Y_1), \ldots, (X_n, Y_n) \in \mathbb{R} \times \mathbb{R}$ be input–output pairs with $n \geqslant 4$. Describe the optimisation problem that a *regression tree algorithm* using a *squared error loss splitting criterion* would take to find the first split point.

(b) Assuming that the inputs are sorted so that $X_1 < \cdots < X_n$, show that the above may be solved in $O(n)$ computational operations.

(c) Now write down the squared error loss empirical risk minimiser $\hat{f}_m : \mathbb{R} \to \mathbb{R}$ over $\mathcal{F} : \{x \mapsto \alpha + x\beta \, : \alpha \in \mathbb{R}, \, \beta \in \mathbb{R}\}$, when trained only on data $(X_1, Y_1), \ldots, (X_m, Y_m)$ for $m \geqslant 2$. [You need not derive it.]

(d) Denote by $\hat{g}_m : \mathbb{R} \to \mathbb{R}$ the equivalent of $\hat{f}_m$ in part (c) when instead training only on $(X_{m+1}, Y_{m+1}), \ldots, (X_n, Y_n)$ for $m \leqslant n-2$. Show carefully how minimising

$$\sum_{i=1}^{m} (Y_i - \hat{f}_m(X_i))^2 + \sum_{i=m+1}^{n} (Y_i - \hat{g}_m(X_i))^2$$

over $m = 2, \ldots, n-2$ may be performed in $O(n)$ computations.

31E Asymptotic Methods

Justifying your steps carefully, use the method of steepest descent to find the first term in the asymptotic approximation of the function:

$$I(x) = \int_C \frac{1}{z^2 + 16} \, e^{x \cosh z} \, dz \,, \quad \text{as} \quad x \to \infty \,,$$

where $x \in \mathbb{R}$ and the integral is over the contour

$$C = \{ z \in \mathbb{C} : \; z = p + iq \,, q = 2 \arctan p \,, \, p \in \mathbb{R} \},$$

taken in the direction of increasing p.

32A Dynamical Systems

For the map $x_{n+1} = F(x_n, \lambda) := \lambda x_n(1 - x_n^2)$ with $\lambda > 0$ and $x_n \in [0, 1]$, show the following:

(i) There is an upper limit on λ if points are not to be mapped outside the domain $[0, 1]$. Find this value.

(ii) For $\lambda < 1$ the origin is the only fixed point and is stable.

(iii) If $\lambda > 1$, then the origin is unstable and a new fixed point x^* exists. This new fixed point x^* is stable for $1 < \lambda < 2$ and unstable for $\lambda > 2$.

(iv) For λ close to but larger than 2, and with $X_n = x_n - x^*$ and $0 < \mu = \lambda - 2 \ll 1$, the map can be locally represented as

$$X_{n+1} = -X_n + \alpha\mu X_n + \beta X_n^2 + \gamma X_n^3 + O(\mu^2), \qquad (*)$$

where α, β and γ are constants that you should evaluate in terms of appropriate derivatives of F. Hence show that the 2-cycle born in the bifurcation at $\lambda = 2$ has points

$$x_{\pm} = x^* \pm \sqrt{\frac{-\alpha\mu}{\gamma + \beta^2}}.$$

[You do not need to substitute the expressions you found for α, β and γ into this formula.]

(v) The 2-cycle is stable for $\lambda > 2$, with $\lambda - 2$ small.

33B Principles of Quantum Mechanics

(a) A composite system is made of two sub-systems with total angular momenta j_1 and j_2, respectively. Let $\mathbf{J} = \{J_x, J_y, J_z\}$ be the angular momentum operator of the composite system and $|j, m\rangle$ a basis of eigenstates of $\mathbf{J}^2$ and J_z.

(i) Write $\mathbf{J}$ and the associated ladder operators $J_\pm$ in terms of the angular momentum operators $\mathbf{J}_{1,2}$ of each sub-system.

(ii) State the possible values of j in terms of j_1 and j_2 and specify under what conditions it is possible to have $j = 0$.

(iii) Write down all the states of definite j and m that have $m \geqslant j_1 + j_2 - 1$, in terms of the states of the sub-systems $|j_1, m_1\rangle$ and $|j_2, m_2\rangle$.

(iv) Given a pure state, define what it means for the state to be a *product state* and what it means for the state to be an *entangled state*. Specify whether each of the states in (iii) is a product state or an entangled state.

(b) Let $j = j_1 + j_2$. For each of the two states of the system $|j, j\rangle$ and $|j, j-1\rangle$ compute the reduced density matrix of subsystem 1 and the associated entanglement entropy. Comment on the value of the entanglement entropy when $j_1 = j_2$.

(c) Explain why, if it exists, the state with $j = 0$ must be of the form

$$|0,0\rangle = \sum_{m=-j_1}^{j_1} \alpha_m \, |j_1, m\rangle_1 \, |j_1, -m\rangle_2 \ .$$

By considering $J_+ \, |0,0\rangle$, determine a relation between α_{m+1} and α_m, and hence find α_m.

[*Units in which $\hbar = 1$ have been used throughout. The states $|j, m\rangle$ obey*

$$J_\pm \, |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)} \, |j, m \pm 1\rangle \ .]$$

34D Applications of Quantum Mechanics

(a) A scalar particle of mass m and charge e is moving in three dimensions in a background electromagnetic field with vector potential $\mathbf{A}(\mathbf{x},t)$ and zero scalar potential. The Hamiltonian is given as

$$\hat{H} \quad = \quad \frac{1}{2m}\left(-i\hbar\nabla + e\mathbf{A}\right)\cdot\left(-i\hbar\nabla + e\mathbf{A}\right) .$$

Specialise to the case of a constant, homogeneous magnetic field $\mathbf{B} = \nabla \times \mathbf{A} = (0,0,B)$ in the z-direction. Suppose further that the x and y coordinates of the particle are constrained to lie in a rectangular region R of the x-y plane with sides of length R_x and R_y, and that the particle has vanishing momentum in the z-direction. By solving the Schrödinger equation in a suitable gauge with periodic boundary conditions in the x- and y-directions, find the energy levels of the system and give the degeneracy of each level. [You may use without proof any results about the spectrum of the quantum harmonic oscillator you may need, and you may assume that R_x and R_y are large compared to other length scales in the problem.]

(b) An electron is a particle of mass m, charge e and spin $1/2$. It is described by a two-component wave function $\vec{\Psi} \in \mathbb{C}^2$ with energy eigenstates obeying a matrix Schrödinger equation

$$\hat{\mathbb{H}}\,\vec{\Psi} \quad = \quad E\vec{\Psi}\,,$$

where

$$\hat{\mathbb{H}} \quad = \quad \hat{H}\,\mathbb{I}_2 \; + \; \frac{e\hbar}{2m}\,\mathbf{B}\cdot\boldsymbol{\sigma}\,,$$

where $\hat{H}$ is the Hamiltonian for the spinless particle given above, $\mathbb{I}_2$ is the (2×2)-unit matrix and $\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$ is a three-component vector whose entries are the Pauli matrices σ_i, for $i = 1,2,3$.

Find the energy levels of a single electron in a constant, homogeneous magnetic field $\mathbf{B} = (0,0,B)$ under the same conditions as in part (a). Give the degeneracy of each energy level.

Now consider N non-interacting electrons occupying these energy levels. Find the ground-state energy $E_{\rm gs}$ of the system as a function of N, identifying any thresholds which occur. Sketch the graph of $E_{\rm gs}$ against N. [*Hint: Recall that electrons are identical fermionic particles obeying the Pauli exclusion principle.*]

35A Statistical Physics

(a) Give *Clausius' statement* of the second law of thermodynamics and *Kelvin's statement* of the second law of thermodynamics. Show that these two statements are equivalent.

Throughout the rest of this question you should consider a classical ideal gas and assume that the number of particles is fixed.

(b) Write down the equation of state for an ideal gas. Write down an expression for its internal energy in terms of the heat capacity at constant volume C_V.

(c) Describe the meaning of an *adiabatic process.* Using the first law of thermodynamics, derive the relationship between p and V for an adiabatic process occurring in an ideal gas.

(d) Consider a cycle involving an ideal gas and consisting of the following four reversible steps:

$A \to B$: Adiabatic compression;
$B \to C$: Expansion at constant pressure with heat in Q_1;
$C \to D$: Adiabatic expansion;
$D \to A$: Cooling at constant volume with heat out Q_2.

(i) Sketch this cycle in the (p, V)-plane and in the (T, S)-plane. Derive equations for the curves DA and BC in the (T, S)-plane.

(ii) Derive an expression for the efficiency, $\eta = W/Q_1$, where W is the work out, in terms of the temperatures T_A, T_B, T_C, T_D at points A, B, C, D, respectively.

36A Electrodynamics

Consider a dielectric medium whose electromagnetic properties are described by the electric displacement $\mathbf{D}$, the magnetisation $\mathbf{H}$, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$.

(a) Write down the Maxwell equations for these four fields in the presence of a free charge density ρ and a free current density $\mathbf{J}$.

(b) Hence establish the identity

$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{E} \cdot \mathbf{J}.$$

(c) Consider a linear dielectric medium with the constitutive relations

$$D_i = \varepsilon_{ij} E_j, \quad B_i = \mu_{ij} H_j$$

where ε_{ij} and μ_{ij} are symmetric matrices, independent of t, representing the anisotropic dielectric response of the material, and the summation convention applies here and below. For a volume V enclosed by the surface S, derive the integral relation

$$\frac{\partial}{\partial t} \int_V \frac{1}{2} (\varepsilon_{ij} E_i E_j + \mu_{ij} H_i H_j) dV + \int_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = - \int_V \mathbf{E} \cdot \mathbf{J} \, dV.$$

In the absence of free currents, interpret the above relation in terms of an energy density ϵ and an energy flux $\mathbf{N}$, clearly identifying each.

(d) Consider a linear dielectric medium with

$$D_i = \varepsilon_{ij} E_j, \quad B_i = \mu \delta_{ij} H_j,$$

where μ and ε_{ij} are independent of space and time, and δ_{ij} is the Kronecker delta. Assuming plane waves

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{e} \sin(\mathbf{k} \cdot \mathbf{x} - \omega t), \quad \mathbf{B}(\mathbf{x}, t) = \mathbf{b} \sin(\mathbf{k} \cdot \mathbf{x} - \omega t)$$

in this medium, show that Maxwell's equations in the absence of free charges and currents imply that the wave vector $\mathbf{k}$, the frequency ω and polarisation $\mathbf{e}$ must satisfy

$$[\mathbf{k} \times (\mathbf{k} \times \mathbf{e})]_i + \omega^2 \mu \varepsilon_{ij} e_j = 0.$$

(e) Show that the energy flux $\mathbf{N}$ identified above, applied to the situation in part (d), points in the direction of wave propagation when the polarisation is an eigenvector of the matrix ε_{ij}.

37B General Relativity

(a) Consider a linearized gravitational plane wave of the form

$$\bar{h}_{\mu\nu} = H_{\mu\nu} e^{ik_\rho x^\rho}$$

where $H_{\mu\nu}$ is independent of x^α, $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$ is the trace-reversed perturbation to the Minkowski metric $\eta_{\mu\nu}$, and we are using Lorentz gauge $\partial^\mu \bar{h}_{\mu\nu} = 0$.

(i) What restrictions are there on k^μ and $H_{\mu\nu}$? Justify your answers.

(ii) Derive the residual gauge symmetry remaining in $H_{\mu\nu}$, even after imposing Lorentz gauge.

[*You may use:* $G_{\mu\nu} = -\frac{1}{2}\partial^\rho \partial_\rho \bar{h}_{\mu\nu} + \partial^\rho \partial_{(\mu} \bar{h}_{\nu)\rho} - \frac{1}{2}\eta_{\mu\nu}\partial^\rho \partial^\sigma \bar{h}_{\rho\sigma}$.]

(b) Suppose that LIGO detects the merger of two black holes, each of which is about 30 solar masses, from an event which takes place approximately a few billion lightyears away.

(i) Estimate the frequency (in Hz) of the gravitational wave source, from the perspective of a hypothetical observer close to the binary system and at rest with respect to it, during the last orbit of the black holes before they merge. In solving this problem you may use the (Newtonian) Kepler's law:

$$T^2 = \frac{4\pi^2}{GM} r^3.$$

Here T is the period and for purposes of estimation you may take $r = 6MG/c^2$, the general relativistic formula for the inner-most stable circular orbit for a test particle in a Schwarzschild geometry. As these assumptions are inexact, do not keep more than one significant figure.

[*You may use:* $c \approx 3.0 \times 10^8$ m/s, $G \approx 6.7 \times 10^{-11}\,\mathrm{m}^3/(\mathrm{kg\,s}^2)$, and the solar mass $M_\odot \approx 2.0 \times 10^{30}\,\mathrm{kg}$.]

(ii) Write down a Big Bang metric suitable for calculations in our universe, which is spatially flat. You may leave the scale factor $a(t)$ as an undetermined function (where t is the proper time).

Let t_e be the time of emission, and t_o be the time of detection. Write down a formula for the frequency of the gravitational wave as it is observed by LIGO, from the perspective of Earth's local reference frame. [For purposes of solving this problem, you may treat the Earth and the binary black hole system as both being at rest relative to the cosmological frame of reference.]

38C Fluid Dynamics II

Consider a two-dimensional wake of constant width $2h$ in an otherwise uniform horizontal flow of speed U. The unperturbed velocity $\mathbf{u} = u\mathbf{e}_x$, where $\mathbf{e}_x$ is a unit vector in the x-direction, is given by

$$u = \begin{cases} U, & y > h \\ 0, & -h < y < h \\ U, & y < -h. \end{cases}$$

The two shear layers at $y = \pm h$ are perturbed symmetrically so that at time t their location is $y = \pm[h + \eta(x,t)]$. The flow is assumed to be irrotational everywhere, the fluid is inviscid and the effects of gravity may be ignored.

(a) Sketch the unperturbed flow and the shape of the deformed shear layers.

(b) State the equation satisfied by the velocity potential ϕ and all the boundary conditions applicable to the three fluid domains ($y < -h - \eta(x,t)$, $-h - \eta(x,t) < y < h + \eta(x,t)$ and $y > h + \eta(x,t)$).

(c) What are the conditions on η and $\partial\eta/\partial x$ necessary in order to linearise the equations and boundary conditions? State the linearised versions of the boundary conditions on ϕ and its derivatives valid under those conditions.

(d) Justify why a full description of the linearised problem is provided by considering solutions of the form

$$\eta(x,t) = \mathrm{Re}\left\{\eta_0 \exp(ikx + \sigma t)\right\},$$

where Re is the real part.

(e) Solve for the dispersion relation linking σ and k with the parameters of the problem. [*Hint: The specified symmetry of the perturbation may allow simplification of the algebra.*] Deduce the conditions on k under which the wake flow is unstable.

(f) In the limit $hk \gg 1$, interpret the result for σ in light of what you know about the Kelvin-Helmholtz instability.

39C Waves

For adiabatic motion of an ideal gas, the pressure p is given in terms of the density ρ by a relation of the form

$$p(\rho) = p_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \tag{$\dagger$}$$

where p_0, ρ_0 and γ are positive constants, with $\gamma > 1$. For such a gas, you are given that the compressive internal energy per unit volume W can be expressed as

$$W(\rho) = \frac{p(\rho)}{\gamma - 1}.$$

(a) For one-dimensional motion with speed u, write down expressions for the mass flux and the momentum flux. Using the expressions for the energy flux $u(p + W + \frac{1}{2}\rho u^2)$ and the mass flux, deduce that if the motion is steady then

$$\frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{1}{2}u^2 = C, \tag{$\star$}$$

for some constant C.

(b) A one-dimensional shock wave propagates at constant speed along a tube containing the gas. Upstream of the shock the gas is at rest with pressure p_0 and density ρ_0. Downstream of the shock the pressure is maintained at the constant value $p_1 = (1 + \beta)p_0$ with $\beta > 0$. Show that

$$\frac{\rho_1}{\rho_0} = \frac{2\gamma + (\gamma + 1)\beta}{2\gamma + (\gamma - 1)\beta}, \tag{$\ddagger$}$$

assuming that ($\star$) holds throughout the flow.

(c) For small β, show that the density ratio ($\ddagger$) from part (b) satisfies approximately the adiabatic relation ($\dagger$), correct to $\mathcal{O}(\beta^2)$.

40C Numerical Analysis

(a) Define the *Rayleigh quotient* of a matrix $A \in \mathbb{R}^{n\times n}$ at a vector $\mathbf{x} \in \mathbb{R}^n$. Describe the method of *Rayleigh quotient iteration* to compute an eigenvalue of a matrix.

In the remainder of the question $A \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{R}$ is a simple eigenvalue of A. $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, with $\|\mathbf{u}\|_2 = \|\mathbf{v}\|_2 = 1$, are respectively the left and right eigenvectors of A associated with the eigenvalue λ. We define $s(\lambda) = 1/|\mathbf{u}^T\mathbf{v}|$ to be the *sensitivity* of the eigenvalue λ.

When A is to be regarded as depending on a parameter t the notation $A(t)$ will be used, with corresponding use of $\lambda(t)$, $\mathbf{u}(t)$ and $\mathbf{v}(t)$.

(b) Let $E \in \mathbb{R}^{n\times n}$ be a perturbation matrix and let $\lambda(t)$ be an eigenvalue of $A(t) = A(0) + tE$ with $t \in \mathbb{R}$. Assuming $\lambda(t)$ is differentiable at $t = 0$, show that

$$|\lambda'(0)| \leqslant \frac{\|E\|_2}{|\mathbf{u}(0)^T\mathbf{v}(0)|}, \tag{1}$$

where $\|E\|_2$ is the operator norm of E.

[*Hint: consider* $\mathbf{u}(0)^T A(t)\mathbf{v}(t)$.]

(c) What can you say about the sensitivity $s(\lambda)$ if A is a symmetric matrix? More generally, what can you say if A is a normal matrix?

(d) Let

$$A = \begin{pmatrix} \lambda_1 & 1 & & \\ & \lambda_2 & 1 & \\ & & \ddots & 1 \\ & & & \lambda_n \end{pmatrix},$$

where $\lambda_1 = 1$, and $\lambda_i = 1 - 1/i$ for $i \geqslant 2$. Show that for the eigenvalue $\lambda = \lambda_1 = 1$, the sensitivity $s(\lambda)$ is at least $n!$.

(e) Consider applying Rayleigh quotient iterations to compute the eigenvalue λ of a matrix A. Upon termination of the algorithm, we obtain $\tilde{\mathbf{v}} \in \mathbb{R}^n$, $\|\tilde{\mathbf{v}}\|_2 = 1$ and $\tilde{\lambda} \in \mathbb{R}$ such that

$$\|A\tilde{\mathbf{v}} - \tilde{\lambda}\tilde{\mathbf{v}}\|_2 = \epsilon$$

where ϵ is the machine precision. Show that $|\tilde{\lambda} - \lambda| \lesssim \epsilon s(\lambda)$.

[*Hint: construct a perturbation matrix E such that $(A + E)\tilde{\mathbf{v}} = \tilde{\lambda}\tilde{\mathbf{v}}$ and use the approximation* $|\lambda(1) - \lambda(0)| \approx |\lambda'(0)|$.]

END OF PAPER