UNIVERSITY OF CAMBRIDGE

MAT2

MATHEMATICAL TRIPOS Part II

Thursday, 08 June, 2023 9:00am to 12:00pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most six questions** from Section I and from any number of questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Number Theory

(a) Prove that for $n \geqslant 1$ we have

$$\frac{2^{2n}}{2n+1} \leqslant \binom{2n}{n} \leqslant (2n)^{\sqrt{2n}} \prod_{p \leqslant 2n,\ p \text{ prime}} p.$$

[The formula for the exact power of p dividing $n!$ may be quoted without proof.]

(b) Deduce that $\sum_{p \leqslant x,\ p \text{ prime}} \log p \geqslant \frac{1}{2}x$ for all x sufficiently large.

(c) A positive integer n is called *decisive* if every integer $1 < a < n$ coprime to n is in fact prime. Prove that there are only finitely many decisive numbers.

2F Topics In Analysis

State Runge's theorem on polynomial approximation.

Which of the following statements are true and which false? Give reasons.

(i) Let $E = \{x + iy : x, y \geqslant 0\}$ and Ω be an open set containing E. Then, if $f : \Omega \to \mathbb{C}$ is analytic, we can find a sequence of polynomials converging uniformly on E to f.

(ii) Let $E = \{x + iy : x, y \geqslant 0\}$ and Ω be an open set containing E. Then, if $f : \Omega \to \mathbb{C}$ is analytic, we can find a sequence of polynomials converging pointwise on E to f.

(iii) Suppose Ω is open, K_1, K_2 are compact subsets of Ω, $f : \Omega \to \mathbb{C}$ is analytic and there exist polynomials $P_{j,n}$ with $P_{j,n} \to f$ uniformly on K_j. Then there exist polynomials P_n with $P_n \to f$ uniformly on $K_1 \cup K_2$.

(iv) Let $I = \{x + iy : 1 \geqslant x \geqslant 0, y = 0\}$. If $f : I \to \mathbb{C}$ is continuous, then we can find polynomials P_n such that $P_n \to f$ uniformly on I.

3I Coding and Cryptography

Let C be a binary linear $[n, m, d]$-code.

Define (i) the *parity check extension* C^+ of C and (ii) the *punctured code* C^- (assuming $n \geqslant 2$). Show that C^+ and C^- are both linear.

What is the *shortening* C' of C (assuming $n \geqslant 2$)? When is C' a linear code?

For the changes to C defined in (i) and (ii), describe the effect of both these changes on the generator and parity check matrices. For the case of (ii) you may assume that $d \geqslant 2$ and you puncture in the last place.

4I Automata & Formal Languages

(a) Let $G = (\Sigma, V, P, S)$ be a formal grammar and let $\Omega = \Sigma \cup V$. Define $\mathcal{L}(G)$.

[You do not need to define the binary relation $\xrightarrow{G}$ on Ω^*.]

(b) Define what it means for two grammars to be *equivalent*.

(c) Define what it means for two grammars to be *isomorphic*.

(d) Fix $\Sigma = \{a, b, c\}$ and consider the following pairs of grammars with start symbol S and given by their respective sets of productions P_0 and P_1; for each pair, determine whether they are equivalent or non-equivalent. Justify your answers.

(i) $P_0 = \{S \to Aa,\ S \to Sb,\ A \to Ab,\ A \to a,\ B \to Aa,\ B \to b\}$,
$P_1 = \{S \to Sb,\ C \to Da,\ C \to b,\ D \to Db,\ D \to a,\ S \to Da\}$.

(ii) $P_0 = \{S \to AB,\ A \to Aa,\ A \to a,\ B \to Bb,\ B \to b,\ AB \to c\}$,
$P_1 = \{S \to XabY,\ X \to Xa,\ X \to a,\ Y \to Yb,\ Y \to b,\ XY \to c\}$.

(iii) $P_0 = \{S \to aAa,\ A \to bAb,\ A \to b\}$,
$P_1 = \{S \to aYa, Y \to ZZ, Z \to aZa, Z \to bZY, Z \to YZ, Y \to bYb, Y \to b\}$.

[You may assume that isomorphic grammars are equivalent.]

5J Statistical Modelling

Write down the density function of a one-parameter exponential family with natural parameter θ and sufficient statistic Y. Define the *deviance* $D(\theta_1, \theta_2)$ from θ_1 to θ_2, and show that it is equal to

$$D(\theta_1, \theta_2) = 2\{(\theta_1 - \theta_2)\mu_1 - K(\theta_1) + K(\theta_2)\},$$

where μ_1 is the mean parameter corresponding to θ_1 and $K(\cdot)$ is the cumulant function of the exponential family.

Derive the deviance from the Poisson distribution with mean μ_1 to the Poisson distribution with mean μ_2, and find the second order Taylor approximation of the deviance as $\mu_2 \to \mu_1$. [*Hint: Recall that if Y follows a Poisson distribution with mean μ, then* $\mathbb{P}(Y = k) = \mu^k e^{-\mu}/k!$, $k = 0, 1, \ldots$.]

6C Mathematical Biology

A gene product with concentration g is produced by a chemical S of concentration s, is autocatalysed and degrades linearly according to the kinetic equation

$$\frac{dg}{dt} = f(g,s) = s + k\frac{g^2}{1+g^2} - g,$$

where $k > 2$ is a constant.

First consider the case $s = 0$. Show that there are two positive steady states, and determine their stability. Sketch the reaction rate $f(g,0)$.

The system starts in the steady state $g = 0$ with $s = 0$. The value of s is then increased to the value s_1, held at this value for a long time, and then reduced to zero. Show that, if s_1 is greater than a value $s_c(k)$, a biochemical switch can be achieved to a state $g = g_* > 0$ whose value you should determine. Give a clear mathematical specification of the value $s_c(k)$. [An explicit formula is not needed.]

For the case $k \gg 1$, use a suitable approximate form of $f(g,s)$ to show that $s_c(k) \simeq Ck^{-1}$ where C is a constant that you should derive.

7E Further Complex Methods

Consider the differential equation

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0.$$

State the conditions on $p(z)$ and $q(z)$ for the point $z = z_0$, with z_0 finite, to be (i) an *ordinary point* and (ii) a *regular singular point*. Derive the corresponding conditions for $z_0 = \infty$.

Determine the most general forms of $p(z)$ and $q(z)$ for which $z = 0$ and $z = \infty$ are regular singular points and all other points are ordinary points. Give the corresponding general form of the solution.

Deduce the further restriction on the form of $p(z)$ and $q(z)$ if $z = 0$ is the only regular singular point and all other points are ordinary points.

8D Classical Dynamics

Consider a 3-dimensional system with phase space coordinates $(\mathbf{q}, \mathbf{p})$.

(a) Define the *Poisson bracket* $\{f, g\}$ of two smooth functions on phase space.

(b) Show that $f(\mathbf{q}, \mathbf{p})$ is conserved along a particle's trajectory if and only if $\{f(\mathbf{q}, \mathbf{p}), H\} = 0$, where H is the Hamiltonian.

(c) Derive a constraint satisfied by a function $f(\mathbf{q}, \mathbf{p})$ given that $\{f(\mathbf{q}, \mathbf{p}), \mathbf{q} \cdot \mathbf{p}\} = 0$. Show that any smooth function obeying $f(\lambda\mathbf{q}, \lambda^{-1}\mathbf{p}) = f(\mathbf{q}, \mathbf{p})$, where λ is a real constant, satisfies this constraint.

9B Cosmology

The equilibrium number density of fermions of mass m at temperature T and chemical potential μ is

$$n = \frac{4\pi g_s}{h^3} \int_0^\infty \frac{p^2 \mathrm{d}p}{\exp\left[\frac{E(p)-\mu}{k_B T}\right] + 1},$$

where g_s is the degeneracy factor, $E(p) = c\sqrt{p^2 + m^2c^2}$, c is the speed of light, k_B is the Boltzmann constant, p is the magnitude of the particle momentum and h is Planck's constant. For a non-relativistic gas with $pc \ll mc^2$ and $k_BT \ll mc^2 - \mu$, show that the number density becomes

$$n = g_s \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} \exp\left[\frac{\mu - m\,c^2}{k_B T}\right] . \qquad (\star)$$

[You may assume that $\int_0^\infty \mathrm{d}x\, x^2\, e^{-x^2/\alpha} = \sqrt{\pi}\alpha^{3/2}/4$ for $\alpha > 0$.]

Before recombination, equilibrium is maintained between neutral hydrogen, free electrons, protons and photons through the interaction

$$p + e^- \leftrightarrow H + \gamma\,.$$

Using the non-relativistic number density $(\star)$, deduce Saha's equation relating the electron and hydrogen number densities,

$$\frac{n_e^2}{n_H} \approx \left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} \exp\left[-\frac{E_{\mathrm{bind}}}{k_B T}\right] ,$$

where $E_{\mathrm{bind}} = (m_p + m_e - m_H)c^2$ is the hydrogen binding energy. State clearly any assumptions made.

10D Quantum Information and Computation

(a) Given two positive integers N and a which are coprime to each other (with $1 < a < N$), define the *order* of $a \bmod N$.

(b) For such a pair of integers (a, N), the *modular exponential function* $f : \mathbb{Z} \to \mathbb{Z}_N$, is defined as $f : k \mapsto a^k \bmod N$, where $\mathbb{Z}_N := \{0, 1, \ldots, N-1\}$. Prove that f is a periodic function and determine its period (clearly stating any theorem that you use).

(c) Suppose that we would like to factorise $N = 33$ and we pick $a = 10$. Following the argument presented in the lecture for Shor's algorithm, show how the order of $a \bmod N$ can be used to factorise N. Find the order of $a \bmod N$ by hand and hence factorise N.

(d) Recall that Shor's algorithm for factoring an integer N involves an application of the quantum Fourier transform on m qubits and a subsequent measurement of these m qubits which yields an integer c, where $0 \leqslant c < 2^m$. Suppose we want to factor the number $N = 21$; we pick $a = 8$, $m = 9$ and get the measurement result $c = 256$. Show how you can find the order of $a \bmod N$ from this measurement result. [You should clearly state any results that you use from the lectures.]

SECTION II

11G Number Theory

Explain what it means for a positive definite integral binary quadratic form to be *reduced*. Let $d < 0$ be an integer with $d \equiv 0$ or $1 \pmod 4$. Define the class number $h(d)$ and prove that $1 \leqslant h(d) < \infty$.

Let q be a prime number with $q \equiv 3 \pmod 8$. Show that $h(-8q) \geqslant 2$. Further show that if $h(-8q) = 2$ then a prime number p greater than q is represented by $x^2 + 2qy^2$ if and only if $p \equiv \pm 1 \pmod 8$ and p is a quadratic residue mod q.

12I Automata & Formal Languages

Let Σ be an alphabet and $\mathbb{W}$ the set of words over Σ. Let $D = (\Sigma, Q, \delta, q_0, F)$ be a deterministic automaton.

(i) Define $\mathcal{L}(D)$, the set of words accepted by the automaton D, precisely defining all auxiliary functions needed for your definition.

(ii) State the *pumping lemma* for the language $\mathcal{L}(D)$. Specify the pumping number precisely in terms of D.

[No proof is required.]

(iii) Let $\Sigma = \{a, b\}$. Consider the regular language

$$L := \{wa^k\,;\; w \in \Sigma^* \text{ with } |w| \leqslant 10 \text{ and } k > 0\}.$$

Show that the minimal deterministic automaton for L has at least ten states.

Let $A \subseteq \mathbb{W}$. Define an equivalence relation on $\mathbb{W}$ by

$$v \sim_A w :\iff \text{for all } u, \text{ we have } vu \in A \text{ if and only if } wu \in A.$$

(iv) Let $A \subseteq \mathbb{W}\backslash\{\varepsilon\}$. Show that A is a regular language if and only if the relation $\sim_A$ has finitely many equivalence classes.

13C Mathematical Biology

Consider the reaction-diffusion system in one spatial dimension $-\infty < x < \infty$,

$$\begin{aligned} \frac{\partial u}{\partial t} &= D\frac{\partial^2 u}{\partial x^2} + f(u) + \rho(u - v), \qquad (1) \\ \epsilon\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + u - v, \qquad (2) \end{aligned}$$

where $D > 0$ is the activator diffusion constant, $\rho > 0$ is a constant, and $0 < \epsilon \ll 1$ so that the inhibitor v is a fast variable relative to the activator u. The nonlinear function $f(u)$ is taken to have the properties $f(0) = 0$ and $f'(0) = -r$ with $0 \leqslant r \leqslant 1$.

(a) Setting $\epsilon = 0$, show that the inhibitor dynamics can be solved to express the Fourier amplitude $\hat{v}(k,t)$ of the inhibitor in terms of the Fourier amplitude $\hat{u}(k,t)$ of the activator.

(b) Using the relation found in part (a), and linearising around the state $u = 0$, find the dynamics of perturbations around $u = 0$ and thus the growth rate $\sigma(k)$ as a function of the wavenumber k.

(c) From the result in (b), show that the threshold of a pattern-forming instability lies along a curve in the $r - \rho$ plane given by

$$\rho_c(r) = \left(\sqrt{r} + \sqrt{D}\right)^2, \qquad (3)$$

along which the critical wavenumber is

$$k_c = \left(\frac{r}{D}\right)^{1/4}. \qquad (4)$$

14B Cosmology

Small density perturbations $\delta_{\mathbf{k}}(t)$ in pressureless matter inside the cosmological horizon obey the following Fourier evolution equation

$$\ddot{\delta}_{\mathbf{k}} + 2\frac{\dot{a}}{a}\dot{\delta}_{\mathbf{k}} - \frac{4\pi G\bar{\rho}_c}{c^2}\delta_{\mathbf{k}} = 0\,,$$

where the overdot indicates differentiation with respect to time t, $a(t)$ the scale factor of the universe, G is Newton's constant, c the speed of light, $\mathbf{k}$ is the co-moving wavevector and $\bar{\rho}_c$ is the background density of the pressureless gravitating matter.

(a) Let $t_{\rm eq}$ be the time of matter-radiation equality. Show that during the matter-dominated epoch, $\delta_{\mathbf{k}}$ behaves as

$$\delta_{\mathbf{k}}(t) = A(\mathbf{k})\left(\frac{t}{t_{\rm eq}}\right)^{2/3} + B(\mathbf{k})\left(\frac{t}{t_{\rm eq}}\right)^{-1}\,,$$

where $A(\mathbf{k})$ and $B(\mathbf{k})$ are functions of $\mathbf{k}$ only.

(b) For a given wavenumber $k \equiv |\mathbf{k}|$, show that the time t_H at which this mode crosses inside the horizon, *i.e.* $c\,t_H \approx 2\pi a(t_H)/k$, is given by

$$\frac{t_H}{t_0} \approx \begin{cases} \left(\frac{k_0}{k}\right)^3\,, & t_H > t_{eq} \\ \dfrac{1}{\sqrt{1+z_{\rm eq}}}\left(\dfrac{k_0}{k}\right)^2\,, & t_H < t_{eq} \end{cases}$$

where t_0 is the age of this universe, $k_0 \equiv 2\pi/(c\,t_0)$, and the matter-radiation equality redshift is given by $1 + z_{\rm eq} = (t_0/t_{\rm eq})^{2/3}$.

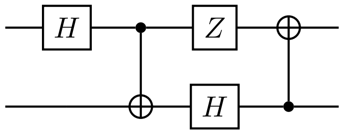
(c) Assume that early in the radiation era there is no significant perturbation growth in $\delta_{\mathbf{k}}$ and that primordial perturbations from inflation are scale-invariant with a constant amplitude at the time of horizon crossing given by $\langle\delta_{\mathbf{k}}(t_H)^2\rangle \approx V^{-1}C/k^3$, where C is a constant and V is a volume. Use the results in parts (a) and (b) to project these perturbations forward to $t_0 \gg t_H$, and show that the power spectrum of perturbations today (at $t = t_0$) is given by

$$P(k) \equiv V\langle\delta_{\mathbf{k}}(t_0)^2\rangle = \begin{cases} \frac{Ck}{k_0^4}\,, & k < k_{eq} \\ \frac{Ck_{\rm eq}}{k_0^4}\left(\frac{k_{\rm eq}}{k}\right)^3\,, & k > k_{\rm eq} \end{cases}$$

where $k_{\rm eq}$ is the wavenumber of modes that entered the horizon at matter-radiation equality.

15D Quantum Information and Computation

Consider the following quantum circuit C:

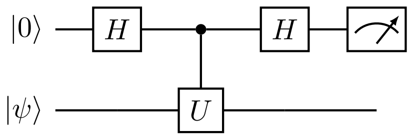


(a) Suppose the state $|0\rangle|0\rangle$ is sent through the circuit. What is the state at the output? Suppose each of the two qubits are measured in the computational basis. What is the distribution of measurement outcomes?

(b) Let V denote the unitary operator corresponding to the circuit C. Draw the quantum circuit corresponding to the inverse operator V^{-1}.

(c) The SWAP gate for two qubits is defined as $\text{SWAP}\,|x\rangle|y\rangle = |y\rangle|x\rangle$, where $x, y \in \{0,1\}$. Show that the SWAP gate can be implemented as a combination of CNOT gates and draw the corresponding quantum circuit.

(d) Let U be a unitary operator with eigenstate $|\psi\rangle$ such that $U|\psi\rangle = e^{i\theta}|\psi\rangle$. Consider the following quantum circuit:



Write down the final state at the end of the algorithm. What is the probability that the outcome 1 is observed when the first register is measured in the computational basis? Suppose we are promised that either $U|\psi\rangle = |\psi\rangle$ or $U|\psi\rangle = -|\psi\rangle$, but we have no other information about U and $|\psi\rangle$. Show that the above circuit can be used to determine which of these is the case with certainty.

16H Logic and Set Theory

In this question we work in a fixed model V of ZFC.

(a) Prove that every set has a transitive closure. [If you apply the Axiom of Replacement to a function-class F, you must explain clearly why F is indeed a function-class.]

(b) State the Axiom of Foundation and the Principle of ϵ-Induction, and show that they are equivalent (in the presence of the other axioms of ZFC).

(c) We say that a set x is *reasonable* if every member of $TC(\{x\})$ is countable. Which of the following are true and which are false? Justify your answers.

(i) A set is reasonable if and only if $TC(\{x\})$ is countable.

(ii) The reasonable sets are all members of V_α, for some α.

(iii) The reasonable sets form a model of ZFC.

[In (c) you may assume any results from the course.]

17H Graph Theory

(a) Let $r \geqslant 2$. Prove Turán's theorem in the form: if G is an n vertex graph that does not contain a K_{r+1} then

$$e(G) \leqslant \left(1 - \frac{1}{r}\right)\frac{n^2}{2}.$$

(b) For $t \leqslant n-1$, show that if G is a connected n vertex graph with $\delta(G) \geqslant t/2$ then G contains a path P_t of length t.

(c) For graphs G, H define the *Ramsey number* $r(G,H)$ to be the minimum n such that every red-blue colouring of the edges of K_n contains either a red copy of G or a blue copy of H. For $s \geqslant 2, t \geqslant 1$, show that

$$r(K_s, P_t) \geqslant (s-1)t + 1.$$

(d) Show further that for $s \geqslant 2, t \geqslant 1$ we have

$$r(K_s, P_t) = (s-1)t + 1.$$

18I Galois Theory

(a) Show that a finite subgroup of the multiplicative group of a field is cyclic.

(b) What is a *primitive n-th root of unity*? Show that if K contains a primitive m-th root of unity and a primitive n-th root of unity, then it contains a primitive N-th root of unity, where N is the least common multiple of m and n.

(c) Define the *cyclotomic polynomials* Φ_n and show that they have integer coefficients. Show also that the reduction of Φ_n modulo a prime p is separable if p does not divide n.

(d) Let K be a field of characteristic zero, L a splitting field for Φ_n over K, and let $G = \mathrm{Gal}(L/K)$ be its Galois group. Write down an injective homomorphism from G into $(\mathbb{Z}/n\mathbb{Z})^\times$, and show that it is surjective if and only if Φ_n is irreducible over K.

(e) Let L be a splitting field for Φ_n over $\mathbb{Q}$. Show that the number of roots of unity in L is n if n is even, and $2n$ if n is odd. [You may assume that Φ_n is irreducible over $\mathbb{Q}$.]

19H Representation Theory

(a) State and prove Burnside's lemma. Deduce that if a finite group G acts 2-transitively on a set X then the corresponding permutation representation $\mathbb{C}X$ decomposes as a direct sum of two non-isomorphic irreducible representations.

(b) Let $G = S_n$ act naturally on the set $X = \{1, \ldots, n\}$. For each non-negative integer r, let X_r be the set of all r-element subsets of X equipped with the natural action of G, and π_r be the character of the corresponding permutation represention. If $0 \leqslant l \leqslant k \leqslant n/2$, show that

$$\langle \pi_k, \pi_l \rangle_G = l + 1.$$

Deduce that $\pi_r - \pi_{r-1}$ is a character of an irreducible representation for each $1 \leqslant r \leqslant n/2$.

What happens for $r > n/2$?

20G Algebraic Topology

Consider the set $X \subset S^3$ given by $X = \{(x_1, x_2, x_3, x_4) \in S^3 \,:\, |x_4| \leqslant \frac{1}{2}\}$ and its boundary $\partial X = \{(x_1, x_2, x_3, x_4) \in S^3 \,:\, |x_4| = \frac{1}{2}\}$. Define Y and ∂Y to be the image of X and ∂X in $\mathbb{RP}^3 = S^3/\sim$, where $x \sim -x$. Show that Y is homotopy equivalent to $\mathbb{RP}^2$. Compute $H_*(\mathbb{RP}^3)$. [You may assume $\mathbb{RP}^3$ admits a triangulation containing Y and ∂Y as subcomplexes, and may use $H_*(\mathbb{RP}^2)$ if you state it precisely.]

Let $f : \partial Y \to \partial Y$ be the identity map, and define Z to be the space obtained by identifying two copies of Y along their boundary: $Z = Y \cup_f Y$. Compute $H_*(Z)$ and $\pi_1(Z, z_0)$, where $z_0 \in Z$. The universal covering space of Z is homeomorphic to a familiar space. What is it?

21F Linear Analysis

Recall that a topological space X is called *normal* if for any pair of non-empty disjoint closed subsets $A, B \subset X$, there is a pair of disjoint open subsets $U_1, U_2 \subset X$ so that $A \subset U_1$ and $B \subset U_2$. Also recall that the Urysohn lemma states that in a normal topological space X, for any pair of non-empty disjoint closed subsets $A, B \subset X$, there is an $f : X \to [0,1]$ continuous so that $f = 0$ on A and $f = 1$ on B.

(a) State and prove the Tietze extension theorem. [You may use the Urysohn lemma.]

(b) Consider a normal topological space X, and $A \subset X$ a non-empty closed subset that can be realised as a countable intersection of open sets. Show that there exists $f : X \to [0,1]$ continuous so that f vanishes on A and on A only.

(c) Consider a normal topological space X, and $A, B \subset X$ a pair of non-empty disjoint closed subsets that can both be realised as countable intersections of open sets. Show that there exists $f : X \to [0,1]$ continuous so that f vanishes on A and on A only, and is equal to 1 on B and on B only.

22F Analysis of Functions

(a) Let $U \subset \mathbb{R}^n$ be bounded and open, and let $m^2 > 0$. Given $f \in L^2(U)$, define what it means for u to be a *weak solution* to

$$\begin{aligned} -\Delta u + m^2 u &= f && \text{in } U \\ u &= 0 && \text{on } \partial U. \end{aligned}$$

Show that for any $f \in L^2(U)$ there is a unique weak solution u and let $Tf = u$. Show that $T : L^2(U) \to L^2(U)$ defines a compact operator. [You may use any theorems from the course if you state them carefully.]

(b) Let $U \subset \mathbb{R}^n$ be bounded and open, and let $(u_k) \subset L^2(U)$ be a sequence such that $u_k \rightharpoonup u$ weakly in $L^2(U)$. Assume that $\sup_k \int_{\{|p| \geqslant t\}} (|\hat{u}_k(p)|^2 + |\hat{u}(p)|^2)\, dp \to 0$ as $t \to \infty$. Show that then $u_k \to u$ in $L^2(U)$.

(c) Given $f \in H^r(\mathbb{R}^n)$, assume that $u \in L^2(\mathbb{R}^n)$ satisfies

$$\Delta^{2022} u + u = f \qquad \text{on } \mathbb{R}^n$$

in distributional sense. For which n is u a function that solves the equation in the classical sense? [You may cite any theorems from the course.]

23F Riemann Surfaces

State the uniformisation theorem.

Write down a list of all Riemann surfaces uniformised by $\mathbb{C}$ and $\mathbb{C}_\infty$, and prove that your list is complete. [You may assume that, if a Riemann surface R is uniformised by a Riemann surface X, then R is conformally equivalent to the quotient of X by a group of conformal equivalences of X acting freely and properly discontinuously. You may also assume standard facts about the groups of conformal equivalences of $\mathbb{C}$ and $\mathbb{C}_\infty$.]

Prove that any domain $D \subseteq \mathbb{C}$ with a complement containing more than one point is uniformised by the open unit disc $\mathbb{D}$.

Suppose there is a holomorphic embedding $\mathbb{C}_* \to R$, where R is a compact Riemann surface. Prove that R is conformally equivalent to the Riemann sphere.

24G Algebraic Geometry

[In this question all algebraic varieties are over $\mathbb{C}$.]

State Hilbert's Nullstellensatz for affine varieties. Suppose that I is a homogeneous ideal such that $\mathbb{V}(I) \subset \mathbb{P}^n$ is empty. What are the possibilities for I?

Let V be a smooth quadric hypersurface in $\mathbb{P}^3$. Construct a pair of disjoint, smooth and projective curves lying on V. Deduce that V is not isomorphic to $\mathbb{P}^2$.

Let W be a smooth projective curve. Prove that every rational map from W to a projective variety is a morphism. Give an example showing that if W is singular, this statement can fail.

Construct an algebraic variety $Z \subset \mathbb{P}^2 \times \mathbb{P}^1$ and a surjective morphism $\pi : Z \to \mathbb{P}^1$ such that there exists a point $p \in \mathbb{P}^1$ whose preimage $\pi^{-1}(p)$ is a smooth projective curve of genus 1, and another point $q \in \mathbb{P}^1$ such that $\pi^{-1}(q)$ has exactly 3 irreducible components.

25G Differential Geometry

(a) Let $S \subset \mathbb{R}^3$ be an oriented surface. Define the Gaussian curvature $K(p)$ and mean curvature $H(p)$ of S at p. Prove that these are Euclidean invariants, i.e. if $E : \mathbb{R}^3 \to \mathbb{R}^3$ is a proper Euclidean motion and $\widetilde{S} = E(S)$ and $\widetilde{K}$, $\widetilde{H}$ denote the Gaussian and mean curvature of $\widetilde{S}$ (with a choice of orientation that you should describe), respectively, then $\widetilde{K}(E(p)) = K(p)$, $\widetilde{H}(E(p)) = H(p)$. Do the Gaussian and mean curvatures depend on the orientation?

(b) Show that there is no Euclidean motion taking a piece of the cylinder to a piece of the plane, and infer that for a general surface S, the property $K = 0$ identically does not imply that there is a Euclidean motion taking S to a piece of the plane. Exhibit similarly two surfaces each with $K = 1$ identically, no respective pieces of which are related by a Euclidean motion, and similarly two surfaces each with $K = -1$ identically.

(c) Let $\mathcal{R} \subset \mathbb{R}^3$ be a compact submanifold of dimension 3 with connected boundary $S = \partial\mathcal{R}$. Note that $S \subset \mathbb{R}^3$ is an orientable surface and can be oriented by the unique normal vector N pointing towards $\mathcal{R}$. Now let $\widetilde{S} \subset \mathcal{R}$ be a surface (without boundary). Suppose $p \in S \cap \widetilde{S}$. Show that $\widetilde{H}(p) \geqslant H(p)$, where H and $\widetilde{H}$ denote the mean curvature of S and $\widetilde{S}$, respectively, where both surfaces are (locally) oriented at p by the N described above. Is it necessarily the case that $\widetilde{K}(p) \geqslant K(p)$? Justify your answer.

26K Probability and Measure

(a) State (without proof) Birkhoff's ergodic theorem. Show that convergence in that theorem holds in $L^1(\mu)$, whenever μ is a probability measure. [You may use convergence results for integrals without proof, provided they are clearly stated.]

(b) Now consider $(0,1]$ equipped with its Borel σ-algebra $\mathcal{B}$ and Lebesgue measure μ. For $A \in \mathcal{B}$, $a \in (0,1] \setminus \mathbb{Q}$, and

$$\theta(x) = x + a \text{ mod } 1, \quad x \in (0,1],$$

determine the μ-almost everywhere limit of $S_n(1_A)/n$ as $n \to \infty$, where

$$S_n(1_A) = 1_A + 1_A \circ \theta + \ldots 1_A \circ \theta^{n-1}.$$

[You may use without proof that θ is ergodic.]

(c) If $A = (a,b]$ for $0 < a < b < 1$, show that convergence in the last limit in fact occurs *everywhere* on $(0,1]$. [*Hint: Use your result from (b) with* $A_k = (a + k^{-1}, b - k^{-1}]$ *for all* k *large enough.*]

27J Applied Probability

(a) Define $M/M/1$ and $M/M/\infty$ queues and state (without proofs) their stationary distributions, as well as all the necessary conditions for their existence. State Burke's theorem for an $M/M/\infty$ queue.

(b) Calls arrive at a telephone exchange as a Poisson process of constant rate λ, and the lengths of calls are independent exponential random variables of parameter μ. Assuming that infinitely many telephone lines are available, set up a Markov chain model for this process.

Show that for large t the distribution of the number of lines in use at time t is approximately Poisson with mean λ/μ.

Let X_t denote the number of lines in use at time t, given that n are in use at time 0. Find $\mathbb{E}s^{X_t}$ for any $s \in [-1, 1]$. Hence or otherwise, identify the distribution of X_t.

[You may use without proof that the probability generating function of a Poisson(λ) random variable is $e^{\lambda(s-1)}$.]

(c) Compute the expected length of the busy period for an $M/M/1$ and an $M/M/\infty$ queue. (The busy period B is the length of time between the arrival of the first customer and the first time afterwards that all servers are free).

[*You may quote any result from the lectures that you need, without proof, provided it is clearly stated.*]

28K Principles of Statistics

Consider a classification problem where data are drawn from two different distributions $N(\mu_0, \Sigma_0)$ or $N(\mu_1, \Sigma_1)$, where $\mu_0, \mu_1 \in \mathbb{R}^p$ and $\Sigma_0, \Sigma_1 \in \mathbb{R}^{p\times p}$ are positive definite matrices.

Let $\pi_0 \in (0, 1)$ and $\pi_1 = 1 - \pi_0$.

(a) Define the *Bayes classifier* δ_{π_0}, and show that the decision boundary is linear when $\Sigma_0 = \Sigma_1$, and otherwise quadratic.

(b) Show that for any (Σ_0, Σ_1), the classifier described in (a) is the unique Bayes rule for the prior (π_0, π_1).

(c) Show that there exists some $\pi^* \in (0, 1)$ such that the Bayes classifier corresponding to the prior $(\pi^*, 1 - \pi^*)$ is minimax. Is the prior least favorable?

[*You may quote any result from the lectures that you need, without proof.*]

29K Stochastic Financial Models

Let $(W_t)_{t \geqslant 0}$ be a Brownian motion, and let a and b be positive constants.

(a) Let $B_0 = 0$ and $B_t = tW_{1/t}$ for $t > 0$. Show that $(B_t)_{t \geqslant 0}$ is a Brownian motion. [You may use without proof a characterisation of Brownian motion as a Gaussian process. You may also use without proof the fact that $W_t/t \to 0$ almost surely as $t \to \infty$.]

(b) Prove that $\mathbb{P}(\sup_{0 \leqslant s \leqslant t}(W_s - as) \leqslant b) = \mathbb{P}(\sup_{u \geqslant 1/t}(W_u - bu) \leqslant a)$.

(c) Use the reflection principle and the Cameron–Martin theorem to show that

$$\mathbb{P}(\sup_{0 \leqslant s \leqslant t}(W_s - as) \leqslant b) = \mathbb{P}(W_t - at \leqslant b) - e^{-2ab}\mathbb{P}(W_t - at \leqslant -b).$$

(d) Let $T = \sup\{t \geqslant 0 : W_t - at > b\}$ with the convention that $\sup \emptyset = 0$. Find $\mathbb{P}(T \leqslant t)$ in terms of the standard normal distribution function Φ.

30E Asymptotic Methods

A stationary Schrödinger equation in one dimension has the form

$$\varepsilon^2 \frac{d^2\psi}{dx^2} = -(E - V(x))\psi\,, \quad \text{for} \quad x \in \mathbb{R}\,, \tag{$*$}$$

where $\varepsilon > 0$ is assumed to be very small and the potential $V(x)$ is given by

$$V(x) = \begin{cases} \frac{1}{4}|x| & \text{for} \quad |x| \leqslant 4 \\ \sqrt{|x|} - 1 & \text{for} \quad |x| \geqslant 4 \end{cases}.$$

The connection formula for the approximate energies E of bound states ψ in $(*)$ is

$$\frac{1}{\varepsilon}\int_a^b (E - V(x))^{1/2}\,dx = (n + \frac{1}{2})\pi\,. \tag{$**$}$$

(a) State the appropriate values of a, b and n.

(b) For $E \geqslant 0$ define

$$f(E) = \int_a^b (E - V(x))^{1/2}\,dx,$$

with a, b as in (a). Find and sketch f, and deduce that for each n and ε, $(**)$ has a unique solution $E = E_n$.

(c) Show that for n fixed and ε sufficiently small, E_n can be determined explicitly and give an expression for it.

(d) Show that as $n \to \infty$ with ε fixed, E_n satisfies

$$E_n \sim c n^{\alpha}\,,$$

and determine the values of c and α.

31A Dynamical Systems

Consider the dependence of the system

$$\dot{x} = (a^2 - x)(a - y^2) \tag{1}$$

$$\dot{y} = x - y \tag{2}$$

on the parameter a. Find the fixed points and plot their location on the (a, x)-plane. Hence, or deduce, that there are bifurcations at $a = 0$ and $a = a^* > 0$ which is to be determined.

Investigate the bifurcation at $a = 0$ by making the substitutions $X = x - a^2$ and $Y = y - a^2$. Find the extended centre manifold in the form $Y(X, a)$ correct to second order. Find the evolution on the extended centre manifold and hence determine the stability of the fixed points.

Use a plot to show which branches of the fixed points in the (a, x)-plane are stable and which are unstable and state, without calculation, the type of bifurcation at a^*. Hence sketch the structure of the (x, y) phase plane close to the bifurcation at a^* where $|a - a^*| \ll 1$ in the cases i) $a < a^*$ and ii) $a > a^*$.

32E Integrable Systems

(a) Compute the group of transformations generated by the vector field

$$V = t\partial_t + x\partial_x \,,$$

and hence, or otherwise, calculate the second prolongation of the vector field V and show that V generates a group of Lie symmetries of the wave equation $u_{tt} - u_{xx} = 0$.

Use the group of symmetries you have just found for the equation $u_{tt} - u_{xx} = 0$ to obtain a group invariant solution for this equation.

(b) Compute the group of transformations generated by the vector field

$$4t^2\partial_t + 4tx\partial_x - (x^2 + 2t)\partial_u$$

and verify that they give rise to a group of Lie symmetries of the equation $u_t = u_{xx} + u_x^2$.

33B Principles of Quantum Mechanics

(a) Consider a composite system of several distinguishable particles. Describe how the multiparticle state is constructed from single-particle states. For the case of two identical particles, describe how considering the interchange symmetry leads to the definition of bosons and fermions.

(b) Consider two non-interacting, identical particles, each with spin 1. The single particle, spin-independent Hamiltonian $H(\mathbf{X}_i, \mathbf{P}_i)$ has non-degenerate eigenvalues E_n and wavefunctions $\psi_n(\mathbf{x}_i)$ where $i = 1, 2$ labels the particle and $n = 0, 1, 2, 3, \ldots$. In terms of these single-particle wavefunctions and single-particle spin states $|1\rangle$, $|0\rangle$ and $|-1\rangle$, write down all of the two-particle states and energies for (i) the ground states and (ii) the first excited states.

(c) For the system in part (b), assume now that E_n is a linear function of n. Find the degeneracy of the N^{th} energy level of the two-particle system for: (i) N even and (ii) N odd.

34D Applications of Quantum Mechanics

Let Λ be a Bravais lattice in three dimensions with primitive vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. Define the *reciprocal lattice* Λ^* and show that it is a Bravais lattice.

An incident particle of mass m and wavevector $\mathbf{k}$ scatters off a crystal which consists of identical atoms located at the vertices of a finite subset $\mathcal{S}$ of the lattice Λ,

$$\mathcal{S} \;=\; \{\mathbf{l} = l_1\mathbf{a}_1 + l_2\mathbf{a}_2 + l_3\mathbf{a}_3 : l_i \in \mathbb{Z},\ -L_i/2 \leqslant l_i \leqslant +L_i/2 \ \text{ for } \ i = 1, 2, 3\},$$

where L_1, L_2 and L_3 are positive even integers. After scattering the particle has wavevector $\mathbf{k}'$ with $|\mathbf{k}| = |\mathbf{k}'| = k$ and the scattering angle θ, with $0 \leqslant \theta \leqslant \pi$, is defined by $\mathbf{k} \cdot \mathbf{k}' = k^2 \cos\theta$. Show that the resulting scattering amplitude is proportional to

$$\Delta\left(\mathbf{k} - \mathbf{k}'\right) \;:=\; \sum_{\mathbf{l} \in \mathcal{S}} \exp\left(i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{l}\right) .$$

For L_1, L_2, $L_3 \gg 1$, show that this quantity is strongly peaked for wavevectors $\mathbf{k}$ and $\mathbf{k}'$ obeying $\mathbf{k} - \mathbf{k}' = \mathbf{q}$ for some $\mathbf{q} \in \Lambda^*$.

Consider the case where Λ is a body centered cubic lattice with primitive vectors

$$\mathbf{a}_1 = \frac{a}{2}\left(\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z\right), \qquad \mathbf{a}_2 = \frac{a}{2}\left(\mathbf{e}_x - \mathbf{e}_y + \mathbf{e}_z\right), \qquad \mathbf{a}_3 = a\,\mathbf{e}_z\,,$$

where $a > 0$ and $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ are, respectively, unit vectors in the x-, y- and z-directions. For scattering at fixed energy $E = \hbar^2 k^2/2m$ with $ka \gg 1$, find the smallest non-zero value of the scattering angle θ for which the scattering amplitude has a strong peak (i.e. a peak such as you found in the previous part of the question).

35A Statistical Physics

(a) State the formula for the Bose-Einstein distribution for the mean occupation numbers n_r of discrete single-particle states r with energies $E_r \geqslant 0$ in a gas of identical ideal Bosons in terms of $\beta = 1/k_B T$ and the chemical potential μ. Write down expressions for the total particle number N and the total energy E when the single-particle states can be treated as continuous with energies $E \geqslant 0$ and density of states $g(E)$.

(b) Consider the bosonic vibrational modes (phonons) in a two-dimensional crystal with dispersion relation $\omega = C|\mathbf{k}|^{\alpha}$, where ω is the frequency, $\mathbf{k}$ is the wavevector, and $C > 0$ and $0 < \alpha < 2$ are constants. The crystal is square with area A.

(i) Show that the density of states is

$$g(\omega) = B\omega^b \,,$$

where B and b are constants that you should determine. [You may assume that the phonons have two polarizations.]

(ii) Calculate the Debye frequency ω_D by identifying the number of single-phonon states with the total number of degrees of freedom $2n$, where n is the number of atoms in the crystal. Find the Debye temperature T_D.

(iii) Derive an expression for the total energy, leaving your answer in integral form with the integral over $x = \beta\hbar\omega$.

(iv) Now consider the case $\alpha = 1/2$. Calculate the heat capacity at constant volume C_V in the limit $T \gg T_D$. Show that $C_V \sim T^d$ in the limit $T \ll T_D$, where d is a real number that you should determine. Comment on these two results.

36A Electrodynamics

The retarded four-potential $A^\mu(\mathbf{x},t) = (\phi/c, \mathbf{A})$ due to a charge density $J^\mu(\mathbf{x}',t')$ is

$$A^\mu(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int \frac{J^\mu(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|} d^3x',$$

where the integral is over all space.

(a) Explain briefly the physical meaning of the above expression and why causality requires $t' = t_{\text{ret}}$, where $t_{\text{ret}} = t - |\mathbf{x}-\mathbf{x}'|/c$.

(b) Consider a particle of charge q moving along the worldline $y^\mu = (ct, \mathbf{y}(t))$ and let $\mathbf{R}(t) = \mathbf{x} - \mathbf{y}(t)$ be the vector from the location of the charge at time t to the field point $\mathbf{x}$. Explain why the implicit equation

$$t_{\text{ret}} + \frac{R(t_{\text{ret}})}{c} = t,$$

determining the retarded potential, can have only one solution.

(c) Hence, or otherwise, obtain the Lienard-Wiechert potentials

$$\phi(\mathbf{x},t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{R - \frac{\mathbf{v}}{c}\cdot\mathbf{R}} \quad \text{and} \quad \mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\frac{q\mathbf{v}}{R - \frac{\mathbf{v}}{c}\cdot\mathbf{R}}$$

for the charge, where $\mathbf{v} = d\mathbf{y}/dt$ is the particle velocity. Clearly specify the time at which the right hand sides are to be evaluated.

(d) For a charge moving without acceleration, show by explicit computation that the resulting potentials satisfy the gauge-fixing condition

$$\frac{1}{c^2}\frac{\partial\phi}{\partial t} + \nabla\cdot\mathbf{A} = 0.$$

37B General Relativity

(a) Let M be the mass of a star and consider a photon with impact parameter b which passes near the star. In this problem, by following the steps below, you will derive the general relativistic formula for the total angle $\delta\phi$ by which the photon bends.

The general relativistic formulae for equatorial null orbits in the Schwarzschild metric (in units where $c = G = 1$) are:

$$\frac{1}{2}\dot{r}^2 + V(r) = \frac{1}{2}E^2, \qquad V(r) = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2},$$

where dot is derivative with respect to proper time, and $L = r^2\dot{\phi}$ is the angular momentum.

(i) Write down the geodesic equation for the trajectory of the photon, parameterized by the ϕ coordinate. Switch to an inverse radial coordinate $y = 1/r$. By differentiating the geodesic equation by ϕ, show that $y'' + y = 3My^2$. Here $'$ denotes $d/d\phi$.

(ii) Solve this equation in the flat space regime ($M = 0$), for a trajectory for which $r \to \infty$ at $\phi = 0, \pi$.

(iii) Using perturbation theory in M identify a differential equation for Δy, the first order perturbation of y due to nonzero M.

(iv) Find the homogeneous and particular solutions for Δy.

(v) Taking $r \to \infty$ at $\phi = 0$, show that the leading order result for the bending of the light ray is:

$$|\delta\phi| \approx \frac{4M}{b}.$$

(b) In Nordström's theory of gravitation, the metric is required to take the form

$$g_{\mu\nu} = \phi^2\eta_{\mu\nu},$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $\phi > 0$ is a dynamical scalar field which approaches the value 1 far from any isolated gravitating system.

Write down the equation satisfied by an affinely parameterised geodesic of the metric $g_{\mu\nu}$. What can you deduce about the bending of light rays around a star of mass M in Nordström's theory? Is this result compatible with observations?

38C Fluid Dynamics II

A two-dimensional lubrication flow occurs between two rigid surfaces in a fluid that has otherwise uniform pressure p_0. The bottom surface at $y = 0$ moves in the horizontal direction with velocity $\mathbf{u} = U\mathbf{e}_x$ while the top surface at $y = h(x)$ moves towards $y = 0$ with velocity $\mathbf{u} = -V\mathbf{e}_y$, with $\mathbf{e}_x$ and $\mathbf{e}_y$ being unit vectors in the x- and y-directions respectively. Both surfaces are of length L in the x direction. Consider the instant when both occupy the region $0 < x < L$.

(a) State *all* conditions involving h, U, V and L ensuring that the flow between the two surfaces is in the lubrication limit.

(b) Solve for the flow in the x direction between the two surfaces.

(c) Use conservation of mass to derive an expression for the pressure gradient between the two surfaces as a function of x.

[*Hint: You may find it convenient to introduce the notation* $\langle f \rangle$ *to denote the mean value of a function* f *over the range* $0 \leqslant x \leqslant L$.]

(d) In the particular case $U = 0$, show that the pressure gradient is necessarily zero somewhere between the two surfaces.

(e) Find the value of U such that the force in the x direction on the bottom surface is zero at the instant considered.

39C Waves

(a) The function $\phi(x,t)$ satisfies the equation

$$\frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x} + \frac{1}{9}\frac{\partial^9 \phi}{\partial x^9} = 0,$$

where $U > 0$ is a constant.

(i) Find the dispersion relation for waves of frequency ω and wavenumber k.

(ii) Sketch both the phase velocity c_p and the group velocity c_g as functions of k.

(iii) Do wave crests move faster or slower than a wave packet?

(b) Suppose that $\phi(x,0)$ is real and given by a Fourier transform as

$$\phi(x,0) = \int_{-\infty}^{\infty} A(k)e^{ikx}dk.$$

(i) Use the method of stationary phase to obtain an approximation for $\phi(Vt,t)$ for fixed $V > U$ and large t.

(ii) If the initial condition is now restricted further to be even, so that $\phi(x,0) = \phi(-x,0)$, deduce an approximation for the sequence of times at which $\phi(Vt,t) = 0$.

(iii) What can be said about $\phi(Vt,t)$ if $V < U$? [Detailed calculation is not required in this case.]

[*Hint: You may assume that* $\int_{-\infty}^{\infty} e^{-au^2}\,du = \sqrt{\frac{\pi}{a}}$ *for* $\mathrm{Re}(a) \geqslant 0, a \neq 0$.]

40C Numerical Analysis

Let A be an $n \times n$ real symmetric positive definite matrix and consider the linear system of equations $A\mathbf{x} = \mathbf{b}$, with $\mathbf{b}, \mathbf{x} \in \mathbb{R}^n$. Let $F(\mathbf{x}) = (1/2)\mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x}$.

(a) Define the *steepest descent method with exact line search* to minimize F. Show that for the 2×2 linear system

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}, \quad \mathbf{b} = \mathbf{0} \in \mathbb{R}^2 \quad (\gamma > 1), \tag{1}$$

with the starting point $\mathbf{x}^{(0)} = (\gamma, 1)$, the k-th iterate of this method satisfies

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^*\|_2}{\|\mathbf{x}^{(0)} - \mathbf{x}^*\|_2} = \left(\frac{\kappa - 1}{\kappa + 1}\right)^k \tag{2}$$

where κ is the *condition number* of A that you should define.

Define the *conjugate gradient method*. If the conjugate gradient method is applied to this example, at most how many iterations will be needed to reach $\mathbf{x}^*$?

(b) Return to the case of general $n \times n$ A as specified at the beginning of the question. The *heavy-ball method* to minimize $F(\mathbf{x})$ is defined by the following iterations

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \alpha \nabla F(\mathbf{x}^{(k)}) + \beta(\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}), \tag{3}$$

for some constants $\alpha, \beta > 0$, with the initial point $\mathbf{x}^{(0)} = 0$. Show that $\mathbf{r}^{(k)} \in \mathcal{K}_k(A, \mathbf{b})$ where $\mathbf{r}^{(k)} = \mathbf{b} - A\mathbf{x}^{(k)}$ is the residual at the kth iterate, and $\mathcal{K}_k(A, \mathbf{b})$ is the kth Krylov subspace of A with respect to $\mathbf{b}$.

(c) Let $\mathbf{e}^{(k)} = \mathbf{x}^* - \mathbf{x}^{(k)}$ be the error for the iterates of the heavy-ball method. Show that we can find a matrix M of size $2n \times 2n$ such that

$$\begin{pmatrix} \mathbf{e}^{(k+1)} \\ \mathbf{e}^{(k)} \end{pmatrix} = M \begin{pmatrix} \mathbf{e}^{(k)} \\ \mathbf{e}^{(k-1)} \end{pmatrix}.$$

Your matrix M should be explicit, and depend only on A, α and β. Assuming A is diagonal, show that M can be made block diagonal with 2×2 blocks by an appropriate permutation of its rows and columns (i.e. there is a permutation matrix P such that PMP^T is block diagonal).

(d) Compute the spectral radius of M for the particular A and $\mathbf{b}$ given in (1) and the choice $\alpha = 1/\gamma$ and $\beta = (1 - \sqrt{1/\gamma})^2$. Compare your result with the rate in (2) when $\gamma \gg 1$. [*Hint: To simplify the algebra you may find it helpful to write α in terms of β.*]

END OF PAPER