UNIVERSITY OF CAMBRIDGE

MAT2

MATHEMATICAL TRIPOS Part II

Tuesday, 06 June, 2023 1:30pm to 4:30pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most six questions** from Section I and from any number of questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Number Theory

State Lagrange's theorem concerning roots of polynomial congruences. Fix a prime number p. We say that d is *good* if the congruence $x^d \equiv 1 \pmod{p}$ has exactly d solutions modulo p. Prove that any divisor of a good number is again good.

Let $n = pq$ where p and q are distinct primes with $\mathrm{GCD}(p-1, q-1) = 10$. For how many bases $b \in (\mathbb{Z}/n\mathbb{Z})^\times$ is n a Fermat pseudoprime to the base b? For how many of these bases is n a strong pseudoprime?

[*The existence of primitive roots may not be assumed without proof.*]

2F Topics In Analysis

In this question we consider Γ, the collection of closed paths γ not passing through 0, that is to say, continuous functions $\gamma : [0,1] \to \mathbb{C} \setminus 0$ with $\gamma(0) = \gamma(1)$.

Define the *winding number* $w(\gamma, 0)$ of $\gamma \in \Gamma$. If $\gamma \in \Gamma$, $\phi : [0,1] \to \mathbb{C}$ is continuous with $\phi(0) = \phi(1)$ and $|\gamma(t)| > |\phi(t)|$ for all $t \in [0,1]$, what can we say about $w(\gamma + \phi, 0)$?

Explain what it means to say that γ_0, γ_1 are *homotopic by paths in* Γ.

State a theorem on the winding number of homotopic paths and use it to prove the fundamental theorem of algebra and the non-existence of retractions for discs.

3I Coding and Cryptography

(a) (i) Consider a source $(X_n)_{n\geqslant 1}$ of random variables taking values in some finite alphabet $\mathcal{A}$. What does it mean for a source to be *Bernouilli*? What does it mean for a source to be *reliably encodable at rate* r? What is the *information rate* of a source?

(ii) Show that the information rate of a Bernouilli source $(X_n)_{n\geqslant 1}$ is at most the expected word length of an optimal code $c : \mathcal{A} \to \{0,1\}^*$ for X_1.

(b) Let $(X_n)_{n\geqslant 1}$ be a source with letters drawn from a finite alphabet $\mathcal{A}$. This source is not necessarily assumed to be Bernouilli. Let $N \geqslant 1$ be an integer and let $Y_i = (X_{(i-1)N+1}, X_{(i-1)N+2}, \ldots, X_{iN})$. Show that the information rate of the source $(Y_n)_{n\geqslant 1}$ is N times that for $(X_n)_{n\geqslant 1}$.

4I Automata & Formal Languages

Let Σ be an alphabet and $\mathbb{W} := \Sigma^*$ be the set of words over Σ.

(a) Define what it means for $A \subseteq \mathbb{W}$ to be *computably enumerable.*

[You do not need to define what it means for a partial function to be computable.]

(b) Prove that for $\varnothing \neq A \subseteq \mathbb{W}$ the following statements are equivalent:

(i) the set A is computably enumerable;

(ii) the set A is the domain of a partial computable function;

(iii) the set A is the range of a partial computable function;

(iv) the set A is the range of a total computable function.

[*You may assume that the truncated computation function is computable, and that the map* $w \mapsto ((w)_0, (w)_1)$ *is a bijection from* $\mathbb{W}$ *to* $\mathbb{W}^2$ *that can be performed by a register machine.*]

5J Statistical Modelling

Explain the following `R` commands in words, then write down the model that is being fitted.

```
> n <- 100
> p <- 2
> X <- matrix(rnorm(n * p), nrow = n, ncol = p)
> Y <- rbinom(n, size = 1, prob = 0.5)
> sum(Y)
[1] 48
> fit1 <- glm(Y ~ X, family = binomial)
> sum(predict(fit1, type = "response"))
[1] 48
```

Explain why the output of the last command should be exactly the same as the output of `sum(Y)` by writing down the likelihood function of the model.

Do you expect the following command to output exactly 48, too? If not, do you expect it to be very different from 48? Justify your answers.

```
> fit2 <- glm(Y ~ X, family = binomial(probit))
> sum(predict(fit2, type = "response"))
```

6C Mathematical Biology

In an SIR model for an infectious disease the population N is divided into susceptible $S(t)$, infected $I(t)$ and recovered (non-infectious) $R(t)$. The disease is assumed to be non-lethal, so the total population does not change in time.

Consider the following SIR model,

$$\frac{dS}{dt} = fR - \beta IS, \quad \frac{dI}{dt} = \beta IS - \nu I, \quad \frac{dR}{dt} = \nu I - fR,$$

and explain the meaning of each of the terms in the equations. Assume that at $t = 0$, $S \simeq N$, while $I, R \ll N$.

(a) Setting $f = 0$, show that if $\beta N < \nu$ no epidemic occurs.

(b) Now take $f > 0$ and suppose that there is an epidemic. Show that the system has a nontrivial fixed point and that it is stable for small disturbances. Show that the eigenvalues of the Jacobian matrix are complex for sufficiently small f but real for sufficiently large f. Give a qualitative sketch of $I(t)$ in the two cases.

7E Further Complex Methods

The Riemann zeta function $\zeta(s)$ is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

which converges for $\mathrm{Re}(s) > 1$.

Show for $\mathrm{Re}(s) > 1$ that

$$(1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{1+e^t}\, dt.$$

Deduce, giving brief justification, an expression for the analytic continuation of $\zeta(s)$ into the region $\mathrm{Re}(s) > 0$.

Hence show that $\zeta(s)$ has a simple pole at $s = 1$ and evaluate the corresponding residue.

8D Classical Dynamics

A rigid body rotates with angular velocity $\boldsymbol{\omega}(t)$ around its centre of mass. Define what is meant by the *fixed space frame* and the *principal body frame*.

Write down an expression for how the body axes change in time. Hence derive Euler's equations for the torque-free motion of a rigid body.

Consider an axisymmetric body with principal moments of inertia $I_1 = I_2 \neq I_3$. Show that Euler's equations imply the angular momentum $\mathbf{L}$, the angular velocity $\boldsymbol{\omega}$ and the body's symmetry axis are always coplanar.

9B Cosmology

The number density n of photons in thermal equilibrium at temperature T takes the form

$$n = \frac{8\pi}{c^3}\int_0^\infty \frac{\nu^2 \mathrm{d}\nu}{\exp(h\nu/k_B T) - 1}\,, \qquad (\star)$$

where h is Planck's constant, k_B is the Boltzmann constant and c is the speed of light.

Using $(\star)$, show that the photon number density n and energy density ρ can be expressed in the form

$$n = \alpha\, T^3 \qquad \text{and} \qquad \rho = \xi\, T^4\,,$$

where the constants α and ξ need not be evaluated explicitly.

At time $t = t_{\rm dec}$ and temperature $T = T_{\rm dec}$, photons decouple from thermal equilibrium. By considering how the photon frequency redshifts as a flat universe expands, show that the form of the equilibrium frequency distribution is preserved if the temperature for $t > t_{\rm dec}$ is defined by

$$T = \frac{a(t_{\rm dec})}{a(t)} T_{\rm dec}\,.$$

10D Quantum Information and Computation

(a) Consider the Bell states

$$|\Phi^+_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \quad \text{and} \quad |\Phi^-_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right). \tag{1}$$

Show that $\langle\Phi^+_{AB}|\, Q \otimes I \,|\Phi^+_{AB}\rangle = \langle\Phi^-_{AB}|\, Q \otimes I \,|\Phi^-_{AB}\rangle$ for any positive semidefinite linear operator Q acting on qubit A.

(b) Suppose you are now given a quantum state which can either be $|\Phi^+_{AB}\rangle$ or $|\Phi^-_{AB}\rangle$ with equal probability.

(i) If you have access to both qubits A and B, can you determine which of the two states you have by doing a measurement on both qubits?

(ii) If you can only access qubit A, can you determine which of the two states you have by doing a measurement on it alone?

(iii) Suppose instead that qubit A is with Alice and qubit B is with Bob. Alice and Bob are at distant locations. They are allowed to do local measurements on the qubits in their possession and can communicate classically with each other. Can they determine the joint state of the two qubits?

(c) Suppose Alice uses the *quantum dense coding protocol* and a third party, Charlie, intercepts the qubit that Alice sends to Bob. Can Charlie infer which of the four bit strings 00, 01, 10 and 11 Alice is trying to send? Justify your answer.

SECTION II

11F Topics In Analysis

(a) State and prove the Baire Category Theorem.

(b) Consider the set $C^\infty([0,1])$ of infinitely differentiable functions on $[0,1]$. Show that

$$d(f,g)=\sum_{r=0}^{\infty}2^{-r}\min\{1,\|f^{(r)}-g^{(r)}\|_\infty\}$$

is a well defined metric on $C^\infty([0,1])$ and that it is complete.

(c) Show that, if we use this metric, then there is a set E of first category for which the following is true. If $f\notin E$, $q\in(0,1)$ is rational and M is a positive integer, then there exists an $m\geqslant M$ such that

$$|f^{(m)}(q)|>m!\times m^m.$$

(d) If $f\notin E$, show that the Taylor series for f has radius of convergence 0 at every rational point $q\in(0,1)$. Explain briefly why this means that, for any point $x\in[0,1]$, there is no Taylor series which converges to f in a neighbourhood of that point.

12I Coding and Cryptography

(a) What is a *one-time pad*?

Suppose that X and Y are independent random variables taking values in $\mathbb{Z}_n$, the integers modulo n. Using Gibbs' inequality, or otherwise, show that

$$H(X+Y) \geqslant \max\{H(X), H(Y)\}.$$

Why is this result of interest in the context of one-time pads? Does this result remain true if X and Y are not independent? Give reasons for your answer.

(b) The notorious spymaster Stan uses a one-time pad to communicate with the even more notorious spy Ollie. The messages are coded in the obvious way, namely, if the pad has C, the third letter of the alphabet and the message has I, the ninth, then the encrypted message has L as the $(3+9)$th. We will work modulo 26. Unknown to Stan and Ollie, the person whom they employ to carry the messages is actually the police agent Eve in disguise. The police are close to arresting Ollie when Eve is given the message

$$LRPFOJQLCUD.$$

Eve knows that the actual message is

$$FLYXATXONCE,$$

and wants to change things so that Ollie deciphers the message as

$$REMAINXHERE.$$

What message should Eve deliver?

(c) Let K be the field with 2^d elements. Recall that the multiplicative group $K^\times$ is a cyclic group; let α be a generator. Let $T : K \to \mathbb{F}_2$ be any non-zero $\mathbb{F}_2$-linear map. You are given that the $\mathbb{F}_2$-bilinear form $K \times K \to \mathbb{F}_2$ such that $(x, y) \mapsto T(xy)$ is non-degenerate (i.e. $T(xy) = 0$ for all $y \in K$ implies $x = 0$).

(i) Show that the sequence $x_n = T(\alpha^n)$ is the output from a linear feedback shift register of length at most d.

(ii) The *period* of $(x_n)_{n\geqslant 0}$ is the least integer $r \geqslant 1$ such that $x_{n+r} = x_n$ for all sufficiently large n. Show that the sequence in (i) has period $2^d - 1$.

13E Further Complex Methods

The functions $g(z)$ and $h(z)$ are defined by

$$g(z)=\int_0^z \frac{1}{(1-t^2)^{1/2}}\,dt \quad \text{and} \quad h(z)=\int_0^z \frac{1}{(1-t^2)^{1/2}}\frac{1}{(1-k^2t^2)^{1/2}}\,dt,$$

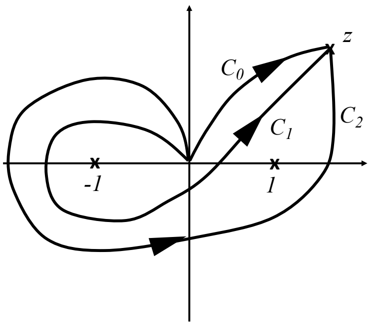
where each integral can be taken along any curve C in the complex t-plane that does not pass through a branch point of the integrand. In both cases the value of the integrand is chosen to be 1 at $t=0$.

(a) First consider $g(z)$. Let $G(z)$ be the value of $g(z)$ evaluated when C is forbidden from crossing the real axis except in the interval $(-1,1)$, with z allowed to lie anywhere in the complex plane except on the parts $(-\infty,-1]$ and $[1,\infty)$ of the real axis. C_0 in the diagram below is such a contour.

(i) Explain why $G(z)$ is a single valued function of z, but $g(z)$ may not be.

(ii) Evaluate $g(z)$ in terms of $G(z)$ when C is each of C_1 and C_2 shown in the diagram below.

(iii) Give, with brief reasoning, all possible values of $g(z)$ as the curve C is varied.



(b) Now consider $h(z)$. Let k be real with $0<k<1$. Let $H(z)$ be the value of $h(z)$ evaluated when C is forbidden from crossing the real axis except in the interval $(-1,1)$, with z allowed to lie anywhere in the complex plane except on the parts $(-\infty,-1]$ and $[1,\infty)$ of the real axis.

(i) Explain why $H(z)$ is a single valued function of z, but $h(z)$ may not be.

(ii) Show, by identifying suitable contours C, that possible values of $h(z)$ include $4K+H(z)$, $2K-H(z)$ and $2iL+H(z)$, where

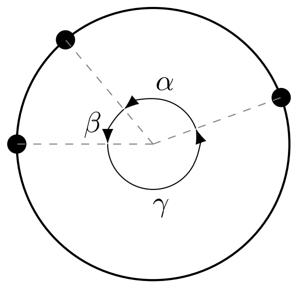
$$K=\int_0^1 \frac{1}{(1-t^2)^{1/2}}\frac{1}{(1-k^2t^2)^{1/2}}\,dt \quad \text{and} \quad L=\int_1^{1/k} \frac{1}{(t^2-1)^{1/2}}\frac{1}{(1-k^2t^2)^{1/2}}\,dt.$$

(c) Deduce that the inverse function $\mathcal{H}(w)$ defined by $h(\mathcal{H}(w))=w$ is a doubly periodic function and give expressions for the two periods.

(d) Assuming that $\mathcal{H}$ is a meromorphic function, explain briefly why it must have at least one pole.

14D Classical Dynamics

Three identical particles, each of mass m, are constrained to move around a fixed circle of radius r that lies in a horizontal plane. You may assume that the particles do not collide. The angles between the locations of the particles are α, β, γ as in the figure, which shows the view from above.



(a) Write down a constraint obeyed by α, β, γ. What degree of freedom is not described by these three angles?

(b) The particles feel the influence of a potential

$$V(\alpha, \beta, \gamma) = V_0(e^{-2\alpha} + e^{-2\beta} + e^{-2\gamma}),$$

where V_0 is a positive constant. Solving your constraint to find $\gamma = \gamma(\alpha, \beta)$, obtain a Lagrangian governing the dynamics of the particles' relative separations as a function of α, β, $\dot{\alpha}$ and $\dot{\beta}$.

(c) Find an equilibrium configuration of the system and show that it is stable. Find three linearly independent normal modes, together with their frequencies, that describe small perturbations about this equilibrium.

(d) The physical system is unchanged by permutations of (α, β, γ). Explain how this is consistent with your answer to part (c).

15D Quantum Information and Computation

(a) Let $\mathbb{Z}_N = \{0, 1, 2, \ldots, N-1\}$ and let QFT_N denote the quantum Fourier transform mod N. What is the action of QFT_N on $|x\rangle$, where $x \in \mathbb{Z}_N$?

(b) Show that $\mathrm{QFT}_N^2 |x\rangle = |-x\rangle$. Hence show that $\mathrm{QFT}_N^4 = \mathbb{I}$. What can you conclude about the eigenvalues of QFT_N?

(c) Let $f : \mathbb{Z}_{16} \to \mathbb{Z}_4$ be a periodic function such that $f(0) = 2$, $f(1) = 1$, $f(2) = 3$, $f(3) = 0$ and $f(x) = f(x-4)$ for all $x \in \mathbb{Z}_{16}$ (so that $f(4) = 2$ etc.).

We want to determine the periodicity of the function f using the quantum Fourier transform. The periodicity determination algorithm acts on two registers and involves two measurements – one being a measurement of the second register and one being a measurement of the first register. Work through all the steps of the periodicity determination algorithm, assuming that the outcome of the first measurement is 1 and the outcome of the second measurement is 12. Does the algorithm succeed?

(d) Now consider the same setup as in part (c) but assume that the outcome of the second measurement is 8. Does the algorithm succeed?

16H Logic and Set Theory

(a) Let α be a non-zero ordinal. Show that there is a greatest ordinal β such that $\omega^\beta \leqslant \alpha$. Deduce that there exist a non-zero natural number n and an ordinal $\gamma < \omega^\beta$ such that $\alpha = \omega^\beta.n + \gamma$.

(b) An ordinal δ is called *additively closed* if whenever $\beta < \delta$ and $\gamma < \delta$ then also $\beta + \gamma < \delta$. Show that a non-zero ordinal is additively closed if and only if it is of the form ω^α for some α.

(c) An ordinal δ is called *multiplicatively closed* if whenever $\beta < \delta$ and $\gamma < \delta$ then also $\beta\gamma < \delta$. Show that an ordinal greater than 2 is multiplicatively closed if and only if it is of the form $\omega^{(\omega^\alpha)}$ for some α.

[*You may assume standard properties of ordinal arithmetic.*]

17H Graph Theory

(a) Define the *chromatic polynomial* $P_G(t)$ of a graph G and prove that it is a polynomial.

(b) Let $f(t) = t^4 - 4t^3 + 5t^2 - 2t$. Explain why f is not the chromatic polynomial of a bipartite graph. Is f the chromatic polynomial of some graph? Justify your answers.

(c) Let G be a connected graph, let $A = A(G)$ be its adjacency matrix and let $d(G)$ denote the diameter of G. Show that the matrices $I, A, A^2, \ldots, A^{d(G)}$ are linearly independent.

(d) Give an infinite family of connected graphs $\{G_n\}$ with adjacency matrices $\{A_n\}$ so that

$$I, A_n, A_n^2, \ldots, A_n^{d(G_n)+1}$$

are linearly dependent, for each n.

18I Galois Theory

Let L/K be a finite extension of fields of characteristic $p > 0$.

(a) Let $x \in L$. What does it mean to say that x is *separable over* K? Show that x is separable over K if and only if its minimal polynomial is not of the form $g(T^p)$ for some $g \in K[T]$.

(b) We say that $x \in L$ is *purely inseparable* over K if for some $n \geqslant 0$, $x^{p^n} \in K$. Show that x is purely inseparable over K if and only if its minimal polynomial is of the form $T^{p^n} - y$, for some $n \geqslant 0$ and some $y \in K$.

(c) Let $g \in K[T]$ be a monic nonconstant polynomial, and $f(T) = g(T^p)$. Assume that L is a splitting field for g over K, and let M be a splitting field for f over L. Show that M is also a splitting field for f over K, and that every root of f in M is purely inseparable over L. Show also that for every $\sigma \in \mathrm{Aut}(L/K)$ there exists a unique automorphism τ of M whose restriction to L equals σ.

(d) Suppose that g is irreducible and separable. Show that $\mathrm{Aut}(M/K)$ acts transitively on the roots of f in M. Deduce that either every root of f lies in L, or every root has degree p over L.

Let $h \in K[T]$ be an irreducible monic factor of f. Show that either $h = f$, or that h is separable. Deduce that f is reducible if and only if every coefficient of g is a p-th power in K.

[*You may assume without proof the uniqueness of splitting fields, and that a nonconstant polynomial f is separable if and only if* $(f, f') = 1$.]

19H Representation Theory

Consider the subset H of $\mathrm{GL}_2(\mathbb{F}_{11})$ consisting of matrices of the form

$$\begin{pmatrix} a^2 & b \\ 0 & 1 \end{pmatrix} \text{ with } a, b \in \mathbb{F}_{11} \text{ and } a \neq 0.$$

Show that H is a non-abelian group of order 55 with 7 conjugacy classes and construct its character table. [*You may assume standard results from the course and that* 2 *is a generator of the cyclic group* $\mathbb{F}_{11}^{\times}$.]

20H Number Fields

(a) The *trace form* for a number field K is the bilinear form $(x, y) := \mathrm{Tr}_{K/\mathbb{Q}}(xy)$.

(i) Prove that the trace form is non-degenerate.

(ii) Let $\alpha_1, \ldots, \alpha_n \in \mathcal{O}_K$ be a basis for $K/\mathbb{Q}$. Let

$$\Delta(\alpha_1, \ldots, \alpha_n) = \det(\mathrm{Tr}_{K/\mathbb{Q}}(\alpha_i \alpha_j)_{i,j}).$$

Show that the minimum absolute value for Δ is a positive integer, obtained precisely when $\alpha_1, \ldots, \alpha_n$ is a $\mathbb{Z}$-basis of $\mathcal{O}_K$.

(b) Let $K = \mathbb{Q}(\sqrt[3]{3})$. Compute the class group of K. You may assume the ring of integers is $\mathbb{Z}[\sqrt[3]{3}]$.

21G Algebraic Topology

Let $p : \widehat{X} \to X$ be a covering map, and suppose that X and $\widehat{X}$ are path connected and locally path connected topological spaces. If $x_0, x_1 \in X$, show that $p^{-1}(x_0)$ and $p^{-1}(x_1)$ have the same cardinality. [You may use any theorems from the course, as long as you state them clearly.]

Define what it means for p to be a *normal covering map.* State an appropriate lifting theorem and use it to prove that if $p : \widehat{X} \to X$ is a *universal* covering map, then it is normal.

Let Σ_g be a surface of genus g and suppose that $p : \widehat{\Sigma}_g \to \Sigma_g$ is a connected covering map of degree $n \in \mathbb{N}$. For which values of g and n must p be normal? Justify your answer. For those values of g and n for which p need not be normal, give an explicit example of a non-normal covering map p.

22F Linear Analysis

(a) Let $(V, \|\cdot\|)$ be a normed vector space over $\mathbb{R}$, and $v, w \in V$. Define

$$S_1^{vw} := \left\{ z \in V \ : \ \|z - v\| = \|z - w\| = \frac{1}{2}\|v - w\| \right\}$$

and then inductively, for $n \geqslant 2$,

$$S_n^{vw} := \left\{ z \in S_{n-1}^{vw} \ : \ \forall\, \tilde{z} \in S_{n-1}^{vw},\ \|z - \tilde{z}\| \leqslant \frac{1}{2}\mathrm{diam}(S_{n-1}^{vw}) \right\},$$

with the definition $\mathrm{diam}(S) := \sup_{z,\tilde{z}\in S} \|z - \tilde{z}\|$. Prove that $\cap_{n\geqslant 1} S_n^{vw} = \{\frac{v+w}{2}\}$.

(b) Let $(V, \|\cdot\|_V)$ and $(\widetilde{V}, \|\cdot\|_{\widetilde{V}})$ be normed vector spaces over $\mathbb{R}$, and $u : V \to \widetilde{V}$ an isometry, i.e. a map with the property that $\|u(v) - u(w)\|_{\widetilde{V}} = \|v - w\|_V$. Using part (a), prove that $u(\frac{v+w}{2}) = \frac{u(v)+u(w)}{2}$ for all $v, w \in V$.

(c) Assume furthermore that the isometry $u : V \to \widetilde{V}$ satisfies $u(0) = 0$. Prove that u is linear.

23F Analysis of Functions

(a) Let $U \subset \mathbb{R}^n$ be open with finite Lebesgue measure. Let $p \in (1, \infty)$ and let $\Lambda \in L^p(U)'$ be positive. Prove there is $\omega \in L^q(U)$ where $1/p + 1/q = 1$ such that

$$\Lambda(f) = \int_U f\omega\, dx \qquad \text{for all } f \in L^p(U).$$

[You may use without proof that $\|g\|_{L^q(U)} = \sup\{\int_U |fg|\, dx : \|f\|_{L^p(U)} \leqslant 1\}$.]

(b) (i) Define the Fourier transform of $f \in L^1(\mathbb{R}^n)$.

(ii) Let $p, q \in (1, \infty)$, and assume $\|\hat{f}\|_{L^q(\mathbb{R}^n)} \leqslant C_{p,q}\|f\|_{L^p(\mathbb{R}^n)}$ for all $f \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Show that q is uniquely determined by p.

(iii) Compute the Fourier transform of $f(x) = |x|^{-1}$ on $\mathbb{R}^3$ up to a multiplicative constant which you do not need to determine. [You may use the Fourier transform of a Gaussian without proof.]

24F Riemann Surfaces

State the valency theorem, and define the *degree* $\deg f$ of an analytic map f of compact Riemann surfaces.

Consider a rational function f with derivative f'. Define the *degree* of f, and prove that $\deg f - 1 \leqslant \deg f' \leqslant 2 \deg f$. Give examples to show that these bounds can be achieved, for every possible value of $\deg f \geqslant 1$.

Consider a non-constant elliptic function g with derivative g'. Define the *degree* of g, and prove that $\deg g + 1 \leqslant \deg g' \leqslant 2 \deg g$. Give examples to show that these bounds can be achieved, for every odd value of $\deg g \geqslant 3$. [You may use properties of standard examples of elliptic functions without proof.]

25G Algebraic Geometry

State the Riemann–Roch theorem for a smooth projective curve X. Using this theorem, calculate the degree of the canonical divisor of X in terms of the genus of X.

Prove that every smooth projective curve of genus g can be embedded in a fixed projective space $\mathbb{P}^n$, where n may depend on g.

State the Riemann–Hurwitz formula. Using this formula or otherwise, construct a smooth projective variety of dimension 2 that contains no curves of genus less than 3.

Let X be a smooth curve of degree d in $\mathbb{P}^2$ and let p be a point not lying on X. Prove that projection away from p gives rise to a morphism

$$\pi : X \to \mathbb{P}^1.$$

Give an upper bound on the number of points of X at which π can be ramified. [You do not need to show that your bound is sharp.]

26G Differential Geometry

(a) For regular curves in $\mathbb{R}^3$, parametrised by arc length s, define *curvature* k and *torsion* τ and derive the Frénet formulas. Indicate carefully all additional assumptions for these to be well defined.

(b) Suppose two regular curves in $\mathbb{R}^3$ both have curvature identically zero and the same arc length. Are they related by a proper Euclidean motion? Justify your answer. Does the answer change if we replace curvature identically zero with curvature identically one?

(c) We say that a quantity $Q(\gamma, s)$, defined for all regular curves γ parametrised by arc length s, is a *pointwise Euclidean invariant of curves* if

$$Q(\gamma, s) = Q(E \circ \gamma, s) \text{ for all proper Euclidean motions } E, \text{and}$$

$$Q(\gamma_{s_0}, s) = Q(\gamma, s - s_0) \text{ for all } s_0 \in \mathbb{R}, \text{ where } \gamma_{s_0}(s) := \gamma(s - s_0).$$

Show that $Q(\gamma, s) := k(s)$ and $Q(\gamma, s) := \tau(s)$, where k and τ refer to the curvature and torsion of the curve γ respectively, are both examples of such pointwise Euclidean invariants of curves.

(d) One can trivially construct other such pointwise Euclidean invariants by applying functions of curvature and torsion, e.g. defining $Q(\gamma, s) := k^2(s)$ or $Q(\gamma, s) := k(s) + \tau(s)$. Are these the only examples, i.e. is it true that if $Q(\gamma, s)$ is any pointwise Euclidean invariant, then $Q(\gamma, s) = f(k(s), \tau(s))$ for some function f (independent of the curve γ)? Justify your answer.

27K Probability and Measure

(a) Denote by $L^1(\mathbb{R}^d)$ the space of Lebesgue integrable functions on $\mathbb{R}^d$. For $f \in L^1(\mathbb{R}^d)$ with Fourier transform $\hat{f} \in L^1(\mathbb{R}^d)$, state (without proof) the Fourier inversion theorem and deduce Plancherel's identity for such f from it. Argue that if f is continuous, then the inversion formula holds everywhere.

(b) Show that the integral

$$g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \frac{4\sin^2(u/2)}{u^2} du, \ x \in \mathbb{R},$$

exists, and vanishes whenever $|x| > 1$. What is $\|g\|_2^2$? Justify your answers.

28J Applied Probability

(a) Define a simple birth process with parameter λ starting with one individual. If X_t denotes the number of individuals at time t in a simple birth process (with $X_0 = 1$), find $\mu(t) := \mathbb{E}X_t$. [You may assume that $\mu(t)$ is a continuous function of t.]

(b) Let $\lambda \in [1, 2]$ and consider the continuous-time Markov chain on $\mathbb{N} = \{0, 1, 2, \ldots\}$ with rates

$$q_{i,i-1} = 2^i, \quad q_{ii} = -(\lambda + 1) \cdot 2^i, \quad q_{i,i+1} = \lambda 2^i \qquad \text{for } i \geqslant 1,$$

and $q_{0,1} = \lambda, q_{0,0} = -\lambda$.

For what values of $\lambda \in [1, 2]$ is X recurrent? For what values of $\lambda \in [1, 2]$ does X have an invariant distribution? For what values of $\lambda \in [1, 2]$ is X explosive? Justify your answers.

[You may assume the recurrence and transience properties of simple random walks on $\mathbb{N}$. You may also assume without proof that for a transient simple random walk on $\mathbb{N}$, $\sup_i \mathbb{E}_i V_i < \infty$, where V_i is the number of visits to the state i.]

(c) Let X be an irreducible continuous-time Markov chain with jump chain Y. Prove or provide a counterexample (with proper justification) to the following: if a state is positive recurrent for X, it is positive recurrent for Y.

[*You may quote any result from the lectures that you need, without proof, provided it is clearly stated.*]

29K Principles of Statistics

(a) Define the correlation $\rho_{X,Y}$ between two random variables (X, Y).

Now suppose $\begin{pmatrix} X \\ Y \end{pmatrix}$ follows a bivariate normal distribution with mean $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and nonsingular covariance matrix $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$.

(b) Derive a formula for an MLE $\widehat{\Sigma} = \begin{pmatrix} \widehat{\Sigma}_{11} & \widehat{\Sigma}_{12} \\ \widehat{\Sigma}_{12} & \widehat{\Sigma}_{22} \end{pmatrix}$ for Σ by solving for a root of the score function. [You may use, without proof, the facts that $\frac{\partial}{\partial A} \log |A| = A^{-1}$ and $\frac{\partial}{\partial A} v^T A v = vv^T$, for a symmetric matrix A and vector v. You do not need to prove that a root of the score function actually corresponds to an MLE.]

(c) Derive an expression for the limiting distribution of $\sqrt{n}\left(\begin{pmatrix} \widehat{\Sigma}_{11} \\ \widehat{\Sigma}_{12} \\ \widehat{\Sigma}_{22} \end{pmatrix} - \begin{pmatrix} \Sigma_{11} \\ \Sigma_{12} \\ \Sigma_{22} \end{pmatrix}\right)$.

[Your answer should be in terms of the entries of Σ. You may use, without proof, the following relations for the bivariate normal:

$$\mathbb{E}[X^4] = 3\Sigma_{11}^2, \quad \mathbb{E}[X^3 Y] = 3\Sigma_{11}\Sigma_{12}, \quad \mathbb{E}[X^2 Y^2] = \Sigma_{11}\Sigma_{22} + 2\Sigma_{12}^2,$$

$$\mathbb{E}[XY^3] = 3\Sigma_{22}\Sigma_{12}, \quad \mathbb{E}[Y^4] = 3\Sigma_{22}^2.]$$

(d) Now consider the plug-in MLE $\widehat{\rho} := \frac{\widehat{\Sigma}_{12}}{\sqrt{\widehat{\Sigma}_{11}\widehat{\Sigma}_{22}}}$. Derive an expression for the limiting distribution of $\sqrt{n}(\widehat{\rho} - \rho_{X,Y})$.

[*You may quote any result from the lectures that you need, without proof.*]

30K Stochastic Financial Models

Let $(M_n)_{n \geqslant 0}$ be a martingale with respect to a filtration $(\mathcal{F}_n)_{n \geqslant 0}$.

(a) Given an $\mathcal{F}_0$-measurable X_0 and a previsible process $(A_n)_{n \geqslant 1}$, let

$$X_n = X_0 + \sum_{k=1}^{n} A_k (M_k - M_{k-1})$$

for $n \geqslant 1$. Assuming that X_n is integrable for each $n \geqslant 0$, show that $(X_n)_{n \geqslant 0}$ is a martingale.

(b) Let T be a stopping time and let

$$Y_n = M_{\min\{n,T\}}$$

for $n \geqslant 0$. Show that $(Y_n)_{n \geqslant 0}$ is a martingale.

(c) Let $(X_n)_{n \geqslant 0}$ be as in part (a) and suppose that both X_n and M_n are square-integrable for each $n \geqslant 0$. Suppose that $\mathrm{Var}(M_n | \mathcal{F}_{n-1}) > 0$ almost surely for each $n \geqslant 1$. Show that

$$A_n = \frac{\mathrm{Cov}(X_n, M_n | \mathcal{F}_{n-1})}{\mathrm{Var}(M_n | \mathcal{F}_{n-1})}$$

for all $n \geqslant 1$.

Let $(\xi_n)_{n \geqslant 1}$ be a sequence of independent random variables such that for all $n \geqslant 1$, $\mathbb{P}(\xi_n = +1) = \frac{1}{2} = \mathbb{P}(\xi_n = -1)$. Suppose that the filtration $(\mathcal{F}_n)_{n \geqslant 0}$ is generated by $(\xi_n)_{n \geqslant 1}$.

(d) Show that there exists a previsible process $(B_n)_{n \geqslant 1}$ such that

$$M_n = M_0 + \sum_{k=1}^{n} B_k \xi_k$$

for all $n \geqslant 0$.

(e) Now show for any bounded stopping time T that

$$\mathbb{E}[M_T^2] = M_0^2 + \mathbb{E}\left[\sum_{k=1}^{T} B_k^2\right].$$

31J Mathematics of Machine Learning

(a) Let $\mathcal{H}$ be a hypothesis class of functions $h : \mathcal{X} \to \{-1, 1\}$ with $|\mathcal{H}| > 2$ and $\mathcal{X} = \mathbb{R}^p$. Define the *shattering coefficient* $s(\mathcal{H}, n)$ and the *VC dimension* $\mathrm{VC}(\mathcal{H})$ of $\mathcal{H}$.

(b) Explain why if $\mathcal{H}_1, \mathcal{H}_2$ are hypothesis classes as above, then $s(\mathcal{H}_1 \cup \mathcal{H}_2, n) \leqslant s(\mathcal{H}_1, n) + s(\mathcal{H}_2, n)$.

Let us use the notation that, for a class $\mathcal{F}$ of functions $f : \mathbb{R}^p \to \mathbb{R}$, we write

$$\mathcal{H}_{\mathcal{F}} := \{h : h(x) = \mathrm{sgn} \circ f(x), \text{ where } f \in \mathcal{F}\}$$

for the class of functions derived through composition with the sgn function.

(c) Now let $\mathcal{F}_1 := \{f : f(x) = x^T \beta, \text{ where } \beta \in \mathbb{R}^p\}$. Stating any results from the course you need, show that

$$s(\mathcal{H}_{\mathcal{F}_1}, n) \leqslant (n+1)^p.$$

(d) Next for a class $\mathcal{G}$ of functions $g : \mathbb{R}^p \to \{-1, 1\}$, define for some fixed $m \in \mathbb{N}$,

$$\mathcal{F}_2 := \left\{ f : f(x) = \sum_{j=1}^{m} \alpha_j g_j(x), \text{ where } g_j \in \mathcal{G},\ \alpha \in \mathbb{R}^m \right\}.$$

Show that if $|\mathcal{G}| < \infty$,

$$s(\mathcal{H}_{\mathcal{F}_2}, n) \leqslant (n+1)^m |\mathcal{G}|^m.$$

Show furthermore that even if $|\mathcal{G}| = \infty$, we have

$$s(\mathcal{H}_{\mathcal{F}_2}, n) \leqslant (n+1)^m s(\mathcal{G}, n)^m.$$

[*Hint: Fix* $x_{1:n} \in \mathcal{X}^n$ *and consider* $\mathcal{G}'$ *with* $|\mathcal{G}'| \leqslant s(\mathcal{G}, n)$ *and* $\mathcal{G}'(x_{1:n}) = \mathcal{G}(x_{1:n})$.]

(e) Finally let $\mathcal{F}_3$ be the class of functions $f : \mathbb{R}^p \to \mathbb{R}$ given by a neural network with a single hidden layer of m nodes and activation function given by sgn. Show that

$$s(\mathcal{H}_{\mathcal{F}_3}, n) \leqslant (n+1)^{(p+1)m}.$$

32E Asymptotic Methods

(a) Let $\phi_n(x) > 0$, for $n = 0, 1, 2, \ldots$, be a sequence of real functions defined on $\{x \in \mathbb{R} : 0 < |x - x_0| < a\}$ which is an asymptotic sequence as $x \to x_0$.

(i) Let $\psi_0(x) = \phi_0(x)$ and

$$\psi_n(x) = \frac{\phi_{n-1}(x)\phi_n(x)}{\phi_{n-1}(x) + \phi_n(x)}\,, \qquad n = 1, 2, 3, \ldots$$

Show that $(\psi_n(x))_{n=0}^{\infty}$ is an asymptotic sequence as $x \to x_0$.

Is it true that $\phi_n(x) \sim \psi_n(x)$ as $x \to x_0$ for every $n = 0, 1, 2, \ldots$? You should either give a proof or a counterexample.

(ii) Let $\chi_0(x) = \phi_0(x)$ and

$$\chi_n(x) = \sqrt{\phi_{n-1}(x)\phi_n(x)}\,, \qquad n = 1, 2, 3, \ldots$$

Show that $(\chi_n(x))_{n=0}^{\infty}$ is an asymptotic sequence as $x \to x_0$.

Is it true that $\phi_n(x) \sim \chi_n(x)$ as $x \to x_0$ for every $n = 0, 1, 2, \ldots$? You should either give a proof or a counterexample.

(b) Let $(\phi_n(x))_{n=0}^{\infty}$ and $(\psi_n(x))_{n=0}^{\infty}$ be two sequences of real functions defined on $\{x \in \mathbb{R} : 0 < |x - x_0| < a\}$ which are asymptotic sequences as $x \to x_0$. Suppose that

$$\phi_n(x) \sim \psi_n(x) \quad \text{as} \quad x \to x_0\,,$$

for $n = 0, 1, 2, \ldots$, and that for some sequence of real numbers $(a_n)_{n=0}^{\infty}$ we have

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x) \quad \text{as} \quad x \to x_0\,.$$

Does there necessarily exist a sequence of real numbers $(b_n)_{n=0}^{\infty}$ such that

$$f(x) \sim \sum_{n=0}^{\infty} b_n \psi_n(x) \quad \text{as} \quad x \to x_0\,?$$

You should either give a proof or a counterexample.

33A Dynamical Systems

(a) Define a Lyapunov function for a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ on $\mathbb{R}^n$ with a fixed point $\mathbf{x}^*$. Explain what it means for a fixed point of the flow to be Lyapunov stable. State and prove Lyapunov's first stability theorem.

(b) Consider the second order differential equation

$$\ddot{x} = F(x) - \mu \dot{x},$$

where $\mu > 0$ and $F(x) = -2x(1-x^2)^2$. Show that there are three fixed points in the $(x, \dot{x})$ plane. Show that one of these is the origin and that it is Lyapunov stable. Show further that the origin is asymptotically stable, and that the ω-limit set of each point in the phase space is one of the three fixed points, justifying your answer carefully.

34E Integrable Systems

Assume $\phi = \phi(x,t)$ is a solution of

$$-\phi_{xx} + u(x,t)\phi = \lambda(t)\phi, \qquad -\infty < x < \infty\,, \tag{S}$$

where $u = u(x,t)$ is smooth. Define $Q = Q(x,t)$ by $Q = \phi_t + u_x\phi - 2(u+2\lambda)\phi_x$ and show that there exists a number α, which you should find, such that

$$\partial_x(\phi_x Q - \phi Q_x) = \phi^2\Big(\dot{\lambda} + \alpha(u_t + u_{xxx} - 6uu_x)\Big) \tag{$*$}$$

where $\dot{\lambda} = \frac{d\lambda}{dt}$.

Now let $u = u(x,t)$ be a smooth solution of the KdV equation $u_t + u_{xxx} - 6uu_x = 0$, which is rapidly decreasing in x, and consider the case when $\phi = \varphi_n$ is the discrete eigenfunction of (S) corresponding to eigenvalue $\lambda_n = -\kappa_n^2 < 0$. Deduce from $(*)$ that $\lambda_n(t) = \lambda_n(0)$. [You may assume that $\kappa_n > 0$ and φ_n is normalized, i.e., $\int_{-\infty}^{\infty} \varphi_n(x,t)^2 dx = 1$ for all times t.]

Deduce further that in this case $Q(x,t) = h_n(t)\varphi_n(x,t)$ for some function $h_n = h_n(t)$ and, by multiplying by φ_n, making use of (S) and integrating, show that $h_n(t) = 0$ and $Q = 0$. Finally, derive from this the time evolution of the discrete normalization $c_n(t)$ which is defined by the asymptotic relation

$$\varphi_n(x,t) \approx c_n(t) e^{-\kappa_n x} \quad \text{as} \quad x \to +\infty\,.$$

[You may assume the differentiated version of this relation also holds.]

35B Principles of Quantum Mechanics

A two-state quantum system has Hamiltonian H_0 with eigenvectors $|-\rangle$ and $|+\rangle$, and corresponding eigenvalues E_- and $E_+ > E_-$. The system is perturbed by the Hermitian operator ΔH with matrix elements

$$\langle +|\,\Delta H\,|-\rangle = i\lambda\,, \qquad\qquad \langle +|\,\Delta H\,|+\rangle = \langle -|\,\Delta H\,|-\rangle = 0,$$

where λ is a real constant.

(i) Starting from the Schrödinger equation for $H_0+\Delta H$ and explicitly deriving any necessary results, determine the corrections to the energy eigenstates and eigenvalues in perturbation theory up to linear order in λ.

(ii) Find the exact eigenstates and eigenvalues and show that they agree with the results of perturbation theory up to linear order in λ.

(iii) Determine the radius of convergence of perturbation theory in λ. [*Hint: the square root function has a branch point when its argument vanishes.*]

36D Applications of Quantum Mechanics

Consider a quantum system with Hamiltonian $\hat{H}$ having a discrete spectrum with a unique groundstate $|\psi_0\rangle$ of energy E_0. For any state $|\psi\rangle$, define the *Rayleigh-Ritz quotient*, $R[\psi]$, and show that it attains its minimum value when $|\psi\rangle = |\psi_0\rangle$.

A particle of mass m moves in one dimension subject to the potential,

$$V(x) \quad = \quad \frac{\hbar^2}{2m}\left(x^6 - 3x^2 + 2\right).$$

Show that the system has an energy eigenstate with (unnormalised) wavefunction,

$$\tilde{\psi}(x) := \exp(-\beta x^n),$$

for a value of β, a positive integer value of n and an energy each of which you should determine.

Estimate the groundstate energy of this system using the variational principle with a Gaussian trial wavefunction of the form

$$\psi_\alpha(x) \quad := \quad \exp\left(-\frac{\alpha}{2}x^2\right)$$

with parameter $\alpha > 0$. Show that the best estimate of the ground-state energy is obtained for the unique value

$$\alpha = \alpha_* = \sqrt{\frac{(p+\sqrt{q})}{2}}\,,$$

where p and q are integers that you should determine. Give the corresponding approximate ground-state energy, E_0^*, in terms of α_*. You should not attempt to evaluate this function numerically. [*Hint: You may use without proof the following definite integral,*

$$\int_{-\infty}^{\infty} x^{2n}\exp\left(-\alpha x^2\right)\,dx \quad = \quad \frac{(2n)!}{n!(4\alpha)^n}\sqrt{\frac{\pi}{\alpha}}. \quad \Bigg]$$

Is your result consistent with the hypothesis that the exact eigenstate $\tilde{\psi}(x)$ found above is the true groundstate? Explain your reasoning carefully.

37A Statistical Physics

A simple one-dimensional model of a rubber molecule consists of a chain of n links, where n is fixed. Each link has a fixed length a and can be oriented in either the positive or negative direction. A unique state i of the molecule is specified by giving the orientation of each link and the molecule's length in this state is l_i. If n_+ links are oriented in the positive direction and n_- in the negative direction, then $n = n_+ + n_-$ and the length of the molecule is $l = (n_+ - n_-)a$. All configurations have the same energy.

(a) What is the range of possible values of l? What is the number of states of the molecule for fixed n_+ and n_-?

(b) Now consider an ensemble with $A \gg 1$ copies of the molecule in which a_i members are in state i. Write down an expression for the mean length L. By introducing Lagrange multipliers τ and α show that the most probable configuration for the $\{a_i\}$ with given L is found by maximising

$$\ln\left(\frac{A!}{\prod_i a_i!}\right) + \tau\sum_i a_i\, l_i - \alpha\sum_i a_i\,.$$

Hence show that the most probable configuration has

$$p_i = e^{\tau l_i}/Z\,,$$

where p_i is the probability for finding an ensemble member in state i and Z is the partition function which should be defined.

(c) Show that Z can be expressed as

$$Z = \sum_l g(l)\, e^{\tau l}$$

where the meaning of $g(l)$ should be explained. Hence show that

$$Z = \sum_{n_+=0}^{n} \frac{n!}{n_+!n_-!}\left(e^{\tau a}\right)^{n_+}\left(e^{-\tau a}\right)^{n_-}\quad,\qquad n_+ + n_- = n\,.$$

(d) Show that the free energy $G = -k_B T \ln Z$ for the system is

$$G = -nk_B T\ln\left(2\cosh\tau a\right)\,,$$

where k_B is the Boltzmann constant and T is the temperature. Hence show that

$$L = -\frac{1}{k_B T}\left(\frac{\partial G}{\partial\tau}\right)\quad\text{and}\quad \tanh\tau a = \frac{L}{na}\,.$$

(e) Why is the tension f in the rubber molecule equal to $k_B T\tau$? [Here f and L are analogous to, respectively, pressure p and volume V in three-dimensional systems, and G is the Gibbs free energy because the setup corresponds to a system with fixed tension rather than with a fixed length.]

(f) Now assume that $na \gg L$. Show that the chain satisfies Hooke's law $f \propto L$. What happens if f is held constant and T is increased?

38B General Relativity

Consider the geometry of 2-dimensional hyperbolic space:

$$ds^2 = a^2(dr^2 + \sinh^2 r\, d\phi^2)$$

where a is a constant. The coordinates have ranges $0 \leqslant r$ and $0 \leqslant \phi < 2\pi$.

(a) For a general metric with components $g_{\alpha\beta}$, give an expression for the Christoffel symbols, $\Gamma^{\alpha}_{\beta\gamma}$, in terms of the metric components and their derivatives. Use this formula to calculate the Christoffel symbols for the metric above.

(b) Using the geodesic equation, show that lines of constant ϕ are always geodesics, but circles of constant $r > 0$ never are.

(c) Calculate both of the nonzero components of the Riemann tensor $R^{\alpha}{}_{\beta\gamma\delta}$.

[*You may use:* $R^{\alpha}{}_{\beta\gamma\delta} := \partial_\gamma \Gamma^{\alpha}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\mu}_{\beta\delta}\Gamma^{\alpha}_{\mu\gamma} - \Gamma^{\mu}_{\beta\gamma}\Gamma^{\alpha}_{\mu\delta}$.]

(d) Show that the Ricci scalar R is constant.

39C Fluid Dynamics II

(a) Consider the incompressible flow of a Newtonian fluid with constant viscosity μ and constant density ρ subject to a body force per unit mass $\mathbf{f}$. Derive the equation for rate of change of kinetic energy in a finite volume Ω with boundary $\partial\Omega$ and give the physical interpretation for each term.

(b) Explain, justifying your arguments with appropriate order-of-magnitude estimates, how the energy balance can be used to estimate the drag on a steadily moving bubble of fixed shape in fluid at rest at infinity, when the Reynolds number is large, without having to solve for the details of the boundary layer around the bubble. [You may ignore all contributions from body forces.]

(c) A two-dimensional circular bubble of radius a, in fluid that is at rest far from the bubble, is moving steadily with velocity $U\mathbf{e}_x$, where $\mathbf{e}_x$ is the unit vector in the x-direction. The flow occurs at high Reynolds number and is assumed to be irrotational outside the boundary layer. [Again you may ignore the effect of any body forces.]

(i) Solve for the irrotational flow outside the boundary layer.

(ii) Using the method in part (b), or otherwise, estimate the drag force exerted on the bubble.

(iii) Give brief reasons why the same approach could not be applied to a rigid body.

[*Hint: In polar coordinates the rate-of-strain tensor has components*

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{r\theta} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right], \quad e_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}.]$$

40C Waves

(a) A uniform elastic solid with wave speeds c_P and c_S (using the usual notation) occupies the region $z < 0$. An SV-wave with unit amplitude displacement

$$\mathbf{u}_I = \mathrm{Re}\{(\cos\theta, 0, -\sin\theta)e^{ik_I(x\sin\theta + z\cos\theta) - i\omega t}\}$$

is incident from $z < 0$ on a rigid boundary at $z = 0$. Find the form and amplitudes of the reflected waves.

(b) Derive a condition on the incident angle θ for the reflected P-wave to be evanescent. Show by explicit calculation that if the P-wave is evanescent:

(i) the reflected SV-wave also has unit amplitude and

(ii) the P-wave has zero acoustic energy flux in the z-direction if time-averaged in an appropriate way, which you should specify carefully.

41C Numerical Analysis

Consider the variable coefficient advection equation

$$\frac{\partial u}{\partial t}(x,t) + c(x)\frac{\partial u}{\partial x}(x,t) = 0 \tag{1}$$

where $x \in (-\infty, \infty)$ and $t \geqslant 0$. Assume that $c(x) > 0$ is 2-periodic, i.e., $c(x+2) = c(x)$. We will seek a 2-periodic solution $u(x,t)$ that satisfies $u(x,t) = u(x+2,t)$ for all t.

(a) Assume $c(x)$ has a finite decomposition in a Fourier basis

$$c(x) = \sum_{n=-d}^{d} \widehat{c}_n e^{i\pi n x} \qquad (\widehat{c}_i = 0 \text{ for } |i| > d).$$

Give an expression for $\widehat{c}_n$ in terms of $c(x)$. Using the fact that $c(x) > 0$ for all x, show that the $(2d+1)\times(2d+1)$ matrix $[\widehat{c}_{n-m}]_{-d \leqslant n,m \leqslant d}$ is Hermitian positive definite.

(b) We seek a solution $u(x,t)$ of (1) of the form

$$u(x,t) = \sum_{n=-d}^{d} \widehat{u}_n(t) e^{i\pi n x}.$$

Let $\widehat{\mathbf{u}}(t) = (\widehat{u}_n(t))_{|n| \leqslant d} \in \mathbb{C}^{2d+1}$. Applying the spectral method to (1) derive an ODE of the form

$$\frac{d\widehat{\mathbf{u}}(t)}{dt} = i\pi B \widehat{\mathbf{u}}(t) \tag{2}$$

for some matrix B of size $(2d+1)\times(2d+1)$ that you should specify.

(c) Explain why the eigenvalues of B are all real. Deduce that the explicit Euler discretization of (2) is unstable.

[*Hint: you can assume, without proof, that if P and Q are two Hermitian matrices and P is positive definite, then the eigenvalues of PQ are all real.*]

(d) Consider the case $c(x) = 2 + \cos(\pi x) - (1/2)\sin(\pi x)$ and $d = 1$. Form the matrix B and compute its eigenvalues.

END OF PAPER