UNIVERSITY OF CAMBRIDGE

MAT2

MATHEMATICAL TRIPOS Part II

Monday, 05 June, 2023 1:30pm to 4:30pm

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most six questions** from Section I and from any number of questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Number Theory

Determine whether or not each of the following equations has a solution in integers x and y. Briefly describe the results and algorithms you use.

(i) $1007x + 2314y = 37$,

(ii) $2508x + 3211y = 55$,

(iii) $5x^2 + 16xy + 13y^2 = 365$.

[When there are solutions, you are not required to find any.]

2F Topics In Analysis

(a) State and prove the theorem of Liouville on approximation of algebraic numbers.

(b) If u, v are coprime positive integers and p, q are coprime positive integers with $q > v$, show that

$$\left|\frac{p}{q} - \frac{u}{v}\right| > \frac{1}{q^2}.$$

(c) Show that, if $a_j \in \mathbb{Q}$, $a_j > 0$ and $\sum_{j=1}^{\infty} a_j$ converges, then we can find a strictly increasing sequence of positive integers $n(j)$ such that $\sum_{j=1}^{\infty} a_{n(j)}$ is transcendental.

3I Coding and Cryptography

(a) Let $\mathcal{A}$ be an alphabet of (finite) cardinality m. What does it mean to say that a code $c : \mathcal{A} \to \{0,1\}^*$ is (i) *prefix-free* or (ii) *optimal*?

Suppose that letters $\mu_1, \ldots, \mu_m$ are sent with probabilities $p_1, \ldots, p_m$. Let c be an optimal prefix-free binary code with word lengths $\ell_1, \ldots, \ell_m$.

Show that if $p_i > p_j$ then $\ell_i \leqslant \ell_j$. Show also that among the codewords of maximal length there must exist two that differ only in the last digit.

(b) Letters $\mu_1, \ldots, \mu_5$ are transmitted with probabilities $0.4, 0.2, 0.2, 0.1, 0.1$. Determine whether there are optimal binary codings with either (i) all but one codeword of the same length, or (ii) each codeword a different length. Justify your answers.

4I Automata & Formal Languages

(a) Define what it means for a grammar to be in *Chomsky normal form*.

(b) Suppose G is a grammar in Chomsky normal form. If $w \in \mathcal{L}(G)$ has $|w| = n$, what is the length of a G-derivation of w? [No justification is required.]

(c) Let $\Sigma = \{a, b\}$, $V = \{S, A, B, C\}$. Consider the grammar $G = (\Sigma, V, P, S)$ in Chomsky normal form given by $P = \{S \to AC, C \to BA, A \to AB, B \to BA, A \to a, B \to b\}$. Show that the word *abbabba* is in $\mathcal{L}(G)$ by providing a G-parse tree for it.

A grammar G is said to be in *weak Chomsky normal form* if all production rules are either of the form $A \to a$, $A \to BC$, or $A \to BCD$, for variables A, B, C, D and letters a.

(d) Give a grammar G' in weak Chomsky normal form that is

(i) equivalent to the grammar G from part (c) and

(ii) there is a G'-derivation for the word *abbabba* of length strictly shorter than the number given in part (b).

Justify your answer.

5J Statistical Modelling

Consider a possibly biased coin. Suppose the probability of flipping a head is $0 < p < 1$ and p is unknown. Let $r > 0$ be given. In a sequence of flips, let X be the total number of tails when r heads are reached. Show that

$$\mathbb{P}(X = x) = \binom{x + r - 1}{x} (1 - p)^x p^r, \ x = 0, 1, \ldots .$$

Show that this is a one-parameter exponential family. Find its natural parameter, sufficient statistic, and cumulant function, and compute the mean and variance of X in terms of p.

6C Mathematical Biology

Consider a birth-death process in which births always give rise to 3 offspring, with rate λ, while the death rate per individual is β. Draw a transition diagram and write down the master equation for this system.

Show that the population mean is given by

$$\langle n \rangle = \frac{3\lambda}{\beta} \left(1 - e^{-\beta t}\right) + n_0 e^{-\beta t},$$

where n_0 is the initial population mean, and that the population variance satisfies

$$\sigma^2 \to \frac{6\lambda}{\beta} \quad \text{as} \quad t \to \infty.$$

7E Further Complex Methods

Starting from the Euler product formula for the gamma function,

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\dots(z+n)},$$

show that

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z/k},$$

where Euler's constant is defined by $\gamma = \lim_{n\to\infty}(1 + \frac{1}{2} + \dots + \frac{1}{n} - \log n)$. You may assume that $\gamma = 0.5772\dots$.

The digamma function $\psi(z)$ is defined by $\psi(z) = d(\log \Gamma(z))/dz$. Show that

$$\psi(z) = -\gamma - \frac{1}{z} + z \sum_{k=1}^{\infty} \frac{1}{(z+k)k}.$$

Use this formula to deduce that, for z real and positive, $\psi'(z) > 0$ and hence that $\psi(z)$ has a single zero on the positive real axis which is located in the interval $(1, 2)$.

8D Classical Dynamics

The Lagrangian for a particle of charge q and mass m in an electromagnetic field is

$$L = \frac{1}{2} m\dot{\mathbf{r}}^2 - q(\phi - \dot{\mathbf{r}} \cdot \mathbf{A}),$$

where $\mathbf{A}(\mathbf{r}, t)$ is the vector potential and $\phi(\mathbf{r}, t)$ is the scalar potential associated with the electromagnetic field.

(a) Determine how L changes under the gauge transformation

$$\phi \mapsto \phi - \frac{\partial f}{\partial t}, \qquad \mathbf{A} \mapsto \mathbf{A} + \boldsymbol{\nabla} f,$$

where $f(\mathbf{r}, t)$ is a smooth function. Why does this change in L not affect the Euler-Lagrange equations?

(b) Show that the Euler-Lagrange equations imply the Lorentz force law.

(c) Now suppose that the electric field vanishes and the magnetic field is constant and uniform. Show that the component of the particle's canonical momentum along the direction of the magnetic field is conserved.

9B Cosmology

(a) A homogeneous and isotropic fluid has an energy density $\rho(t)$ and pressure $P(t)$. Use the relation $\mathrm{d}E = -P\,\mathrm{d}V$ for the energy E to derive the continuity equation in a universe with scale factor $a(t)$,

$$\dot{\rho} + 3\,\frac{\dot{a}}{a}\,(\rho + P) = 0\,,$$

where the overdot indicates differentiation with respect to time t. *[Hint: recall that the physical volume $V(t) = a(t)^3 V_0$, where V_0 is the co-moving volume.]*

(b) Given that the scale factor $a(t)$ satisfies the Raychaudhuri equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho + 3P)\,,$$

where G is Newton's constant and c is the speed of light, show that the quantity

$$Q = \frac{8\pi G}{3c^2}\rho\, a^2 - \dot{a}^2\,,$$

is time independent.

(c) The pressure P is related to ρ by the equation of state

$$P = \omega\,\rho, \quad |\omega| < 1\,.$$

Given that $a(t_0) = 1$, find $a(t)$ for an *expanding* universe with $Q = 0$, and hence show that $a(t_\star) = 0$ for some $t_\star < t_0$.

10D Quantum Information and Computation

(a) Assume that you are given a device that is able to clone arbitrary quantum states. Consider two states $|\phi\rangle, |\psi\rangle$ with $|\phi\rangle \neq |\psi\rangle$. Show how the given device can be used to distinguish between these states with arbitrarily high success probability. [You may use without proof any results from the course provided these are clearly stated.]

(b) Assume you are given a device that is able to distinguish the states $|\phi\rangle$ and $|\psi\rangle$ perfectly. Show how this can be used to clone these states. [You can assume that you are able to prepare any computational basis state and implement any unitary operator U.]

(c) Let $\{|\phi_0\rangle, |\phi_1\rangle\}$ and $\{|\psi_0\rangle, |\psi_1\rangle\}$ be two sets of states. Show that there exists a unitary operator U and states $|e_0\rangle$ and $|e_1\rangle$ such that

$$U|\phi_0\rangle|0\rangle = |\psi_0\rangle|e_0\rangle$$
$$U|\phi_1\rangle|0\rangle = |\psi_1\rangle|e_1\rangle$$

if and only if $|\langle\phi_0|\phi_1\rangle| \leqslant |\langle\psi_0|\psi_1\rangle|$.

[*Hint: You can use the fact that for sets of states* $\{|\xi_0\rangle, |\xi_1\rangle\}$ *and* $\{|\eta_0\rangle, |\eta_1\rangle\}$ *with* $\langle\xi_0|\xi_1\rangle = \langle\eta_0|\eta_1\rangle$ *there exists a unitary operator* U *such that* $U|\xi_0\rangle = |\eta_0\rangle$ *and* $U|\xi_1\rangle = |\eta_1\rangle$.]

SECTION II

11I Coding and Cryptography

(a) What is a *binary symmetric channel* (BSC) with error probability p? Write down its channel matrix. Why can we assume that $p < \frac{1}{2}$? State Shannon's second coding theorem and use it to compute the capacity of this channel.

(b) Codewords 00 and 11 are sent with equal probability through a BSC with error probability p. Compute the mutual information between the codeword sent and the first digit received as output. Show that the extra mutual information gained on receipt of the second digit is $H(2p(1-p)) - H(p)$ bits. [Here $H(p)$ denotes the entropy of a random variable which takes the value 1 with probability p and 0 with probability $1-p$.]

(c) Consider a ternary alphabet and a channel that has channel matrix

$$\begin{pmatrix} 1-2\alpha & \alpha & \alpha \\ \alpha & 1-2\alpha & \alpha \\ \alpha & \alpha & 1-2\alpha \end{pmatrix}.$$

Calculate the capacity of the channel.

12I Automata & Formal Languages

Let Σ be an alphabet, $\mathbb{W}$ the set of words over Σ, $A, B \subseteq \mathbb{W}$, and $\mathcal{C}$ any set of subsets of $\mathbb{W}$.

(i) Define what $A \leqslant_{\mathrm{m}} B$ means.

(ii) Define what it means for a set A to be $\mathcal{C}$-*complete.*

(iii) Define what it means for A to be in Σ_1.

(iv) Define the *halting problem* $\mathbf{K}$.

(v) Prove that the halting problem $\mathbf{K}$ is Σ_1-complete.

A set $P \subseteq \mathbb{W}$ is in Π_2 if and only if there is a computable partial function $f : \mathbb{W} \times \mathbb{W} \dashrightarrow \mathbb{W}$ such that for all $w \in \mathbb{W}$, we have that $w \in P$ if and only if for all $v \in \mathbb{W}$, $f(w, v)\downarrow$.

(vi) We define $\mathbf{Tot} \subset \mathbb{W}$ to be the set $\{v \,;\, \mathrm{W}_v = \mathbb{W}\}$. Show that $\mathbf{Tot}$ is Π_2-complete.

[*In this entire question, you are allowed to use the fact that truncated computation functions are computable, provided that you give a precise and correct definition of the function used. You may use the partial function $f_{w,1}$ without providing a definition.*]

13J Statistical Modelling

Let X be a fixed $n \times p$ design matrix with full column rank. Let H be the projection matrix onto the column space of X. Suppose the n-vector of response Y satisfies $Y \sim \mathrm{N}(\mu, \sigma^2 I_n)$ where the n-vector μ is fixed but unknown. Let $Y^* \sim \mathrm{N}(\mu, \sigma^2 I_n)$ be another random vector that has the same distribution as Y but is independent of Y.

(i) Show that

$$\mathbb{E}(\|HY - Y^*\|^2) = \|(I - H)\mu\|^2 + (n + p)\sigma^2.$$

Explain why the above identity is an example of the bias-variance tradeoff. [You may use without proof the fact that H is a projection matrix with rank p.]

(ii) Suppose σ^2 is known. Show that Mallows' C_p, given by

$$C_p = \|Y - HY\|^2 + 2p\sigma^2,$$

is an unbiased estimator of $\mathbb{E}(\|HY - Y^*\|^2)$.

For the rest of this question, suppose $\mu = X\beta$ for some unknown p-vector β and σ^2 is unknown.

(iii) Write down the $(1 - \alpha)$-level confidence ellipsoid for β.

(iv) Recall *Cook's distance* for the observation (X_i, Y_i) (where X_i^T is the ith row of X) is a measure of the influence of (X_i, Y_i) on the fitted values. Give the precise definition of Cook's distance and give its interpretation in terms of the confidence ellipsoid for β.

(v) In the model above with $n = 100$ and $p = 4$, you notice that one observation has Cook's distance 3.1. Would you be concerned about the influence of this observation? Justify your answer.

[*Hint: You may find some of the following facts useful:*

1. *If* $Z \sim \chi^2_4$, *then* $\mathbb{P}(Z \leqslant 1.06) = 0.1$, $\mathbb{P}(Z \leqslant 7.78) = 0.9$.
2. *If* $Z \sim F_{4,96}$, *then* $\mathbb{P}(Z \leqslant 0.26) = 0.1$, $\mathbb{P}(Z \leqslant 2.00) = 0.9$.
3. *If* $Z \sim F_{96,4}$, *then* $\mathbb{P}(Z \leqslant 0.50) = 0.1$, $\mathbb{P}(Z \leqslant 3.78) = 0.9$.]

14E Further Complex Methods

The Laguerre differential equation is

$$zy'' + (1-z)y' + \lambda y = 0,$$

where λ is a real constant.

Show that $z = 0$ is a regular singular point of the Laguerre equation. Briefly explain why in the neighbourhood of $z = 0$ the equation has only one solution, $y_1(z)$, that takes the form of a power series, up to multiplication by a constant. A second, linearly independent, solution is $y_2(z)$. What do you expect to be the leading term in an expansion of $y_2(z)$ in the neighbourhood of $z = 0$?

Seek solutions to the Laguerre equation of the form

$$y(z) = \int_\gamma e^{zt} f(t) dt,$$

determining the form required for the function $f(t)$ and the conditions required on the contour γ.

Assume that $\mathrm{Re}(z) > 0$. Consider separately each of the cases:

(i) $\lambda < 0$ and λ non-integer;

(ii) $\lambda > 0$ and λ non-integer;

(iii) λ equal to a negative integer;

(iv) λ equal to a non-negative integer.

Show that in each of these cases one possible choice of γ, say γ_1, is a finite closed contour, and another, say γ_2, is a contour starting at a finite value of t and extending to $-\infty$. Provide a sketch giving a clear specification of these contours in each of the cases (i)-(iv).

Show that the $y(z)$ obtained from the finite closed contour γ_1 is a constant multiple of the solution $y_1(z)$ and that in case (iv) this solution is a polynomial in z. What can you say about the form of this solution in case (iii)?

15B Cosmology

In a homogeneous and isotropic universe, the scale factor $a(t)$ obeys the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K\,c^2}{a^2} = \frac{8\,\pi\,G}{3c^2}\rho\,,$$

where K is a constant curvature parameter and ρ is the energy density which, together with the pressure P, satisfies the continuity equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0\,. \qquad (\star)$$

(a) Use the equations to show that the rate of change of the Hubble parameter $H = \dot{a}/a$ satisfies

$$\dot{H} + H^2 = -\frac{4\pi G}{3c^2}(\rho + 3P)\,.$$

(b) Suppose that an *expanding* universe is filled with radiation (with energy density ρ_r and pressure $P_r = \rho_r/3$) as well as a cosmological constant component (with density ρ_Λ and pressure $P_\Lambda = -\rho_\Lambda$). Both radiation and cosmological constant components satisfy the continuity equation $(\star)$ separately.

Given that the energy densities of these two components are measured today ($t = t_0$) to be

$$\rho_{r0} = \beta\frac{3c^2 H_0^2}{8\pi G} \quad \text{and} \quad \rho_{\Lambda 0} = \frac{3c^2 H_0^2}{8\pi G} \quad \text{with constant} \quad \beta > 0 \quad \text{and} \quad a(t_0) = 1\,,$$

show that the curvature parameter must satisfy $Kc^2 = \beta\,H_0^2$. Hence, derive the following relations for the Hubble parameter H and its time derivative:

$$H^2 = \frac{H_0^2}{a^4}(\beta - \beta\,a^2 + a^4)\,,$$
$$\dot{H} = -\beta\frac{H_0^2}{a^4}(2 - a^2)\,.$$

(c) Show qualitatively that universes with $a(0) = 0$ and $\beta > 4$ will recollapse to a Big Crunch in the future. *[Hint: you may find it useful to sketch $a^4 H^2$ and $a^4 \dot{H}$ versus a^2 for representative values of β.]*

(d) For $\beta = 4$, find an explicit solution for the scale factor $a(t)$ satisfying $a(0) = 0$. Find the limiting behaviours of this solution for large and small t.

16H Logic and Set Theory

(a) State and prove the Compactness Theorem for first-order logic. State and prove the Upward Lowenheim-Skolem Theorem. State the Downward Lowenheim-Skolem Theorem, and explain briefly why it is true.

(b) For a language L and an L-structure A, an *automorphism* of A is a bijection from A to itself that preserves all the functions and relations of L. An L-structure is *rigid* if it has no automorphism apart from the identity map.

(i) If a theory T in a language L has arbitrarily large finite non-rigid models, show that T also has an infinite non-rigid model.

(ii) If a theory T in a language L has arbitrarily large finite models, and every finite model of T is rigid, does it follow that every infinite model of T is rigid? Justify your answer.

[*You may assume the Completeness Theorem for first-order logic.*]

17H Graph Theory

(a) By considering the random graph $G(n,p)$, with $p = (1/2)n^{-2/3}$, show that for every $k \geqslant 1$ there exists a graph G that contains no $K_{3,3}$ and has $\chi(G) > k$.

(b) (i) For $p = n^{-2/3}\log n$, let $G \sim G(n,p)$. Show that

$$\lim_{n\to\infty} \mathbb{P}(G \text{ contains at least 100 copies of } K_4) = 1.$$

(ii) Let $\{H_1, \ldots, H_r\}$ be vertex disjoint copies of K_4, in a graph G. (Recall that this means that $V(H_i) \cap V(H_j) = \emptyset$, for all $i \neq j$.)

Show that

$$\lim_{n\to\infty} \mathbb{P}(G \text{ contains at least 100 vertex disjoint copies of } K_4) = 1.$$

[*Hint: you may wish to consider the number of K_4's in the graph and also the number of pairs of K_4's that are not disjoint.*]

18I Galois Theory

(a) What does it mean to say that a finite extension L/K is *normal*? Show that L/K is normal if and only if L is a splitting field of some polynomial over K.

(b) Let $M/L/K$ be finite extensions. Which of the following statements are true, and which are false? Give a proof or counterexample in each case.

i) If M/K is normal then M/L is normal.

ii) If M/K is normal then L/K is normal.

iii) If M/L and L/K are normal, then M/K is normal.

(c) Let L be a splitting field of $T^4 - 7$ over $\mathbb{Q}$. Show that $\mathrm{Gal}(L/\mathbb{Q})$ is the dihedral group of order 8. Determine all the subfields of L, and express each of them in the form $\mathbb{Q}(x)$ for some $x \in L$. Which of them are normal extensions of $\mathbb{Q}$?

19H Representation Theory

What is a *representation* of a finite group G? What does it mean to say that a representation is *irreducible*? What is the *degree* of a representation? What does it mean to say that two representations are *isomorphic*?

Consider the dihedral group D_{2n} of order $2n$ for $n \geqslant 3$. Show directly that every irreducible complex representation of D_{2n} has degree at most 2.

For odd $n \geqslant 3$, explicitly construct $\frac{n+3}{2}$ pairwise non-isomorphic irreducible complex representations of D_{2n}. Justify your answer.

For even $n \geqslant 4$, explicitly construct $\frac{n+6}{2}$ pairwise non-isomorphic irreducible complex representations of D_{2n}. Justify your answer.

20H Number Fields

(a) (i) The zeta function $\zeta_K(s)$ of a number field K is the infinite sum $\zeta_K(s) = \sum_{\mathfrak{a} \leqslant \mathcal{O}_K} N(\mathfrak{a})^{-s}$. Show that it factors 'formally' as an infinite product, where the product has a term for each prime ideal of $\mathcal{O}_K$.

[You do not need to show ζ_K converges.]

(ii) Now let $K = \mathbb{Q}(\sqrt{d})$, where $d \neq 0, 1$ and d is square free. Show the zeta function factors 'formally', $\zeta_K(s) = \zeta_{\mathbb{Q}}(s) L(\chi, s)$ where

$$L(\chi, s) = \prod_{p \text{ prime}} (1 - \chi(p) p^{-s})^{-1}$$

for an explicit function χ, which you should determine in terms of how the ideal (p) factorises in $\mathcal{O}_K$.

['Formally' means the terms match up, i.e. you do not need to discuss convergence.]

(b) Let $p \equiv 11 \pmod{12}$ be a prime, and $K = \mathbb{Q}(\sqrt{-p})$.

Show that 3 splits completely in $\mathcal{O}_K$.

Let $\mathfrak{p}_1$ be a factor of the prime ideal (3) of $\mathcal{O}_K$. Define the *class group* Cl_K of K, and explain what it means for the ideal $\mathfrak{p}_1$ of $\mathcal{O}_K$ to have order n in Cl_K.

Suppose $n > 0$ is such that $p > 3^{n+2}$. Show the order of $\mathfrak{p}_1$ in Cl_K does not equal n if n is odd.

21G Algebraic Topology

State the universal property which characterizes an amalgamated free product of groups. State the Seifert-van Kampen theorem.

Suppose that $\{U_1, U_2\}$ is an open cover of a topological space X, that $U_1 \cap U_2$ is path connected and that $x_0 \in U_1 \cap U_2$. If $i_k : U_k \to X$ is the inclusion, prove that $\pi_1(X, x_0)$ is generated by $i_{1*}(\pi_1(U_1, x_0))$ and $i_{2*}(\pi_1(U_2, x_0))$. [You may use the Lebesgue covering lemma if you state it clearly.]

Consider the Mobius band $M = I^2/\sim$, where $(0, x) \sim (1, 1 - x)$. Identify its boundary $\partial M = (I \times \{0, 1\})/\sim$ with S^1. Note that if $f : \partial M \to X$, the space obtained by *attaching a Mobius band to X using f* is $X \cup_f M = (X \amalg M)/\sim$, where now $\sim$ is the smallest equivalence relation containing $x \sim f(x)$ for all $x \in \partial M$. Now let Y be the space obtained by attaching two Mobius bands to $T^2 = S^1 \times S^1$ using the maps $f_1, f_2 : S^1 \to T^2$ given by $f_1(z) = (z, z)$ and $f_2(z) = (z^2, z^3)$. Give a two-generator one-relator presentation of $\pi_1(Y, y_0)$ for some $y_0 \in Y$. Show that this group is non-abelian.

22F Linear Analysis

(a) State the open mapping theorem and the closed graph theorem, and prove that the former implies the latter.

(b) Let V be a Banach space. Give the definition of the *dual space* V^*, and prove that V^* is a Banach space.

(c) Let V be a Banach space over the real field, and let $T : V \to V^*, v \mapsto T_v$ be a linear map between these two Banach spaces that satisfies $T_v(v) \geqslant 0$ for all $v \in V$. Prove that T is continuous.

23F Analysis of Functions

(a) State and prove the Sobolev trace theorem that maps $H^s(\mathbb{R}^n)$ into a suitable Sobolev space over $\mathbb{R}^{n-1}$.

(b) Show that there is no bounded linear operator $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^{n-1})$ satisfying $Tu = u|_{\mathbb{R}^{n-1}\times\{0\}}$ for all $u \in C(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$.

(c) For $u \in C_c^\infty(\mathbb{R}^2)$, prove that

$$\int_{\mathbb{R}^2} |u|^4 \, dx \leqslant C \int_{\mathbb{R}^2} |u|^2 \, dx \int_{\mathbb{R}^2} |\nabla u|^2 \, dx,$$

for a constant C independent of u. [*Hint: First show that, for all* $(x, y) \in \mathbb{R}^2$, $|u(x, y)|^2 \leqslant 2 \int_{\mathbb{R}} |u(x, t)||\nabla u(x, t)| \, dt$.]

24F Riemann Surfaces

Let D be a domain in $\mathbb{C}$. What is a *germ* on D? Define the *space of germs* $\mathcal{G}$ over D. Briefly describe the topology, the forgetful map $\pi : \mathcal{G} \to D$ and the complex structure on $\mathcal{G}$, all without proof. Define the evaluation map $\mathcal{E} : \mathcal{G} \to \mathbb{C}$, and prove that $\mathcal{E}$ is analytic.

Let D be the result of removing the eighth roots of unity from $\mathbb{C}$, and consider the function element $w = \sqrt{z^8 - 1}$ defined over D. Give an explicit gluing construction of the component R of the space of germs corresponding to w. You should construct the evaluation map and the forgetful map on R, and exhibit an analytic embedding $\Phi : R \hookrightarrow \mathcal{G}$. [You do not need to prove that the image of Φ is a component of $\mathcal{G}$.]

Assume that R can be embedded into a compact Riemann surface $\overline{R}$ by adding finitely many points. Assume, furthermore, that the forgetful map π extends to a meromorphic function $\bar{\pi} : \overline{R} \to \mathbb{C}_\infty$. How many points are in $\overline{R} \setminus R$? What is the genus of $\overline{R}$? [You may use standard theorems from the course, as long as you state them carefully.]

25G Algebraic Geometry

Let X be an irreducible affine variety. Define the tangent space to X at a point $p \in X$. What does it mean for X to be *smooth*?

Prove that any irreducible affine cubic has at most one singular point.

Prove that the set of smooth points of any irreducible variety is dense in the Zariski topology.

Let $X \subset \mathbb{P}^n$ be a smooth irreducible hypersurface. Recall there is a natural map

$$\pi : \mathbb{A}^{n+1} \setminus \{\underline{0}\} \to \mathbb{P}^n.$$

Let $Y \subset \mathbb{A}^{n+1}$ be the closure of $\pi^{-1}(X)$. Prove that Y contains at most one singular point. Give examples to show that Y can be smooth and that Y can be singular.

26G Differential Geometry

(a) Let X, Y be smooth manifolds and $f : X \to Y$ be a smooth map. What does it mean for y_0 to be a *regular value* of f? Give an example of a smooth map f that has a regular value, together with a regular value of f, justifying your answer. State Sard's theorem.

(b) Let X and Y be compact manifolds of dimension n and $f : X \to Y$ be a smooth map. Define the degree mod 2 of f, quoting carefully (but without proof) the results from the course necessary to make this well defined.

(c) Let $S \subset \mathbb{R}^3$ be a surface and $p \in S$. Define the exponential map $\exp_p$, explaining carefully its domain. Explain also briefly why the exponential map is smooth. Give an explicit example where the domain of $\exp_p$ is not T_pS, and an example where $\exp_p$ is not surjective, justifying carefully your answers.

(d) Let $S \subset \mathbb{R}^3$ be a compact surface, and let V be a smooth vector field on S. Consider the map $\phi : S \to S$ defined by $\phi(p) = \exp_p(V(p))$. Explain why this map is well-defined and smooth. What is its degree mod 2?

27K Probability and Measure

(a) State and prove Dynkin's lemma.

(b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Show that if $\mathcal{A}_1, \mathcal{A}_2$ are π-systems contained in $\mathcal{F}$ such that

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2) \quad \text{for all } A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2,$$

then the generated σ-algebras $\sigma(\mathcal{A}_1)$ and $\sigma(\mathcal{A}_2)$ are independent.

28J Applied Probability

(a) Arrivals of the Number 1 bus form a Poisson process of rate 1 bus per hour, and arrivals of the Number 8 bus form an independent Poisson process of rate 8 buses per hour. What is the probability that exactly three Number 8 buses pass by while I am waiting for a Number 1?

(b) Let $(N_t, t \geqslant 0)$ be a Poisson process of constant intensity λ on $\mathbb{R}_+$. Conditional on the event $N_t = n$, show that the jump times $(J_1, J_2, \ldots, J_n)$ are distributed as the order statistics of n i.i.d. $U[0,t]$ random variables.

(c) As above, let $N = (N_t, t \geqslant 0)$ be a Poisson process with intensity $\lambda > 0$ and let $(X_i)_{i \geqslant 1}$ be a sequence of i.i.d. random variables, independent of N. Show that if $g(s,x) : \mathbb{R}^2 \mapsto \mathbb{R}$ is a measurable function and J_i are the jump times of N, then for any $\theta \in \mathbb{R}$,

$$\mathbb{E}\left[\exp\left\{\theta \sum_{i=1}^{N_t} g(J_i, X_i)\right\}\right] = \exp\left\{\lambda \int_0^t (\mathbb{E}(e^{\theta g(s,X_1)}) - 1) ds\right\}.$$

(d) Define the age process (the time since the last renewal) $A(t) = t - J_{N_t}$, where J_n is the n-th jump time of the Poisson process N. Show that

$$\mathbb{E}A(t) = (1 - e^{-\lambda t})/\lambda.$$

[You may use without proof that $\mathbb{E}U_{(i)} = \frac{i}{n+1}$, where $U_{(i)}$ is the i-th order statistic of a sample of n i.i.d. $U[0,1]$ random variables.]

29K Principles of Statistics

Consider a parametric model $\{f(\cdot,\theta) : \theta \in \Theta\}$ satisfying the usual regularity conditions.

(a) Let $\widehat{\theta}_n$ denote a maximum likelihood estimator based on i.i.d. samples $X_1, \ldots, X_n$ from the distribution P_{θ_0}. What is the limiting distribution of $\sqrt{n}(\widehat{\theta}_n - \theta_0)$?

For the remainder of this question, let $\Theta = \mathbb{R}^p$.

(b) Define the *Wald statistic* $W_n(\theta)$ based on a parameter $\theta \in \Theta$. Suppose $\theta_0 \in \Theta$. What is the limiting distribution of $W_n(\theta_0)$ based on i.i.d. samples from P_{θ_0}? State an asymptotically valid confidence region for θ_0 with coverage probability $1 - \alpha$.

(c) Suppose $k \leqslant p$. For a fixed $\theta^* \in \Theta$, suppose we wish to test the null hypothesis

$$H_0 : \theta_i = \theta_i^*, \quad \forall 1 \leqslant i \leqslant k,$$

i.e., the first k coordinates of θ are equal to the first k coordinates of θ^*. Define a test statistic (a generalization of the Wald statistic) whose limiting distribution under H_0 is a chi-squared distribution with k degrees of freedom, and rigorously prove the correctness of the limiting distribution. Deduce an asymptotically valid level-α hypothesis test based on the statistic.

[*You may quote any result from the lectures that you need, without proof.*]

30K Stochastic Financial Models

Let U be a smooth, increasing and concave function on $\mathbb{R}$.

(a) Given a vector space $\mathcal{X}$ of random variables, define a function F by

$$F(y) = \sup_{X \in \mathcal{X}} \mathbb{E}[U(X + y)],$$

and suppose that for all constants $y \in \mathbb{R}$ the supremum is achieved by some $X_y \in \mathcal{X}$. Show that F is increasing and concave.

(b) Given a constant m and a random variable Z such that $\mathbb{E}(Z) = 0$, define a function G by

$$G(s) = \mathbb{E}[U(m + sZ)].$$

Show that G is concave. Show that G is decreasing on $[0, \infty)$.

Now consider a market with interest rate $r \geqslant 0$ and d risky assets, such that the vector of time-n prices S_n evolves as

$$S_n = (1 + r)S_{n-1} + \xi_n.$$

Assume that the sequence $(\xi_n)_{n \geqslant 1}$ of $\mathbb{R}^d$-valued random vectors is IID and generates the filtration. Given initial wealth X_0, an investor's time-n wealth X_n evolves as

$$X_n = (1 + r)X_{n-1} + \theta_n^\top \xi_n$$

for $n \geqslant 1$, where the trading strategy $(\theta_n)_{n \geqslant 1}$ is previsible. Given a time horizon $N \geqslant 1$, the investor seeks a trading strategy to maximise $\mathbb{E}[U(X_N)]$.

(c) Write down the Bellman equation for the investor's value function V. Assuming that an optimal portfolio exists at each time-step, show that $V(n, \cdot)$ is increasing and concave for all $0 \leqslant n \leqslant N$.

(d) Now assume that for all $n \geqslant 1$, ξ_n has the multi-variate Gaussian $N(b, \Sigma)$ distribution, where $b \neq 0$ and Σ is positive definite. Assuming the existence of a unique optimal time-n portfolio θ_n^* for each $1 \leqslant n \leqslant N$, show that there exist non-negative random variables λ_n such that $\theta_n^* = \lambda_n \Sigma^{-1} b$.

[You may use the mutual fund theorem or the Gaussian integration-by-parts formula without proof.]

31J Mathematics of Machine Learning

(a) What does it mean for a set $C \subseteq \mathbb{R}^d$ to be *convex*?

(b) What does it mean for a function $f : C \to \mathbb{R}$ to be *strictly convex*? Show that any minimiser of f must be unique.

(c) Define the *projection* $\pi_C(x)$ of a point $x \in \mathbb{R}^d$ onto a closed convex set C. Briefly explain why this is unique. [Standard results about convex functions may be used without proof, and you need not show that $\pi_C(x)$ always exists.]

(d) Prove that $\pi \in C$ is the projection of x onto a closed convex set C if

$$(x-\pi)^T(z-\pi) \leqslant 0 \qquad \text{for all } z \in C.$$

(e) Let C be a closed convex set given by

$$C := \left\{ \begin{pmatrix} v \\ s \end{pmatrix} \in \mathbb{R}^p \times \mathbb{R} : \ \|v\|_2 \leqslant s \right\}.$$

Using part (d) or otherwise, show that if $(u,t) \in \mathbb{R}^p \times \mathbb{R}$ satisfy $\|u\|_2 \geqslant |t|$ then

$$\pi_C\left(\begin{pmatrix} u \\ t \end{pmatrix}\right) = \frac{1}{2}\left(1 + \frac{t}{\|u\|_2}\right)\begin{pmatrix} u \\ \|u\|_2 \end{pmatrix}.$$

What is $\pi_C\left(\begin{pmatrix} u \\ t \end{pmatrix}\right)$ when $\|u\|_2 \leqslant -t$?

(f) Let C be as in (e) and let $(X_i, Y_i) \in \mathbb{R}^{p+1} \times \mathbb{R}$ for $i = 1, \ldots, n$ be data formed of input–output pairs. Write down the projected gradient descent procedure for finding the empirical risk minimiser with squared error loss over the hypothesis class $\mathcal{H} = \{h : h(x) = \beta^T x, \text{ where } \beta \in C\}$, giving explicit forms for any gradients or projections involved.

32A Dynamical Systems

(a) State and prove Dulac's theorem. State the Poincaré-Bendixson theorem.

(b) Consider the system

$$\dot{x} = r - x(1+s) + x^2 y \tag{1}$$
$$\dot{y} = sx - x^2 y, \tag{2}$$

where r and s are positive numbers. Show that there is a unique fixed point. Show that for a suitable choice of α to be determined, with $0 < \alpha < r$, trajectories enter the closed region bounded by $x = \alpha$, $y = s/\alpha$, $x + y = r + s/\alpha$ and $y = 0$. Deduce that when $s - 1 > r^2$ the system has a periodic orbit.

33E Integrable Systems

Let $q = q(x,t)$ and $r = r(x,t)$ be complex valued functions and consider the matrices (U,V) defined by

$$U(\lambda) = \begin{pmatrix} i\lambda & iq \\ ir & -i\lambda \end{pmatrix}, \quad V(\lambda) = 2i\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2i\lambda \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} + \begin{pmatrix} 0 & q_x \\ -r_x & 0 \end{pmatrix} - i \begin{pmatrix} rq & 0 \\ 0 & -rq \end{pmatrix}.$$

Derive the zero curvature equation as the consistency condition for the system of equations

$$\Psi_x = U\Psi\,, \qquad \Psi_t = V\Psi$$

and show that it holds precisely when q, r satisfy a system of the form

$$ir_t + r_{xx} + aqr^2 = 0\,, \tag{1}$$

$$iq_t - q_{xx} - arq^2 = 0\,, \tag{2}$$

where a is a real number which you should determine. Show that if $r = \overline{q}$ this system reduces to the *nonlinear Schrödinger equation*

$$ir_t + r_{xx} + a|r|^2 r = 0\,, \tag{NLS1}$$

and find a similar reduction to the equation

$$ir_t + r_{xx} - a|r|^2 r = 0\,. \tag{NLS2}$$

Write these equations in Hamiltonian form. Search for solutions to (NLS1) and (NLS2) of the form $e^{-iEt}f(x)$ with real constant E and smooth, rapidly decreasing real-valued f. In each case either find such a solution explicitly, or explain briefly why it is not expected to exist.

[*Hint: you may use without derivation the indefinite integral*

$$\int \frac{dy}{\sqrt{\lambda^2 y^2 - y^4}} = -\frac{1}{\lambda}\operatorname{sech}^{-1}\frac{y}{\lambda}\,.]$$

34B Principles of Quantum Mechanics

(a) Write down the Hamiltonian for a quantum harmonic oscillator of frequency ω in terms of the creation and annihilation operators $A^\dagger$ and A. You may work in units where $\hbar = 1$. Define the *number operator* N and state all commutation relations amongst A, $A^\dagger$ and N. Show that the eigenvalues of N are: (i) real, (ii) non-negative and (iii) integers.

(b) Consider a system of two independent harmonic oscillators of frequency $\omega = 1$ and $\omega = 2$. The $\omega = 1$ oscillator has creation and annihilation operators $A^\dagger$ and A, while the $\omega = 2$ oscillator has creation and annihilation operators $B^\dagger$ and B.

(i) Find the five lowest eigenvalues of the Hamiltonian H_0 of the combined system and determine the degeneracy of each of them.

(ii) The system is perturbed so that it is now described by the new Hamiltonian $H = H_0 + \lambda H'$, where $\lambda \in \mathbb{R}$ and $H' = A^\dagger A^\dagger B + AAB^\dagger$. Using degenerate perturbation theory, calculate to order λ the energies of the eigenstates associated with the level $E_0 = \frac{9}{2}$. Write down the perturbed eigenstates, to order λ, associated with these perturbed energies. By explicit evaluation show that they are in fact exact eigenstates of H with these energies as eigenvalues.

[In part (b) you may use without proof that for the harmonic oscillator

$$A\,|n\rangle = \sqrt{n}\,|n-1\rangle \qquad \text{and} \qquad A^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,.]$$

35D Applications of Quantum Mechanics

(a) A beam of particles of mass m and energy E, moving in one dimension, scatters off a potential barrier $V(x)$ which is localised near the origin $x = 0$ and is reflection invariant, $V(x) = V(-x)$ for all x. With reference to the asymptotic form of the wave function as $x \to \pm\infty$, define the corresponding *reflection* and *transmission coefficients*, denoted r and t respectively, and write down the *S-matrix* $\mathcal{S}$.

For the case $V(x) = V_0\,\delta(x)$, where $\delta(x)$ denotes the Dirac δ-function, determine r and t as functions of the energy E, and show explicitly that $\mathcal{S}$ is a unitary matrix.

(b) A particle of mass m and energy E moves in one dimension subject to a potential $\tilde{V}(x)$ obeying $\tilde{V}(x + a) = \tilde{V}(x)$ for all x. Define the corresponding *Floquet matrix* $\mathcal{M}$. Explain briefly how the Floquet matrix determines the resulting energy spectrum of continuous bands separated by forbidden regions. [You may state without proof any results from the course you might need.]

Determine $\mathcal{M}$ as a function of E for the case $\tilde{V}(x) = V_0 \sum_{n=-\infty}^{+\infty} \delta(x - na)$. Find algebraic equations which determine all the edges of the allowed energy bands. For each edge express $\exp(-ika)$ at the edge in terms of r and t. Here $r = r(E)$ and $t = t(E)$ are the reflection and transmission coefficients determined in part (a), and $E = \hbar^2 k^2/2m$ with $k > 0$.

36A Statistical Physics

(a) What is meant by the *microcanonical*, *canonical* and *grand canonical ensembles*? Under what conditions is the choice of ensemble irrelevant?

(b) Consider a classical particle of mass m moving non-relativistically in two-dimensional space enclosed inside a circle of radius R and attached by a spring to the centre. The particle therefore moves in a potential

$$V(r) = \begin{cases} \frac{1}{2}\kappa r^2 & \text{for } r < R\,, \\ \infty & \text{for } r \geqslant R\,, \end{cases}$$

where κ is the spring constant and $r^2 = x^2 + y^2$. The particle is coupled to a heat reservoir at temperature T.

(i) Calculate the partition function for the particle.

(ii) Calculate the average energy $\langle E \rangle$ and the average potential energy $\langle V \rangle$ of the particle.

(iii) Compute $\langle E \rangle$ in the two limits $\frac{1}{2}\kappa R^2 \gg k_B T$ and $\frac{1}{2}\kappa R^2 \ll k_B T$. How do these two results compare with what is expected from equipartition of energy?

(iv) Compute the partition function for a collection of N identical non-interacting such particles.

37A Electrodynamics

Consider spacetime with coordinates $x^\mu = (ct, \mathbf{x})$ and metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, where $\mu, \nu = 0, 1, 2, 3$ and c is the speed of light. An electromagnetic field described by the vector potential $A_\mu(x)$ fills spacetime, and a particle of mass m and charge q moves through it along the worldline $x^\mu(\lambda)$, where λ is a parameter along the worldline.

(a) Explain using the requirements of Lorentz invariance and gauge invariance why the action

$$S = -mc \int (-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu)^{\frac{1}{2}}\, d\lambda + q \int A_\mu(x)\dot{x}^\mu d\lambda$$

is suitable for describing the relativistic mechanics of the particle, where $\dot{x}^\mu = dx^\mu/d\lambda$.

(b) By varying the action with respect to a worldline with fixed end points, obtain the Euler-Lagrange equations of motion

$$m\frac{du^\mu}{d\tau} = qF^\mu{}_\nu u^\nu$$

where $u^\mu(\tau) = dx^\mu/d\tau$ is the four-velocity, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor and τ is the proper time.

(c) Show that the rate of change of the particle energy $\epsilon = \gamma mc^2$ satisfies

$$\frac{d\epsilon}{dt} = q\mathbf{E} \cdot \mathbf{v}$$

where $\mathbf{v} = d\mathbf{x}/dt$ is the particle velocity, $\mathbf{E}$ is the electric field and $\gamma = 1/\sqrt{1 - v^2/c^2}$.

(d) Hence, or otherwise, derive the following expression for the acceleration of the particle

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m\gamma}[\mathbf{E} + \mathbf{v} \times \mathbf{B} - \frac{1}{c^2}\mathbf{v}(\mathbf{v} \cdot \mathbf{E})],$$

where $\mathbf{B}$ is the magnetic field. Derive the non-relativistic limit of the above expression and comment on its relationship with the Lorentz force law.

38B General Relativity

A Klein-Gordon scalar field ϕ satisfies the equation of motion $\nabla^\alpha \nabla_\alpha \phi = m^2 \phi$ where m is a constant. Its stress-energy tensor takes the form:

$$T_{\mu\nu} = \frac{1}{2}\Big[\nabla_\mu \phi \nabla_\nu \phi + g_{\mu\nu}\big(A \nabla_\rho \phi \nabla^\rho \phi + B\phi^2\big)\Big]. \qquad (*)$$

(a) Using the fact that the stress-energy tensor is covariantly conserved, determine the value of the parameters A and B.

(b) Using the Einstein equation, write an expression for the Ricci curvature $R_{\mu\nu}$ in terms of ϕ and its derivatives, in a $D > 2$ dimensional spacetime. Simplify your answer as much as possible.

(c) Now consider a general stress-energy tensor of the form $(*)$, with A and B not necessarily given by the values you have found above. The stress-energy tensor is said to satisfy the *weak energy condition* if

$$T_{\mu\nu} X^\mu X^\nu \geqslant 0$$

for all timelike vectors X^μ. Find the most general constraints on A and B such that $(*)$ satisfies the weak energy condition, and show that your answer to part (a) satisfies these constraints.

[Hint: you may find it useful to work in normal coordinates and furthermore to choose these coordinates such that $X^\mu = \delta_0{}^\mu$.]

39C Fluid Dynamics II

(a) In an incompressible Stokes flow, show that the Laplacian of the vorticity is zero. If the flow is also two-dimensional, deduce the equation satisfied by the streamfunction $\psi(x, y)$.

(b) A stationary two-dimensional rigid disk of radius a, centred at the origin, is subject to an external shear flow $\mathbf{u}_\infty = \gamma y \mathbf{e}_x$, where γ is a constant and $\mathbf{e}_x$ is the unit vector in the x-direction.

(i) What are the equations and boundary conditions satisfied by ψ for the flow outside the disk?

(ii) Solve for ψ, ensuring that your solution satisfies all the required boundary conditions.

(iii) Compute the hydrodynamic torque exerted by the shear flow on the disk.

[*Hint: in polar coordinates,*

$$(\nabla \times \mathbf{u}) \cdot \mathbf{e}_z = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta},$$

$$\nabla^2 f(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2},$$

$$\sigma_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right].]$$

40C Waves

(a) Starting from the equations governing sound waves linearized about a state with density ρ_0 and sound speed c_0, derive the acoustic energy equation, giving expressions for the kinetic energy density K, the potential energy density due to compression W and the wave-energy flux $\mathbf{I}$.

(b) The radius $R(t)$ of a sphere oscillates according to

$$R(t) = a + \mathrm{Re}(\epsilon e^{i\omega t}),$$

where ϵ and ω are real, with $0 < \epsilon \ll a$.

(i) Find an expression for the velocity potential $\phi(r,t)$ in the region outside the sphere.

(ii) Show that for an appropriate time-average, which you should define carefully, the time-averaged rate of working by the surface of the sphere is

$$2\pi a^2 \rho_0 \omega^2 \epsilon^2 c_0 \frac{\omega^2 a^2}{c_0^2 + \omega^2 a^2}.$$

(iii) Calculate the value at $r = a$ of the dimensionless ratio $c_0 \langle K + W \rangle / |\langle \mathbf{I} \rangle|$, where angle brackets denote the time average used above.

(iv) Comment briefly on the limits $c_0 \ll \omega a$ and $c_0 \gg \omega a$, explaining their physical meaning and considering the relative magnitudes of the time-averaged kinetic energy, potential energy and acoustic energy flux.

41C Numerical Analysis

Consider the diffusion equation in 2D on a square domain $(x, y) \in [0, 1]^2$

$$\frac{\partial u}{\partial t}(x, y, t) = \nabla^2 u(x, y, t), \tag{1}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian. We assume zero Dirichlet boundary conditions $u(x, 0, t) = u(x, 1, t) = u(0, y, t) = u(1, y, t) = 0$ for all $t \geqslant 0$.

We discretize the domain $[0, 1]^2$ by a regular grid (ih, jh) where $0 \leqslant i, j \leqslant m + 1$ and $h = 1/(m + 1)$.

(a) Show that if we discretize the Laplacian operator ∇^2 by the five-point finite-difference scheme, we get an ordinary differential equation of the form

$$\frac{d\mathbf{u}(t)}{dt} = \frac{1}{h^2}(A_x + A_y)\mathbf{u}(t) \qquad \mathbf{u}(t) \in \mathbb{R}^{m^2}, \tag{2}$$

where $\mathbf{u}_{ij} \approx u(ih, jh)$, and A_x and A_y are two matrices of size $m^2 \times m^2$ that correspond respectively to discretizations of $\partial^2/\partial x^2$ and $\partial^2/\partial y^2$. You should verify that your matrices A_x and A_y commute, i.e., $A_x A_y = A_y A_x$.

(b) Consider the following time-stepping scheme for (2), where $k > 0$ is the time step and $\mu = k/h^2$:

$$\begin{cases} \mathbf{u}^{n+1/2} & = \mathbf{u}^n + \mu A_y \mathbf{u}^{n+1/2} \\ \mathbf{u}^{n+1} & = \mathbf{u}^{n+1/2} + \mu A_x \mathbf{u}^{n+1/2}. \end{cases}$$

(i) Explain why $\mathbf{u}^{n+1}$ can be computed from $\mathbf{u}^n$ using at most $\mathcal{O}(m^2)$ arithmetic operations.
(ii) Show that $\mathbf{u}^{n+1} = C\mathbf{u}^n$ for some matrix C that you should make explicit. Deduce conditions on μ for the method to be stable.

[*Hint: For (ii), you can use the fact that A_x and A_y are diagonalizable in the same orthogonal basis of eigenvectors $(\mathbf{v}^{(p,q)})_{1 \leqslant p,q \leqslant m}$ where $\mathbf{v}^{(p,q)} \in \mathbb{R}^{m^2}$, and that $A_x \mathbf{v}^{(p,q)} = \lambda_p \mathbf{v}^{(p,q)}$ and $A_y \mathbf{v}^{(p,q)} = \lambda_q \mathbf{v}^{(p,q)}$ and $\lambda_p = -4\sin^2(p\pi h/2)$.*]

(c) We consider the following modified discretization method to compute $\mathbf{u}^{n+1}$ from $\mathbf{u}^n$:

$$\begin{cases} \widetilde{\mathbf{u}}^{n+1/2} & = \mathbf{u}^n + \mu A_y \widetilde{\mathbf{u}}^{n+1/2} \\ \widetilde{\mathbf{u}}^{n+1} & = \widetilde{\mathbf{u}}^{n+1/2} + \mu A_x \widetilde{\mathbf{u}}^{n+1/2} \\ \mathbf{u}^{n+1} & = \widetilde{\mathbf{u}}^{n+1} + \mu A_x(\mathbf{u}^{n+1} - \mathbf{u}^n). \end{cases}$$

By writing the method as $\mathbf{u}^{n+1} = D\mathbf{u}^n$ for some matrix D, and analyzing the eigenvalues of D, show that this method is stable for any choice of $\mu > 0$.

END OF PAPER