UNIVERSITY OF CAMBRIDGE

MAT1

MATHEMATICAL TRIPOS Part IB

Friday, 09 June, 2023 1:30pm to 4:30pm

PAPER 4

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most four** questions from Section I and **at most six** questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Linear Algebra

Let V be a finite-dimensional real vector space. What is a *non-degenerate bilinear form* on V?

If $B_1(-,-)$ is a non-degenerate bilinear form on V and $B_2(-,-)$ is a bilinear form on V, which may be degenerate, show that there is a linear map $\alpha : V \to V$ such that

$$B_2(v,w) = B_1(v,\alpha(w)) \text{ for all } v,w \in V.$$

Show that

$$\{w \in V \ : \ B_2(v,w) = 0 \text{ for all } v \in V\} = \mathrm{Ker}(\alpha).$$

[*You may use any results on dual vector spaces provided they are clearly stated.*]

2G Analysis and Topology

Let (f_n) be a sequence of continuous real-valued functions on a topological space X. Assume that there is a function $f\colon X \to \mathbb{R}$ such that every $x \in X$ has a neighbourhood U on which (f_n) converges to f uniformly. Show that f is continuous at every $x \in X$. Further show that (f_n) converges to f uniformly on every compact subset of X.

3G Complex Analysis

Define what it means for two domains in $\mathbb{C}$ to be *conformally equivalent*.

For each of the following pairs of domains, determine whether they are conformally equivalent. Justify your answers.

(i) $\mathbb{C} \setminus \{0\}$ and $\{z \in \mathbb{C} : \ 0 < |z| < 1\}$;

(ii) $\mathbb{C}$ and $\{z \in \mathbb{C} : \ \mathrm{Im}(z) > 0\}$;

(iii) $\{z \in \mathbb{C} : \ \mathrm{Im}(z) > 0, \ |z| < 1\}$ and $\{z \in \mathbb{C} : \ \mathrm{Im}(z) > 0\}$.

4D Quantum Mechanics

(a) Prove Ehrenfest's theorem in one-dimensional quantum mechanics:

$$\frac{d}{dt}\langle \hat{O} \rangle_\psi = \frac{i}{\hbar}\langle [\hat{H}, \hat{O}] \rangle_\psi + \langle \frac{\partial \hat{O}}{\partial t} \rangle_\psi,$$

where $\hat{O}$ is a Hermitian operator, $\hat{H}$ is the Hamiltonian and

$$\langle \hat{O} \rangle_\psi = \int \psi^*(x,t) \hat{O} \psi(x,t) dx$$

is the expectation value of the operator $\hat{O}$ in a state determined by the wave function $\psi(x,t)$.

(b) Using Ehrenfest's theorem prove that

$$m\frac{d}{dt}\langle \hat{x} \rangle_\psi = \langle \hat{p} \rangle_\psi, \qquad \frac{d}{dt}\langle \hat{p} \rangle_\psi = -\langle \frac{dU}{dx} \rangle_\psi, \qquad \frac{d}{dt}\langle \hat{H} \rangle_\psi = 0,$$

where $U(x)$ is the scalar potential. Compare with similar expressions in classical mechanics.

5D Electromagnetism

Consider a system of electric charges distributed in such a way that there is a charge $-Q$ at the point $(x,y,z) = (0,0,d)$, a charge $+NQ$, with N a positive integer, located at the origin of coordinates and a charge $-MQ$ for a positive integer M at the point $(0,0,-d)$.

(a) Compute the electric potential at a distance $\mathbf{r}$ and expand in powers of $1/r$. Identify the monopole, dipole and quadrupole terms in the expansion.

(b) For which values of N and M do monopole and/or dipole terms cancel? If the monopole term cancels, what can be said about the limits for which $d \to 0$ but either Qd or Qd^2 are constants?

(c) For the case where the monopole and dipole terms cancel, compute the force on a particle of charge $-Q$ located at $\mathbf{r} = (x,0,0)$. Is the force attractive or repulsive?

6B Numerical Analysis

Consider the inner product

$$\langle g, h\rangle = \int_a^b g(x)h(x)w(x)\,dx \qquad (*)$$

on $C[a,b]$, where $w(x) > 0$ for $x \in (a,b)$. Define $\|g\|^2 = \langle g, g\rangle$. Let $Q_0, Q_1, Q_2, \ldots$ be orthogonal polynomials with respect to the inner product $(*)$, and let $f \in C[a,b]$.

(a) Prove that the polynomial $p_n^* \in \mathcal{P}_n$ that minimises the squared distance $\|f - p\|^2$ among all $p \in \mathcal{P}_n$ is given by

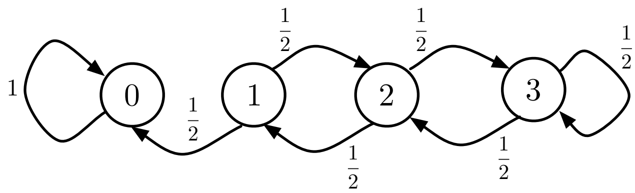
$$p_n^*(x) = \sum_{k=0}^{n} \frac{\langle f, Q_k\rangle}{\langle Q_k, Q_k\rangle} Q_k(x).$$

(b) Hence, show that

$$\|f\|^2 = \|f - p_n^*\|^2 + \|p_n^*\|^2.$$

7H Markov Chains

Consider the Markov chain in the figure below.



(a) Let $g(i) = \mathbb{E}_i[T_0]$ be the expected time to get absorbed in state 0 starting from state i. Find $g(1), g(2)$ and $g(3)$.

(b) Suppose the Markov chain is initialised in state 1. What is the probability it will visit 3 before getting absorbed in 0?

(c) Suppose the Markov chain is initialised in state 1. What is the expected number of visits to state 3 before the chain gets absorbed in 0?

SECTION II

8F Linear Algebra

If V and W are finite-dimensional vector spaces and $\gamma : V \to W$ is a linear map, what is the *matrix representation* of γ with respect to bases $\mathcal{B}$ of V and $\mathcal{C}$ of W?

If $\alpha, \beta : V \to V$ are linear maps, what does it mean to say that they are *conjugate*? How is this interpreted in terms of matrices representing α and β with respect to a basis $\mathcal{B}$ of V?

Let V be a vector space and $\beta : V \to V$ be a linear isomorphism. Write $\mathcal{L}(V, V)$ for the vector space of linear maps from V to V, and define a function by

$$\begin{aligned} \phi_\beta : \mathcal{L}(V, V) &\longrightarrow \mathcal{L}(V, V) \\ \alpha &\longmapsto \beta^{-1}\alpha\beta. \end{aligned}$$

Show that ϕ_β is a linear isomorphism, and that if β is conjugate to β' then ϕ_β is conjugate to $\phi_{\beta'}$.

Assuming that V is a 2-dimensional complex vector space, determine the Jordan Normal Form of ϕ_β in terms of that of β.

9E Groups, Rings and Modules

State and prove Eisenstein's criterion. Show that if p is a prime number then $f(X) = X^{p-1} + X^{p-2} + \ldots + X^2 + X + 1$ is irreducible in $\mathbb{Z}[X]$. Let $\zeta \in \mathbb{C}$ be a root of f. Prove that $\mathbb{Z}[\zeta] \cong \mathbb{Z}[X]/(f)$. [Any form of Gauss' lemma may be quoted without proof.]

Now let $p = 3$. Show that $\mathbb{Z}[\zeta]$ is a Euclidean domain. Prove that if n is even then there is exactly one conjugacy class of matrices $A \in \mathrm{GL}_n(\mathbb{Z})$ such that $A^2 + A + I = 0$. What happens if n is odd? You should carefully state any theorems that you use.

10G Analysis and Topology
Define the notions of *compact space, Hausdorff space* and *homeomorphism.*

Let X be a topological space and R be an equivalence relation on X. Define the quotient space X/R and show that the quotient map $q\colon X \to X/R$ is continuous. Let Y be another topological space and $f\colon X \to Y$ be a continuous function such that $f(x) = f(y)$ whenever xRy in X. Show that the unique function $F\colon X/R \to Y$ with $F \circ q = f$ is continuous.

Show that the quotient of a compact space is compact. Give an example to show that the quotient of a Hausdorff space need not be Hausdorff.

Let $f\colon X \to Y$ be a continuous bijection from the compact space X to the Hausdorff space Y. Carefully quoting any necessary results, show that f is a homeomorphism.

Let $X = [0,1]^2$ be the closed unit square in $\mathbb{R}^2$. Define an equivalence relation R on X by $(x_1, y_1)R(x_2, y_2)$ if and only if one of the following holds:

(i) $x_1 = x_2$ and $y_1 = y_2$, or

(ii) $\{x_1, x_2\} = \{0, 1\}$ and $y_1 = y_2$, or

(iii) $y_1 = y_2 \in \{0, 1\}$.

Show that the quotient space X/R is homeomorphic to the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.

11E Geometry

(a) Show that the Möbius maps commuting with $z \mapsto 1/\overline{z}$ are of the form

$$z \mapsto \frac{az+b}{\overline{b}z+\overline{a}}$$

where $a, b \in \mathbb{C}$ with $|a|^2 - |b|^2 \neq 0$. Which of these maps preserve the unit disc?

(b) Write down the Riemannian metric on the disc model $\mathbb{D}$ of the hyperbolic plane. Describe the geodesics passing through O and prove that they are length minimising curves. Deduce that every geodesic is part of a circle or line preserved by the transformation $z \mapsto 1/\overline{z}$. [You may assume that the maps in part (a) that preserve the unit disc are isometries.]

(c) Let $P \in \mathbb{D}$ be a point at a hyperbolic distance $\rho > 0$ from O. Let ℓ be the hyperbolic line passing through P at right angles to OP. Show that ℓ has Euclidean radius $1/\sinh\rho$ and centre at a distance $1/\tanh\rho$ from O.

(d) Consider a hyperbolic quadrilateral with three right angles, and angle θ at the remaining vertex v. Show that

$$\cos\theta = \tanh a \tanh b$$

where a and b are the hyperbolic lengths of the sides incident with v.

12B Complex Methods

Let $B : [0, \infty) \to \mathbb{R}^{n\times p}$ be a $n \times p$ matrix-valued function. The Laplace transform $\mathcal{L}\{B\}$ of B is defined componentwise on the matrix element functions of B.

(a) Show that if A is a constant $n \times n$ matrix and $B : [0, \infty) \to \mathbb{R}^{n\times p}$ is an $n \times p$ matrix-valued function, then $\mathcal{L}\{AB\} = A\mathcal{L}\{B\}$.

(b) Consider the ODE given by

$$y'(t) = Ay(t) + g(t), \quad y(0) = y_0 \in \mathbb{R}^n, \quad t \geqslant 0, \qquad (*)$$

where A is a constant $n \times n$ matrix, and $g : [0, \infty) \to \mathbb{R}^n$ is a vector-valued function whose Laplace transform $G(s) = \mathcal{L}\{g\}(s)$ exists for all but one $s \in \mathbb{C}$. Show that

$$Y(s) = (sI - A)^{-1}(y_0 + G(s)),$$

and that

$$\mathcal{L}\{e^{tA}\}(s) = (sI - A)^{-1},$$

for all s that are not eigenvalues of A, where $Y = \mathcal{L}\{y\}$ is the Laplace transform of the solution y of $(*)$. You may assume that y exists and is the unique solution to the ODE for all $t \geqslant 0$ with solution $y(t) = e^{tA}y_0$ when $g = 0$.

(c) Consider the ODE

$$y'(t) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} y(t) + \begin{bmatrix} e^{2t} \\ -2t \end{bmatrix}, \quad y(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad t \geqslant 0.$$

Determine the integer values $n \in \mathbb{N}$ such that $\lim_{t\to\infty} e^{-nt}y(t)$ exists and is a finite and nonzero vector in $\mathbb{R}^2$.

13C Variational Principles

(a) Consider a functional of the form

$$\mathcal{L}[u,v] = \iint_\Omega f(x,y,u,v,u_x,u_y,v_x,v_y)\,\mathrm{d}x\mathrm{d}y,$$

where u and v are functions of x and y [we use the notation a_b to denote the partial derivative $\partial a/\partial b$]. Assuming small variations $u \to u + \delta u$ and $v \to v + \delta v$ and using integration by parts, derive the two Euler–Lagrange equations satisfied by u and v in Ω associated with an extremum of $\mathcal{L}$ (you may ignore all contributions from boundary terms).

(b) An elastic material deforms in two dimensions with a displacement field $\mathbf{u}(\mathbf{x}) = [u(x,y), v(x,y)]$, that minimises the total elastic energy

$$\mathcal{J} = \iint_\Omega \left[\frac{1}{2}\mu(\nabla\mathbf{u} : \nabla\mathbf{u}^T) + \frac{1}{2}(\lambda+\mu)(\nabla\cdot\mathbf{u})^2\right]\mathrm{d}x\mathrm{d}y$$

where $\nabla\mathbf{u}$ is the displacement gradient tensor, defined as

$$\nabla\mathbf{u} = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix},$$

where μ and λ are two material constants and where we use the notation $\mathbf{A} : \mathbf{B}$ to refer to the trace of the matrix product $\mathbf{AB}$.

(i) Show that

$$\mathcal{J} = \iint_\Omega \left[\left(\frac{\lambda}{2}+\mu\right)(u_x^2+v_y^2) + \frac{\mu}{2}(u_y^2+v_x^2) + (\lambda+\mu)u_x v_y\right]\mathrm{d}x\mathrm{d}y.$$

(ii) Derive the two Euler–Lagrange equations satisfied by u and v and show that they can be combined into a single equation for $\mathbf{u}$.

(iii) In the one-dimensional limit where $v = 0$, $\partial u/\partial y = 0$ with boundary conditions $u(0) = 0$, $u(L) = \Delta$, show that the solution to the equation obtained in (ii) is linear in x.

14A Methods

(a) Using Fourier transforms with respect to x, express in integral form the general solution $\theta(x,t)$ to the (unforced) heat equation with initial data $\Theta(x)$ and diffusivity $D > 0$:

$$\frac{\partial \theta}{\partial t} = D\frac{\partial^2 \theta}{\partial x^2}; \; \theta(x,0) = \Theta(x).$$

[You may quote the convolution theorem for Fourier transforms without proof.]

(b) By constructing an appropriate Green's function, express in integral form the general solution $\theta_f(x,t)$ to the forced heat equation with homogeneous initial data:

$$\frac{\partial \theta_f}{\partial t} - D\frac{\partial^2 \theta_f}{\partial x^2} = f(x,t); \; \theta_f(x,0) = 0,$$

for some function $f(x,t)$.

(c) Now consider the combined problem:

$$\frac{\partial \theta_c}{\partial t} - D\frac{\partial^2 \theta_c}{\partial x^2} = -A\delta\left(x + 2\sqrt{D}\right)\delta(t-1); \; \theta_c(x,0) = \delta\left(x - 2\sqrt{D}\right),$$

where A is a positive real constant. Determine $\theta_c(x,t)$, and hence deduce that $\theta_c(0,2) = 0$ if

$$A = \sqrt{\frac{e}{2}}.$$

[*The following convention is used in this question:*

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx \;\; \textit{and} \;\; f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}\,dk.$$

You may also quote the transform pair

$$g(x,t) = \frac{1}{\sqrt{4\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right); \; \tilde{g}(k,t) = e^{-Dk^2 t},$$

as well as any relevant properties of the δ-function without proof.]

15D Quantum Mechanics

(a) Using the canonical commutation relations $[\hat{x}_i, \hat{p}_j] = \mathrm{i}\hbar\delta_{ij}$ with $i, j = 1, 2, 3$, show that the angular momentum operators $\hat{L}_i = \epsilon_{ijk}\hat{x}_j\hat{p}_k$ satisfy the commutation relations:

$$[\hat{L}_i, \hat{L}_j] = \mathrm{i}\hbar\epsilon_{ijk}\hat{L}_k, \qquad [\hat{L}_i, \hat{x}_j] = \mathrm{i}\hbar\epsilon_{ijk}\hat{x}_k, \qquad [\hat{L}_i, \hat{p}_j] = \mathrm{i}\hbar\epsilon_{ijk}\hat{p}_k.$$

Using these relations show that $[\hat{L}^2, \hat{L}_i] = 0$ where $\hat{L}^2 = \hat{L}_i\hat{L}_i$. Show further that for a spherically symmetric system $[\hat{L}^2, \hat{H}] = 0$, where the Hamiltonian $\hat{H}$ takes the form $\hat{H} = \frac{\hat{p}^2}{2m} + U(\hat{r})$. Can the operators $\hat{H}, \hat{L}^2, \hat{L}_3$ be simultaneously diagonalised? Justify your answer.

(b) Consider the Schrödinger equation for the Hydrogen atom in which the potential energy is $U(r) = -\frac{q^2}{r}$. Concentrating on the wave function with zero eigenvalues for both $\hat{L}_3$ and $\hat{L}^2$, the equation for the radial component of the wave function, $R(r)$, reduces to:

$$R'' + \frac{2}{r}R' + \left(\frac{\beta}{r} - \gamma^2\right) R = 0,$$

where $\beta = \frac{2mq^2}{\hbar^2}$ and $\gamma^2 = -\frac{2mE}{\hbar^2}$, with E denoting the energy.

(i) Considering the $r \to \infty$ limit, explain why $R \sim e^{-\gamma r}$.

(ii) Consider then the series solution

$$R(r) = f(r)e^{-\gamma r}, \qquad f(r) = \sum_n a_n r^n.$$

Derive the recurrence relation

$$a_n = \frac{2\gamma n - \beta}{n(n+1)} a_{n-1},$$

then argue why the energy is quantised and determine the ground state energy.

(iii) Using the ground state wave function $R(r) = Ce^{-\gamma r}$, determine the normalisation factor C and estimate the expectation value of the radius $\langle r \rangle_R$. Compare with the Bohr radius.

16C Fluid Dynamics

(a) A body of fluid has a free surface given by $z = \eta(x, y, t)$ in Cartesian coordinates and the fluid velocity is denoted by $\mathbf{u} = (u, v, w)$. Applying the kinematic boundary condition at the free surface, derive the relationship between the value of w at the free surface and $\mathrm{D}\eta/\mathrm{D}t$.

(b) An inviscid fluid is confined in a box with sides at $x = 0, L$ and $y = 0, L$. The fluid is semi-infinite in the $-z$ direction and is bounded above by a free surface at $z = \eta(x, y, t)$. The fluid is forced to oscillate by applying a prescribed variation in the air pressure just above the free surface,

$$p(x, y, t) = p_0 \cos(\pi x/L) \cos(2\pi y/L) \cos(\omega t),$$

with ω a prescribed constant frequency.

(i) Assuming irrotational flow and small-amplitude motion of the interface, state the equation satisfied by the velocity potential ϕ in the fluid and state all the boundary conditions.

(ii) Show that a separable solution for ϕ of the form

$$\phi = Z(z) \cos(\pi x/L) \cos(2\pi y/L)\, F(t)$$

is consistent with the dynamic boundary condition and that it satisfies the boundary conditions at $x = 0, L$ and $y = 0, L$.

(iii) Solve for the function $Z(z)$.

(iv) Using the kinematic boundary condition, show that the shape of the interface is of the form

$$\eta(x, y, t) = \cos(\pi x/L) \cos(2\pi y/L)\, H(t),$$

and derive the relationship between $H(t)$ and $F(t)$.

(v) Use the dynamic boundary condition to solve for $H(t)$ and $F(t)$.

(vi) Deduce that the amplitudes H and F do not remain bounded for a specific value of the frequency ω which you should determine, and briefly interpret this phenomenon physically.

17H Statistics

An ecologist takes data $\{(x_i, y_i)\}_{i=1}^n$, where $x_i \geqslant 0$ is the size of an area and $y_i \in \mathbb{N}$ is the number of moss plants in the area. For fixed $\{x_i\}_{i=1}^n$, we model the data by $Y_i \sim \text{Poisson}(\theta x_i)$, where the Y_i are independent of each other.

(a) Write down a linear model relating the y_i to the x_i. Derive a formula for the least squares estimator $\hat{\theta}_{LS}$. Is the estimator biased?

(b) Compute the maximum likelihood estimator $\hat{\theta}_{MLE}$. Is the estimator biased?

(c) Compare the variances of $\hat{\theta}_{LS}$ and $\hat{\theta}_{MLE}$.

(d) Suppose we wish to test the hypotheses $H_0 : \theta = 1$ versus $H_1 : \theta = 2$. Describe a hypothesis test with test statistic $\hat{\theta}_{MLE}$, which has approximate size 0.05 when $\sum_{i=1}^n x_i$ is large. Describe a hypothesis test with test statistic $\hat{\theta}_{LS}$, which has approximate size 0.05 when each x_i is large. *[Hint: A Poisson*(λ) *distribution may be approximated by a* $\mathcal{N}(\lambda, \lambda)$ *distribution when* λ *is large.]*

18H Optimisation

Let A be the $m \times n$ payoff matrix of a two-person, zero-sum game. What is Player I's optimization problem?

Write down a sufficient condition that a vector $p \in \mathbb{R}^m$ is an optimal mixed strategy for Player I in terms of the optimal mixed strategy for Player II and the value of the game.

If $m = n$ and A is an invertible, symmetric matrix such that $A^{-1}e \geqslant 0$, where $e = (1, 1, \ldots, 1)^\top \in \mathbb{R}^m$, show that the value of the game is $(e^\top A^{-1} e)^{-1}$.

Consider the following game: Players I and II each have three cards labelled 1, 2, and 3. Each player chooses one of their cards, independently of the other player, and places it in the same envelope. If the sum of the numbers in the envelope is smaller than or equal to 4, then Player II pays Player I the sum (in £), and otherwise Player I pays Player II the sum. (For instance, if Player I chooses card 3 and Player II chooses card 2, then Player I pays Player II £5.) What is the optimal strategy for each player?

END OF PAPER