UNIVERSITY OF CAMBRIDGE

MAT1

MATHEMATICAL TRIPOS Part IB

Thursday, 08 June, 2023 1:30pm to 4:30pm

PAPER 3

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most four** questions from Section I and **at most six** questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Groups, Rings and Modules

Let F be a finite field of order q. Let $G = \mathrm{GL}_2(F)/Z$ where $Z \leqslant \mathrm{GL}_2(F)$ is the subgroup of scalar matrices. Define an action of $\mathrm{GL}_2(F)$ on $F \cup \{\infty\}$ and use this to show that there is an injective group homomorphism

$$\phi : G \to S_{q+1}.$$

Now let $F = \mathbb{F}_2[\omega]/(\omega^2 + \omega + 1) = \{0, 1, \omega, \omega + 1\}$ be the field with $q = 4$ elements (where $\mathbb{F}_2 = \{0, 1\}$ is the field with 2 elements). Compute the order of G, find a Sylow 2-subgroup P of G, and show that $\phi(P) \leqslant A_5$.

2E Geometry

Let $\mathbb{H}$ be the hyperbolic upper half plane. Explain how the Riemannian metric $\frac{dx^2+dy^2}{y^2}$ on $\mathbb{H}$ can be used to compute lengths, angles and areas.

Consider the triangle in $\mathbb{H}$ with vertices at $e^{i\alpha}$, $e^{i\beta}$ and ∞, where $0 < \alpha < \beta < \pi$. Compute its area, and deduce the Gauss–Bonnet theorem for a hyperbolic polygon.

3B Complex Methods

Let $f = u + iv$ be an analytic function in a connected open set $D \subset \mathbb{C}$, where $u(x, y)$ and $v(x, y)$ are real-valued functions on D, with $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$, for $z \in D$.

(a) Show that $f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$, and state the Cauchy–Riemann equations.

(b) Suppose there are real constants a, b and c such that $a^2 + b^2 \neq 0$ and

$$au(x, y) + bv(x, y) = c, \qquad z \in D.$$

Show that f is constant on D.

4C Variational Principles

Consider a function f: $\mathbb{R}^n \to \mathbb{R}$, not necessarily differentiable. What does it mean for f to be *convex* in a domain D?

If f is once differentiable, state an equivalent condition involving ∇f at two points $\mathbf{x}$ and $\mathbf{y}$ in D.

If f is twice differentiable, state an equivalent condition involving the Hessian $\mathbf{H}$.

Compute the largest domain on which the function $f(x, y) = x^3 + y^3 + Axy$ is convex in $\mathbb{R}^2$ (A is a constant) and sketch it.

5A Methods

Calculate the Green's function $G(x;\xi)$ given by the solution to

$$\frac{d^2G}{dx^2} - G = \delta(x-\xi); \; G(0;\xi) = 0 \text{ and } G(x;\xi) \to 0 \text{ as } x \to \infty,$$

where $\xi \in (0,\infty)$, $x \in (0,\infty)$ and $\delta(x)$ is the Dirac δ-function.

Use this Green's function to calculate an explicit solution $y(x)$ to the boundary value problem

$$\frac{d^2y}{dx^2} - y = e^{-2x},$$

where $x \in (0,\infty)$, $y(0) = 0$ and $y(x) \to 0$ as $x \to \infty$.

6D Quantum Mechanics

Consider the one-dimensional, time-independent Schrödinger equation:

$$\frac{d^2\chi(x)}{dx^2} + \frac{2m}{\hbar^2}[E - U(x)]\chi(x) = 0, \qquad x \in \mathbb{R}.$$

(a) Explain the meaning of the functions $\chi(x), U(x)$ and parameters $E, m, \hbar$.

(b) Solutions of this equation describing bound states correspond to $\chi(x) \to 0$ for $x \to \pm\infty$. Are there bound states for a potential that asymptotes to a constant U_0 (that is $U(x) \to U_0$ as $x \to \pm\infty$) for the cases $E > U_0 > 0$ and $0 < E < U_0$?

(c) Show, by contradiction or otherwise, that the energy spectrum of bound states is non-degenerate.

7C Fluid Dynamics

A two-dimensional cylinder of radius a is stationary in a uniform flow of velocity $U\mathbf{e}_x$. The flow is assumed to be steady, inviscid, two-dimensional and irrotational. There is no circulation around the cylinder.

Using a velocity potential, solve for the flow $\mathbf{u}(r,\theta)$ around the cylinder. Use Bernoulli's equation to compute the pressure on its surface as a function of the polar angle θ.

8H Markov Chains

A gang of thieves decides to commit a robbery every week. The gang only robs one of three possible targets: Art museums, Banks, or Casinos, which they conveniently denote by $\{A, B, C\}$. The places they rob follows a Markov chain with the following transition probability matrix:

$$P = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 3/4 & 0 & 1/4 \\ 3/8 & 1/8 & 1/2 \end{pmatrix}.$$

(a) Find the stationary distribution of this Markov chain.

(b) Is the Markov chain reversible?

(c) Since this spate of robberies had been going on for a long time (i.e., the Markov chain is in stationarity), the police approach Detective Holmes for assistance. Detective Holmes arrives at the crime scene, which happens to be a bank. Detective Holmes asks the police, "What is the probability that these thieves robbed a bank two weeks ago, as well?" The police, not having taken Part IB Markov Chains, are stumped. Please help the police by finding this probability.

SECTION II

9F Linear Algebra

Let V be a finite-dimensional real inner product space, and $\alpha : V \to V$ be a linear map. What does it mean to say that α is *self-adjoint*?

If $\alpha : V \to V$ is self-adjoint, prove that there is an orthonormal basis for V consisting of eigenvectors of α.

Let P_n denote the vector space of real polynomials of degree at most n. Show that

$$\langle f, g \rangle = \int_0^\infty f(x)g(x)e^{-x}\,dx$$

defines an inner product on this vector space, and that the linear map $\alpha : P_n \to P_n$ given by

$$\alpha(f) = xf'' + (1-x)f'$$

is self-adjoint with respect to this inner product.

Show that α has eigenvalues $0, -1, -2, -3, \ldots, -n$. When $n = 2$ determine corresponding eigenvectors.

[*Hint: You may use the identity* $\int_0^\infty x^n e^{-x}\,dx = n!$.]

10E Groups, Rings and Modules

(a) Let R be a unique factorisation domain (UFD) with field of fractions F. What does it mean to say that a polynomial $f \in R[X]$ is *primitive*? Assuming that the product of two primitive polynomials is primitive, prove that for $f \in R[X]$ primitive the following implications hold.

(i) f irreducible in $R[X] \implies f$ irreducible in $F[X]$.

(ii) f prime in $F[X] \implies f$ prime in $R[X]$.

Deduce that $R[X]$ is a UFD. [You may use any standard characterisation of a UFD, provided you state it clearly.]

(b) A rational function $f \in \mathbb{C}(X, Y)$ is *symmetric* if $f(X, Y) = f(Y, X)$. Show that if $f \in \mathbb{C}(X, Y)$ is symmetric then it can be written as $f = g/h$ where $g, h \in \mathbb{C}[X, Y]$ are coprime and symmetric.

11G Analysis and Topology

Let $f\colon U \to \mathbb{R}^n$ be a function where U is an open subset of $\mathbb{R}^m$, and let $a \in U$. Define what it means that f *is differentiable at* a and define *the derivative of* f *at* a. Define what it means that f *is continuously differentiable at* a. Show that a linear map $\mathbb{R}^m \to \mathbb{R}^n$ is continuously differentiable at every point of $\mathbb{R}^m$.

State and prove the mean value inequality. Let U be an open, connected subset of $\mathbb{R}^m$. Let $f\colon U \to \mathbb{R}^n$ be a differentiable function such that $Df|_a$ is the zero map for all $a \in U$. Show that f is a constant function.

State the inverse function theorem. Consider the curve C in $\mathbb{R}^2$ defined by the equation

$$x^2 + y + \cos(xy) = 1.$$

Show that there exist an open neighbourhood U of $(0,0)$ in $\mathbb{R}^2$, an open interval I in $\mathbb{R}$ containing 0 and a continuous function $g\colon I \to \mathbb{R}$ such that $U \cap C$ is the graph of g, *i.e.*,

$$\{(x,y) \in \mathbb{R}^2 : \, x \in I, \; y = g(x)\} = U \cap C.$$

12E Geometry

Let $\sigma : V \to \Sigma$ be a smooth parametrisation of an embedded surface $\Sigma \subset \mathbb{R}^3$, and let $\gamma : (a,b) \to \Sigma;\, t \mapsto \sigma(u(t), v(t))$ be a smooth curve. Show by differentiating $\sigma_u \cdot \gamma'$ and $\sigma_v \cdot \gamma'$ that γ satisfies the geodesic equations if and only if $\gamma''(t)$ is normal to the surface. Deduce that geodesics are parametrised at constant speed.

Now assume in addition that Σ is a surface of revolution. Let $\rho(t)$ be the distance from $\gamma(t)$ to the axis of revolution, and let $\theta(t)$ be the angle between γ and the parallel at $\gamma(t)$. Prove that if γ is a geodesic then it satisfies the Clairaut relation

$$\rho(t) \cos\theta(t) = \text{constant}.$$

On the hyperboloid $\Sigma = \{x^2 + y^2 = z^2 + 1\}$ give examples of

(i) a curve parametrised at constant speed, which satisfies the Clairaut relation, but is *not* a geodesic,

(ii) a plane that meets Σ in a pair of disjoint geodesics,

(iii) a plane that meets Σ in a pair of geodesics that intersect at right angles.

Are there any geodesics entirely contained in the region $z > 0$? Are there any geodesics $\gamma \subset \Sigma$ with $\phi(\gamma) = \gamma$ for every isometry $\phi : \Sigma \to \Sigma$? Justify your answers.

13G Complex Analysis

State Rouché's theorem. State the open mapping theorem and prove it using Rouché's theorem. Show that if f is a non-constant holomorphic function on a domain Ω, then $|f|$ has no local maximum on Ω.

Let Ω be a bounded domain in $\mathbb{C}$, and let $\overline{\Omega}$ denote the closure of Ω. Let $f\colon \overline{\Omega} \to \mathbb{C}$ be a continuous function that is holomorphic on Ω. Show that if $|f(z)| \leqslant M$ for all $z \in \partial\Omega$, then $|f(z)| \leqslant M$ for all $z \in \Omega$, where $\partial\Omega = \overline{\Omega} \setminus \Omega$ is the boundary of Ω.

Consider the unbounded domain $\Omega = \{z \in \mathbb{C} : \operatorname{Re} z > 1\}$. Let $f\colon \overline{\Omega} \to \mathbb{C}$ be a continuous function that is holomorphic on Ω. Assume that f is bounded both on Ω and on its boundary $\partial\Omega$. Show that if $|f(z)| \leqslant M$ for all $z \in \partial\Omega$, then $|f(z)| \leqslant M$ for all $z \in \Omega$. [*Hint: Consider for large* $n \in \mathbb{N}$ *and for a large disc* $D(0,R)$ *the function* $z \mapsto \big(f(z)\big)^n/z$ *on* $D(0,R) \cap \Omega$.] Is the boundedness assumption of f on Ω necessary? Justify your answer.

14A Methods

(a) You are given that $f(x)$, $g(x)$ and $h(x)$ are all absolutely integrable functions with absolutely integrable Fourier transforms $\tilde{f}(k)$, $\tilde{g}(k)$ and $\tilde{h}(k)$ such that

$$\tilde{h}(k) = [\tilde{f}(k)][\tilde{g}(k)],$$

i.e. that $\tilde{h}(k)$ is the product of $\tilde{f}(k)$ and $\tilde{g}(k)$. Express $h(x)$ in terms of an integral expression involving $f(x)$ and $g(x)$.

(b) If $p'(x) = g(x)$, express $\tilde{p}(k)$ in terms of $\tilde{g}(k)$. [You may assume that the transforms are well-defined.]

(c) Express the inverse transforms of $\cos ka$ and $\sin ka$ in terms of the δ-function, where a is a positive constant.

(d) Consider the following wave problem for $u(x,t)$:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2};\ u(x,0) = f(x),\ \frac{\partial}{\partial t}u(x,0) = g(x).$$

Use parts (a)-(c) to construct d'Alembert's solution:

$$u(x,t) = \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(\xi)\, d\xi. \qquad (\star)$$

[No credit will be given for using any other approach to derive ($\star$). You may assume the expression derived in part (a) applies.]

(e) Consider the specific case

$$f(x) = 0;\ g(x) = \begin{cases} x & \text{for } |x| \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $t > 1$, identify a region of the x-t plane including the line $x = 0$ where $u(x,t) = 0$. Briefly interpret this result physically. [*Hint: You may find it useful to consider the lines* $x = 1 - t$ *and* $x = -1 + t$.]

[*The following convention is used in this question:*

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\, dx \ \textit{and}\ \ f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}\, dk.]$$

15D Electromagnetism

Consider a steady electric current density $\mathbf{J}(\mathbf{r})$ and the corresponding magnetic vector potential $\mathbf{A}(\mathbf{r})$.

(a) Show that each component of $\mathbf{A}(\mathbf{r})$ in Cartesian coordinates satisfies a Poisson equation $\nabla^2\mathbf{A} = -\mu_0\mathbf{J}$ and write down the general integral expression for $\mathbf{A}(\mathbf{r})$ in terms of $\mathbf{J}(\mathbf{r})$. Explain why you can assume $\nabla \cdot \mathbf{A} = 0$.

(b) Use the expression for the vector potential to derive the Biot–Savart law:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} d^3\mathbf{r}'.$$

(c) Consider a circular loop of wire of radius R in the x-y plane with a circulating current I. Using the Biot–Savart law, determine the direction and magnitude of the corresponding magnetic field $\mathbf{B}(\mathbf{r})$ at a point on the z-axis. What is the magnetic field at the centre of the loop?

(d) If there is a second parallel loop of radius $2R$ with centre in the z-axis at a distance D from the first loop and current $2I$ circulating in the opposite direction, find the point between the wires at which the magnetic field vanishes.

16C Fluid Dynamics

(a) Starting from the Euler equation for an inviscid fluid with no body force, derive the unsteady Bernoulli equation relating the pressure and the velocity potential in a time-dependent irrotational, incompressible flow.

(b) A liquid occupies the two-dimensional annular region $a(t) < r < b(t)$ between a gas bubble occupying $0 \leqslant r < a(t)$ and an infinite gas in $r > b(t)$. The flow is incompressible, irrotational and radially symmetric.

(i) If the radius of the gas bubble is prescribed (i.e. the function $a(t)$ is known), solve for the potential flow in the liquid. Deduce the time-variation of $b(t)$ and interpret your result physically.

(ii) The pressure in the gas in $r > b$ is a constant p_∞. Compute the time-varying pressure $p(r,t)$ in the liquid at $r = a(t)$.

(iii) Assuming small perturbations for the bubble radius $a(t) = a_0[1+\epsilon(t)]$ with $|\epsilon| \ll 1$, deduce the linearised variation of the radius $b(t)$. Find the linearised variation of the pressure $p(a,t)$.

(iv) The pressure $p_0(t)$ in the bubble is uniform in space and satisfies $p_0 V =$ *const*, where $V(t)$ is the volume of the bubble. Deduce the relationship between ϵ and $p(a,t) - p_\infty$.

(v) Show that the bubble undergoes oscillations and compute its frequency ω.

17B Numerical Analysis

Consider $C[a,b]$ equipped with the inner product $\langle f,g\rangle = \int_a^b f(x)g(x)w(x)\mathrm{d}x$, where $w(x) > 0$ for $x \in (a,b)$. Let $\mathcal{P}_n$ denote the set of polynomials of degree less than or equal to n. For $f \in C[a,b]$ consider the quadrature formulas

$$I(f) = \int_a^b f(x)w(x)\mathrm{d}x \approx \sum_{i=0}^{n} a_i^{(n)} f(x_i^{(n)}) = I_n(f), \quad n = 0,1,2,\ldots \tag{$*$}$$

with weights $a_i^{(n)} \in \mathbb{R}$ and nodes $x_i^{(n)} \in [a,b]$, which are exact on all polynomials $q \in \mathcal{P}_n$.

(a) Prove that the quadrature formula $(*)$ is exact for all $q \in \mathcal{P}_{n+1+k}$ if and only if the polynomial $Q_{n+1}(x) = \prod_{i=0}^{n}(x - x_i^{(n)})$ is orthogonal (with respect to $\langle\cdot,\cdot\rangle$) to all polynomials of degree k.

(b) Prove that no quadrature formula $(*)$ could be exact on polynomials of degree $2n+2$.

(c) Prove that if $(*)$ is exact on $\mathcal{P}_{2n}$, then $a_i^{(n)} > 0$.

(d) Show that if $a_i^{(n)} > 0$ for all i and n, then

$$I_n(f) \to I(f), \qquad n \to \infty.$$

[*Hint: Use the Weierstrass theorem: for any* $\epsilon > 0$ *there exists* $n \in \mathbb{N}$ *and a polynomial* $p_n \in \mathcal{P}_n$ *such that* $|f(x) - p_n(x)| < \epsilon$, *for* $x \in [a,b]$.]

18H Statistics

(a) Define a *uniformly most powerful (UMP) test* when $X \sim f(\cdot|\theta)$ for $\theta \in \Theta$, and the two hypotheses correspond to

$$H_0 : \theta \in \Theta_0 \subseteq \Theta$$
$$H_1 : \theta \in \Theta_1 \subseteq \Theta.$$

(b) Let $f(x|\theta)$ be the logistic location probability density function

$$f(x|\theta) = \frac{e^{(x-\theta)}}{(1+e^{(x-\theta)})^2}, \qquad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

(i) Based on one observation X, find the most powerful size-α test of $H_0 : \theta = 0$ versus $H_1 : \theta = 1$. You may use any results from the lectures without proof provided you state them clearly.

(ii) Prove that the test in part (i) is UMP of size α for testing $H_0 : \theta \leqslant 0$ versus $H_1 : \theta > 0$.

19H Optimisation

Let $S \subset \mathbb{R}^3$ be the set of all $(x_1, x_2, x_3) \in \mathbb{R}^3$ satisfying the following linear inequalities:

$$0 \leqslant x_1, x_2, x_3 \leqslant 1,$$
$$x_1 + x_2 + x_3 \leqslant 2.5.$$

(a) Show that S is a non-empty convex set.

(b) What is meant by an *extreme point* of a convex set? Find all extreme points of S.

(c) Suppose we want to solve the following linear program:

$$\begin{aligned} \text{maximise} \quad & x_1 + 2x_2 + 4x_3 \\ \text{subject to} \quad & (x_1, x_2, x_3) \in S. \end{aligned}$$

What is the solution to this problem and where is it attained?

(d) Suppose the simplex method is initialised at $(0, 0, 0)$ to solve the above linear program. Recall that depending on the choices of pivot elements made at each step, many different outcomes are possible. Here, an outcome denotes the path the simplex method takes over the basic feasible solutions of the problem.

What is the smallest number of steps in which the simplex method can find the solution? What is the largest number of steps in which the simplex method can find the solution? Calculate the total number of distinct outcomes possible when the simplex method is initialised at $(0, 0, 0)$.

It may be helpful to draw a picture.

END OF PAPER