UNIVERSITY OF CAMBRIDGE

MAT1

# MATHEMATICAL TRIPOS Part IB

Wednesday, 07 June, 2023 9:00am to 12:00pm

## PAPER 2

## Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most four** questions from Section I and **at most six** questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

## At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

**Every cover sheet must also show your Blind Grade Number and desk number.**

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

**STATIONERY REQUIREMENTS**
Gold cover sheets
Green main cover sheet

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

## SECTION I

### 1E Groups, Rings and Modules

Let $R$ be a commutative ring. Show that the following statements are equivalent.

(i) There exists $e \in R$ with $e^2 = e$ and $e \neq 0, 1$.

(ii) $R \cong R_1 \times R_2$ for some non-trivial rings $R_1$ and $R_2$.

Let $R = \{(a, b) \in \mathbb{Z}^2 \mid a \equiv b \pmod 2\}$. Show that $R$ is a ring under componentwise operations. Is $R$ an integral domain? Is $R$ isomorphic to a product of non-trivial rings?

### 2G Analysis and Topology

Show that a topological space $X$ is connected if and only if every continuous integer-valued function on $X$ is constant.

Let $\mathcal{A}$ be a family of connected subsets of a topological space $X$ such that $\bigcup_{A \in \mathcal{A}} A = X$. Assume that $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{A}$. Prove that $X$ is connected.

Deduce, or otherwise show, that if $X$ and $Y$ are connected topological spaces, then $X \times Y$ is also connected in the product topology.

### 3A Methods

Expand $f(x) = x^3 - \pi^2 x$ as a Fourier series on $-\pi < x < \pi$.

Use the series and Parseval's theorem for Fourier series (which you may quote without proof) to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

### 4D Electromagnetism

Define what is meant by a *capacitor* and by *capacitance*.

Consider a cylindrical capacitor consisting of two concentric cylinders of length $L$, linear charge density $\lambda$ and radii $a$ and $b > a$, respectively. Assuming that $L \gg b$ and that end effects may be neglected, compute the electric field $E$ between the cylinders, the potential difference $V$, the capacitance $C$ and the energy $U$ stored in this system. Verify that $U = \frac{1}{2} Q V$ where $Q$ is the total charge.

**5C Fluid Dynamics**

A three-dimensional flow has a velocity field $\mathbf{u}(\mathbf{x}) = \mathbf{\Gamma}\cdot\mathbf{x}+\mathbf{U}_0$, where $\mathbf{\Gamma}$ is a constant second-rank tensor and $\mathbf{U}_0$ is a constant vector, with components

$$\mathbf{\Gamma} = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}, \quad \mathbf{U}_0 = \begin{pmatrix} P \\ Q \\ R \end{pmatrix}.$$

(a) What are the conditions on the components of $\mathbf{\Gamma}$ and $\mathbf{U}_0$ for the flow to be:

(i) incompressible?

(ii) irrotational?

(b) In the case where

$$\mathbf{\Gamma} = \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{U}_0 = \begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix}, \quad (\alpha \neq 0),$$

compute the streamline passing through the point $(1,0,0)$.

**6H Statistics**

Let $\sigma > 0$ be a fixed, known constant, and let $X_1, \ldots, X_n$ be i.i.d. $\mathcal{N}(\theta, \sigma^2)$ random variables.

(a) Compute a maximum likelihood estimator $\hat{\theta}_{MLE}$ for $\theta$.

(b) What is a 95% confidence interval for $\theta$ based on $\hat{\theta}_{MLE}$?

Now suppose $\theta$ has a prior distribution $\theta \sim \mathcal{N}(\mu, \nu^2)$, where $\mu \in \mathbb{R}$ and $\nu > 0$ are both known. The posterior distribution of $\theta$, given the observations $\{X_1, \ldots, X_n\}$ is known to be

$$\mathcal{N}\left(\frac{\frac{n\bar{X}}{\sigma^2} + \frac{\mu}{\nu^2}}{\frac{n}{\sigma^2} + \frac{1}{\nu^2}}, \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\nu^2}}\right), \quad \text{where} \quad \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

(c) What is a 95% credible interval for $\theta$ based on the posterior?

(d) Compare the answers to parts (b) and (c) as $n \to \infty$.

**7H Optimisation**

Solve the following optimisation problem using the Lagrange sufficiency theorem:

$$\begin{aligned} \text{minimise} \quad & x^2 + y^4 + z^6 \\ \text{subject to} \quad & x + 2y + 3z = 6. \end{aligned}$$

Does strong duality hold for this problem?

Let $\phi$ be the value function $\phi(b) = \inf\{x^2 + y^4 + z^6 : x + 2y + 3z = b\}$. Evaluate the derivative $\phi'(6)$.

## SECTION II

**8F Linear Algebra**

What is the *characteristic polynomial* of a square matrix $A$?

State and prove the Cayley–Hamilton theorem for square complex matrices.

For square matrices $X$ and $Y$ let us write $[X, Y] = XY - YX$. Given another square matrix $Z$, show that $[X, YZ] = [X, Y]Z + Y[X, Z]$.

Suppose now that $A$ and $B$ are square complex matrices such that $[B, A]$ commutes with $A$, i.e. $[[B, A], A] = 0$. Show that for any polynomial $\varphi(t)$ we have

$$[B, \varphi(A)] = \varphi'(A)[B, A],$$

where $\varphi'(t)$ denotes the derivative of $\varphi$. For a polynomial $f(t)$, whose $k$th derivative is denoted by $f^{(k)}(t)$, satisfying $f(A) = 0$, show by induction that $f^{(k)}(A)[B, A]^{2^k-1} = 0$. Deduce that some power of the matrix $[B, A]$ is zero.

**9E Groups, Rings and Modules**

(a) Let $P$ be a Sylow $p$-subgroup of a group $G$, and let $Q$ be any $p$-subgroup of $G$. Prove that $Q \leqslant gPg^{-1}$ for some $g \in G$. State the remaining Sylow theorems.

(b) Let $G$ be a group acting faithfully and transitively on a set $X$ of size 7. Suppose that

(i) for every $x \in X$ we have $\mathrm{Stab}_G(x) \cong S_4$,

(ii) for every $x, y \in X$ distinct we have $\mathrm{Stab}_G(x) \cap \mathrm{Stab}_G(y) \cong C_2 \times C_2$.

Determine the order of $G$ and its number of Sylow $p$-subgroups for each prime $p$. [*Hint: For one of the primes p it may help to use the following fact, which you may assume. If H is a subgroup of $S_p$ of order p then the normaliser of H in $S_p$ has order $p(p-1)$.*]

Deduce that no proper normal subgroup of $G$ has order divisible by 3 or order divisible by 7. Hence or otherwise prove that $G$ is simple.

**10G Analysis and Topology**

Define the notion of *uniform convergence* for a sequence $(f_n)$ of real-valued functions on an arbitrary set $S$ and the notion of *uniform continuity* for a function $h\colon M \to N$ between metric spaces.

Let $C_0(\mathbb{R}^d)$ denote the set of all continuous functions $f\colon \mathbb{R}^d \to \mathbb{R}$ satisfying $f(x) \to 0$ as $\|x\| \to \infty$, i.e. for all $\varepsilon > 0$ there exists $K > 0$ such that $|f(x)| < \varepsilon$ whenever $\|x\| > K$ (where $\|x\|$ denotes the usual Euclidean length of $x$). Briefly explain why every function in $C_0(\mathbb{R}^d)$ is bounded. Prove that $C_0(\mathbb{R}^d)$ is a complete metric space in the uniform metric. Is it true that every member of $C_0(\mathbb{R}^d)$ is uniformly continuous? Give a proof or counterexample.

Let $\varepsilon\colon \mathbb{R} \to [0,\infty)$ be a continuous function with $\varepsilon(0) = 0$. For $n \in \mathbb{N}$ define $f_n\colon \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \sqrt{x^2 + \varepsilon(x/n)}$. Must $(f_n)$ converge pointwise? Must $(f_n)$ converge uniformly? Do your answers change if we further assume that for some $M \geqslant 0$ and for all $t \in \mathbb{R}$ we have $\varepsilon(t) \leqslant M|t|$? Justify your answers.

**11F Geometry**

Let $U \subset \mathbb{R}^2$ and $f : U \to \mathbb{R}$ be a smooth function. Derive a formula for the first and second fundamental forms of the surface in $\mathbb{R}^3$ parametrised by

$$\begin{aligned}\sigma : U &\longrightarrow \mathbb{R}^3\\ (u,v) &\longmapsto (u,v,f(u,v))\end{aligned}$$

in terms of $f$. State a formula for the Gaussian curvature in terms of the first and second fundamental forms, and hence give a formula for the Gaussian curvature of this surface.

Let $\Sigma \subset \mathbb{R}^3$ be a smooth surface and $P \subset \mathbb{R}^3$ be a plane. Supposing that $\Sigma$ is tangent to $P$ along a smooth curve $\gamma \subset \mathbb{R}^3$ and otherwise lies on one side of $P$, show that the Gaussian curvature of $\Sigma$ is zero at all points on $\gamma$.

**12B Complex Analysis OR Complex Methods**

(a) Suppose that $f : \mathbb{C} \to \mathbb{C}$ is analytic, and is bounded in the half-plane $\{z : \mathrm{Re}(z) > 0\}$. Prove that, for any real number $c > 0$, there is a positive real constant $M$ such that

$$|f(z_1) - f(z_2)| \leqslant M|z_1 - z_2|$$

whenever $z_1, z_2 \in \mathbb{C}$ satisfy $\mathrm{Re}(z_1) > c$, $\mathrm{Re}(z_2) > c$, and $|z_1 - z_2| < c$.

(b) Let the functions $g, h : \mathbb{C} \to \mathbb{C}$ both be analytic.

(i) State Liouville's Theorem.

(ii) Show that if $g$ is not constant, then $g(\mathbb{C})$ is dense in $\mathbb{C}$.

(iii) Show that if $|h(z)| \leqslant |\mathrm{Re}(z)|^{-1/2}$ for all $z \in \mathbb{C}$, then $h$ is constant.

**13C Variational Principles**

(a) For a functional of the form

$$\mathcal{L}[y] = \int_a^b F(x, y, y', y'')\,\mathrm{d}x,$$

derive the Euler–Lagrange equation satisfied by the solution $y(x)$ leading to a stationary value of $\mathcal{L}$. Show that all boundary terms cancel if the solution is assumed to have fixed values for $y$ and $y'$ at the end points.

(b) A diving board of length $L$ at a swimming pool takes the shape $y(x)$ that minimises the energy

$$\mathcal{E} = \int_0^L \left[\frac{1}{2}A(y'')^2 + \rho g y\right]\mathrm{d}x,$$

where $A > 0$ is the bending rigidity, $\rho > 0$ the mass density and $g > 0$ the acceleration due to gravity ($A, \rho, g$ are constants).

(i) Derive the ODE satisfied by $y(x)$.

(ii) The board is clamped at the origin (i.e. $y(0) = 0$, $y'(0) = 0$) while at $x = L$, it is torque free (i.e. $y''(L) = 0$) and a vertical force of magnitude $F$ is applied to it (i.e. $-Ay'''(L) = F$). Solve for $y(x)$ and show that it may be written as $y(x) = y_0 + y_F$, where $y_0$ is the solution when $F = 0$ and $y_F$ is proportional to $F$.

(iii) Compute the vertical displacement at the end of the board, $\Delta = y(L)$, and show that it can be written as $\Delta = h_0 + h$, where $h_0$ is the displacement when $F = 0$ and $h$ is proportional to $F$.

(iv) For the solution in part (ii) compute the corresponding value of the energy $\mathcal{E}$ and show that it can be written as $\mathcal{E} = E_0 + E$, with $E_0$ independent of $F$ and $E$ quadratic in $F$.

(v) Relate $\frac{\mathrm{d}E}{\mathrm{d}F}$ to $h$ and interpret your result.

**14A Methods**

(a) Laplace's equation in plane polar coordinates has the form

$$\nabla^2\phi = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\phi(r,\theta) = 0.$$

Using separation of variables, show that the general solution is:

$$\phi(r,\theta) = a_0 + c_0 \ln r + \sum_{n=1}^{\infty}\left(a_n r^n + c_n r^{-n}\right)\cos n\theta + \sum_{n=1}^{\infty}\left(b_n r^n + d_n r^{-n}\right)\sin n\theta,$$

for arbitrary real constants $a_i$, $b_i$, $c_i$ and $d_i$.

Which (if any) constants must be zero for the solution to be regular in:

(i) the interior of a disc centred at the origin?

(ii) the exterior of a disc centred at the origin?

(iii) an annular region centred at the origin?

(b) Consider $2\pi$-periodic functions $f(\theta)$ such that

$$f(\theta) = \sum_{n=1}^{\infty} A_n \cos n\theta,$$

for some coefficients $A_n$.

(i) Solve Laplace's equation $\nabla^2\phi = 0$ in the annulus $1 < r < e^2$ with boundary conditions:

$$\phi(r,\theta) = \begin{cases} f(\theta) - 1, & r = 1 \\ f(\theta) + 1, & r = e^2, \end{cases}$$

for general $f(\theta)$.

(ii) Calculate the explicit solution for the specific choice:

$$f(\theta) = \begin{cases} \frac{\pi}{2} - \theta, & 0 \leqslant \theta < \pi \\ -\frac{3\pi}{2} + \theta, & \pi \leqslant \theta < 2\pi. \end{cases}$$

**15D Quantum Mechanics**

(a) Consider the Schrödinger equation for the wave function $\psi(\mathbf{r},t)$ corresponding to a particle subject to a real potential energy $U(\mathbf{r},t)$. Defining the probability density $\rho(\mathbf{r},t)=|\psi(\mathbf{r},t)|^2$ and probability current density

$$\mathbf{J}(\mathbf{r},t)=-\frac{i\hbar}{2m}\left[\psi^*\nabla\psi-(\nabla\psi)^*\,\psi\right],$$

derive and interpret the continuity equation $\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{J}=0$.

(b) Consider the one-dimensional Schrödinger equation with a step potential

$$U(x)=\begin{cases}0 & x<-a\\ U_0 & -a<x<a\\ 0 & x>a,\end{cases}$$

where $a>0, U_0>0$.

(i) Using matching conditions at $x=\pm a$, find the transmitted wave function $\psi(x,t)$ and probability density $\rho(x,t)$ in the region $x>a$, for an incident wave corresponding to a particle of mass $m$ and energy $E=U_0/2$ moving towards the potential barrier from $x<-a$. Express the results in terms of the quantity $k=\sqrt{2mE}/\hbar$.

(ii) Compute the ratio between the transmitted and the incident current densities and interpret the result in terms of the continuity equation.

**16D Electromagnetism**

Consider a relativistic particle of mass $m$ and charge $q$ in the presence of a constant electric field $\mathbf{E}$ and constant magnetic field $\mathbf{B}$.

(a) Write down the covariant relativistic generalisation of the Lorentz force law, explaining each of the terms. Decompose the equation in terms of the temporal and spatial components. Compare with the non-relativistic version of the law.

(b) Find the time variation of the energy $\mathcal{E}(t)$ in terms of the electric field and the particle's velocity.

(c) For $\mathbf{B} = \mathbf{0}$ and $\mathbf{E} = (0, 0, E)$ find the particle's energy $\mathcal{E}(t)$ and position $z(t)$ as functions of time $t$ assuming that the particle was initially at the origin with momentum $\mathbf{p_0} = (p_0, 0, 0)$ (and energy $\mathcal{E}_0 = \sqrt{m^2c^4 + c^2p_0^2}$ ) where $c$ is the speed of light. [*Hint: Recall* $\mathcal{E}^2 - c^2p^2 = m^2c^4$.]

(d) Determine the trajectory of the particle in the $x$-$z$ plane, $z(x)$, expressing the result in terms of the constants $q, m, E, p_0, \mathcal{E}_0$. [*Hint: Recall* $dz/dx = p_z/p_x$. ]

(e) Determine the limiting behaviour of $z(t)$ for both large and small $t$ and compare the latter with the well-known non-relativistic result of a particle with constant acceleration $a = qE/m$. What are $z(x)$ and $x(t)$ in this case?

**17B Numerical Analysis**

Consider an ODE of the form

$$y' = f(y), \quad y(0) = y_0 \in \mathbb{R}, \qquad (*)$$

where $y(t)$ exists and is unique for $t \in [0, T]$ and $T > 0$.

(a) For a numerical method approximating the solution of $(*)$, define the linear stability domain. What does it mean for such a numerical method to be A-stable?

(b) Let $a \in \mathbb{R}$ and consider the Runge–Kutta method—producing a sequence $\{y_n\}_{n \leqslant N}$, where $N = \lfloor \frac{T}{h} \rfloor$ and $h > 0$ is the step-size—defined by

$$\begin{aligned} k_1 &= f\left(y_n + \frac{1}{4}hk_1 + \left(\frac{1}{4} - a\right)hk_2\right), \\ k_2 &= f\left(y_n + \left(\frac{1}{4} + a\right)hk_1 + \frac{1}{4}hk_2\right), \\ y_{n+1} &= y_n + \frac{1}{2}h(k_1 + k_2), \quad n = 0, 1, \ldots, N-1. \end{aligned}$$

Determine the values of the parameter $a \in \mathbb{R}$ for which the Runge–Kutta method is A-stable.

**18H Markov Chains**

Let $(X_n)_{n \geqslant 0}$ be a Markov chain with finite state space $S$. Let $(Y_n)_{n \geqslant 0}$ denote another Markov chain on the same state space $S$. Let $P$ denote the transition probability matrix of $(X_n)_{n \geqslant 0}$ and $Q$ denote the transition probability matrix of $(Y_n)_{n \geqslant 0}$. You are given that for each $i, j \in S$,

$$P_{ij} > 0 \implies Q_{ij} > 0.$$

For each of the following statements provide a proof or counterexample:

(a) If $(X_n)_{n \geqslant 0}$ is irreducible, then $(Y_n)_{n \geqslant 0}$ is also irreducible.

(b) If every state in $(X_n)_{n \geqslant 0}$ is aperiodic, then every state in $(Y_n)_{n \geqslant 0}$ is also aperiodic.

(c) If $(X_n)_{n \geqslant 0}$ has no transient states, then $(Y_n)_{n \geqslant 0}$ also has no transient states.

(d) For $i \in S$, let $\mu_i$ denote the mean of the first return time to $i$ starting from $i$, in the Markov chain $(X_n)_{n \geqslant 0}$, and let $\eta_i$ denote the mean of the first return time to $i$ starting from $i$, in the Markov chain $(Y_n)_{n \geqslant 0}$. Then $\eta_i \leqslant \mu_i$.

## END OF PAPER