UNIVERSITY OF CAMBRIDGE

MAT1

MATHEMATICAL TRIPOS Part IB

Tuesday, 06 June, 2023 9:00am to 12:00pm

PAPER 1

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most four** questions from Section I and **at most six** questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Linear Algebra

Let V and W be finite-dimensional real vector spaces, and $\mathcal{L}(V, W)$ denote the vector space of linear maps from V to W. Prove that the dimensions of these vector spaces satisfy

$$\dim(\mathcal{L}(V, W)) = \dim(V) \cdot \dim(W).$$

If $A \leqslant V$ and $B \leqslant W$ are vector subspaces, let

$$X = \{\phi \in \mathcal{L}(V, W) \,:\, \phi(A) \leqslant B\},$$

which you may assume is a vector subspace of $\mathcal{L}(V, W)$. Prove a formula for the dimension of X in terms of the dimensions of V, W, A and B.

If S and T are vector subspaces of V such that $V = S + T$, let

$$Y = \{\phi \in \mathcal{L}(V, V) \,:\, \phi(S) \leqslant S \text{ and } \phi(T) \leqslant T\},$$

which you may assume is a vector subspace of $\mathcal{L}(V, V)$. Prove a formula for the dimension of Y in terms of the dimensions of V, S, and T.

2F Geometry

What is a *topological surface*?

Consider

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \,:\, x^2 + y^2 + z^2 = 1\},$$

which you may assume is a topological surface. For the equivalence relation $\sim$ on S^2 generated by $(x, y, z) \sim (-x, -y, -z)$, show that $S^2/\sim$ is a topological surface. For the equivalence relation $\approx$ on S^2 generated by $(x, y, z) \approx (-x, -y, z)$, show that $S^2/\approx$ is homeomorphic to S^2.

3B Complex Analysis OR Complex Methods

(a) What is the Laurent series of $e^{1/z}$ about $z_0 = 0$?

(b) Let $\rho > 0$. Show that for all large enough $n \in \mathbb{N}$, all zeros of the function

$$f_n(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \ldots + \frac{1}{n!z^n}$$

lie in the open disc $\{z : |z| < \rho\}$.

4C Variational Principles

Briefly explain how to use a Lagrange multiplier to find the extrema of a function $f(\mathbf{x})$ subject to a constraint $g(\mathbf{x}) = 0$.

Find the maximum volume of a cuboid of side lengths $x \geqslant 0$, $y \geqslant 0$, and $z \geqslant 0$ whose space diagonal has length L.

5B Numerical Analysis

Given a matrix $A \in \mathbb{R}^{m\times n}$ and a vector $\boldsymbol{y} \in \mathbb{R}^m$ where $m \geqslant n$, consider the problem of finding $\boldsymbol{c}^* \in \mathbb{R}^n$ that minimises $\|A\boldsymbol{c} - \boldsymbol{y}\|_2$ for $\boldsymbol{c} \in \mathbb{R}^n$, where $\|\cdot\|_2$ is the standard Euclidean norm.

(a) Prove that $\boldsymbol{c}^*$ is a solution to the above minimisation problem if and only if $A^T A\boldsymbol{c}^* = A^T\boldsymbol{y}$.

(b) Show that if A is of full rank, then $\boldsymbol{c}^*$ is unique.

6H Statistics

(a) Define the *generalized likelihood ratio test statistic* and state Wilks' Theorem.

(b) The following experiment was conducted in the late 1800s to determine whether the use of carbolic acid was helpful in amputations. Out of 75 amputations, with and without carbolic acid, the following data were collected:

	Carbolic acid used	Carbolic acid not used
Patient lived	34	19
Patient died	6	16

Describe a hypothesis test to determine whether the use of carbolic acid affects the rate of patient mortality: What is the null hypothesis, and what is the alternative? What is an appropriate statistic and what is the critical region for a test of size α? You need not calculate the value of the statistic.

7H Optimisation

What is the *minimum-cost flow problem* on a graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set E? Your answer should be in terms of

- a cost matrix $C \in \mathbb{R}^{n \times n}$,
- a vector $b \in \mathbb{R}^n$ whose i-th entry is the amount of flow that enters vertex i,
- a lower bound on the flow given by a matrix $\underline{M} \in \mathbb{R}^{n \times n}$, and
- an upper bound on the flow given by a matrix $\overline{M} \in \mathbb{R}^{n \times n}$.

Show that we can always assume $\underline{M} = 0$ by constructing an equivalent problem to the general problem above. Explain why the problems are equivalent.

SECTION II

8F Linear Algebra

For each of the following statements give a proof or counterexample.

(a) If A and B are 3×3 complex matrices with the same characteristic polynomial and the same minimal polynomial, then they are conjugate.

(b) There are three mutually non-conjugate complex matrices with characteristic polynomial $(2-t)^2(1-t)^5$ and minimal polynomial $(2-t)^2(1-t)^2$.

(c) If $\alpha : V \to V$ is a linear isomorphism from a finite-dimensional complex vector space to itself such that some iterate α^N with $N > 0$ is diagonalisable, then α is diagonalisable.

(d) A real matrix which is diagonalisable when considered as a complex matrix is also diagonalisable as a real matrix.

(e) Two real matrices which are conjugate when considered as complex matrices are also conjugate as real matrices.

9E Groups, Rings and Modules

Let R be a Noetherian integral domain with field of fractions F. Prove that the following statements are equivalent.

(i) R is a principal ideal domain.

(ii) Every pair of elements $a, b \in R$ has a greatest common divisor which can be written in the form $ra + sb$ for some $r, s \in R$.

(iii) Every finitely generated R-submodule of F is cyclic.

(iv) Every R-submodule of R^n can be generated by n elements.

Show that any integral domain that is isomorphic to $\mathbb{Z}^n$ as a group under addition is Noetherian as a ring. Find an example of such a ring that does *not* satisfy conditions (i)-(iv). Justify your answer.

10G Analysis and Topology

Define the terms *Cauchy sequence* and *complete metric space*. Prove that every Cauchy sequence in a metric space is bounded.

Show that a metric space (M, d) is complete if and only if given any sequence (F_n) of non-empty, closed subsets of M satisfying

- $F_n \supset F_{n+1}$ for all $n \in \mathbb{N}$ and
- $\operatorname{diam} F_n = \sup\{d(x, y) : x, y \in F_n\} \to 0$ as $n \to \infty$,

the intersection $\bigcap_{n \in \mathbb{N}} F_n$ is non-empty.

State the contraction mapping theorem.

Let (Λ, ρ) and (M, d) be non-empty metric spaces, and assume that (M, d) is complete. Let $T\colon \Lambda \times M \to M$ be a function with the following properties:

- there exists $0 \leqslant k < 1$ such that $d(T(\lambda, x), T(\lambda, y)) \leqslant kd(x, y)$ for all $\lambda \in \Lambda$ and all $x, y \in M$;
- for each $x \in M$, the function $\Lambda \to M$, given by $\lambda \mapsto T(\lambda, x)$, is continuous.

Show that there is a unique function $x^*\colon \Lambda \to M$ such that $T(\lambda, x^*(\lambda)) = x^*(\lambda)$ for all $\lambda \in \Lambda$. Show further that the function x^* is continuous.

11F Geometry

Define in terms of allowable parametrisations what it means to say that a subset $S \subset \mathbb{R}^3$ is a *smooth surface*.

Let $\phi : \mathbb{R} \to (0, \infty)$ be a smooth function. Show that

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 = \phi(z)^2\}$$

is a smooth surface in $\mathbb{R}^3$.

Suppose $a < b$ and $r > 0$ are such that for all $a \leqslant a' < b' \leqslant b$ we have

$$\operatorname{Area}(\{(x, y, z) \in \Sigma \ : \ a' \leqslant z \leqslant b'\}) = 2\pi r \cdot (b' - a').$$

Show that ϕ must satisfy $r^2 = \phi(t)^2 + \phi(t)^2\phi'(t)^2$ for $a \leqslant t \leqslant b$. Assuming that $\phi(t) < r$ for $a \leqslant t \leqslant b$, show that the graph of the function $\phi|_{[a,b]}$ lies on a circle of radius r.

12G Complex Analysis OR Complex Methods

(a) Let $f(z) = -\sum_{n=1}^{\infty} \frac{(1-z)^n}{n}$ for $|z-1| < 1$. By differentiating $z\exp(-f(z))$, show that f is an analytic branch of logarithm on the disc $D(1,1)$ with $f(1) = 0$. Use scaling and the function f to show that for every point a in the domain $D = \mathbb{C} \setminus \{x \in \mathbb{R} : x \geqslant 0\}$, there is an analytic branch of logarithm on a small neighbourhood of a whose imaginary part lies in $(0, 2\pi)$.

(b) For $z \in D$, let $\theta(z)$ be the unique value of the argument of z in the interval $(0, 2\pi)$. Define the function $L \colon D \to \mathbb{C}$ by $L(z) = \log|z| + i\theta(z)$. Briefly explain using part (a) why L is an analytic branch of logarithm on D. For $\alpha \in (-1, 1)$ write down an analytic branch of z^{α} on D.

(c) State the residue theorem. Evaluate the integral

$$I = \int_0^{\infty} \frac{x^{\alpha}}{(x+1)^2}\, dx$$

where $\alpha \in (-1, 1)$.

13A Methods

(a) Let $y_0(x)$ be a non-trivial solution of the Sturm–Liouville problem

$$\mathcal{L}(y_0;\lambda_0) = 0;\ \ y_0(0) = y_0(1) = 0,$$

where

$$\mathcal{L}(y;\lambda) = \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \left[q(x) + \lambda w(x)\right] y.$$

Show that, if $y(x)$ and $f(x)$ are related by

$$\mathcal{L}(y;\lambda_0) = f,$$

with $y(x)$ satisfying the same boundary conditions as $y_0(x)$, then

$$\int_0^1 y_0 f\, dx = 0. \qquad (\star)$$

(b) Now assume that y_0 is normalised so that

$$\int_0^1 w y_0^2\, dx = 1,$$

and consider the problem

$$\mathcal{L}(y;\lambda) = y^{m+1};\ \ y(0) = y(1) = 0,$$

where m is a positive integer. By choosing f appropriately in $(\star)$ deduce that, if

$$\lambda - \lambda_0 = \epsilon^m \mu \text{ and } y(x) = \epsilon y_0(x) + \epsilon^2 y_1(x),$$

where $0 < \epsilon \ll 1$ and $\mu = O(1)$, then

$$\mu = \int_0^1 y_0^{m+2}\, dx + O(\epsilon).$$

14D Quantum Mechanics

Consider a physical observable O represented by a Hermitian operator $\hat{O}$ acting on a Hilbert space $\mathcal{H}$. We define the uncertainty $\Delta_\psi O$ in a measurement of O on a state ψ as $(\Delta_\psi O)^2 = \langle \hat{O}^2 \rangle_\psi - \langle \hat{O} \rangle_\psi^2$ with the expectation value defined as $\langle \hat{O} \rangle_\psi = (\psi, \hat{O}\psi)$.

(a) Using the Schwartz inequality $|(\phi,\psi)|^2 \leqslant (\phi,\phi)(\psi,\psi)$ for two states ϕ,ψ, prove the generalised uncertainty relation for the observables A, B:

$$(\Delta_\psi A)(\Delta_\psi B) \geqslant \frac{1}{2}\Big|\Big(\psi, [\hat{A},\hat{B}]\psi\Big)\Big|, \qquad (\dagger)$$

where $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ is the commutator of $\hat{A}$ and $\hat{B}$.

(b) Given the two Hermitian operators $\hat{X}$ and $\hat{Y}$ and a real parameter λ, we define

$$f(\lambda) = \Big\langle \big(\hat{X} - i\lambda\hat{Y}\big)\big(\hat{X} + i\lambda\hat{Y}\big) \Big\rangle_\psi .$$

Minimising $f(\lambda)$ and using the fact that $f(\lambda) \geqslant 0$, provide an alternative derivation of the uncertainty relation $(\dagger)$.

(c) For the position and momentum operators, $\hat{x}$ and $\hat{p} = -i\hbar\frac{\partial}{\partial x}$, respectively, find their commutator $[\hat{x},\hat{p}]$ and derive the Heisenberg uncertainty relation $\Delta_\psi x \Delta_\psi p \geqslant \frac{1}{2}\hbar$.

(d) Show that a Gaussian wave function $\psi(x) = Ce^{-\alpha x^2}$ solves the one-dimensional Schrödinger's equation for a quadratic potential $U(x) = kx^2$ with $k > 0$. Determine the value of the constants α, C and the energy E in terms of k and the particle's mass m. Show that this wave function saturates the Heisenberg uncertainty relation ($\Delta_\psi x \Delta_\psi p = \frac{1}{2}\hbar$). Furthermore, show that in order to saturate this Heisenberg relation, the wave function has to be Gaussian. [*Hint: You may use* $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$ *and* $\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \sqrt{\frac{\pi}{4a^3}}$.]

15D Electromagnetism

Write down Maxwell's equations in free space for the electric field $\mathbf{E}(\mathbf{x}, t)$ and magnetic field $\mathbf{B}(\mathbf{x}, t)$ in the presence of an electric charge density $\rho(\mathbf{x}, t)$ and current density $\mathbf{J}(\mathbf{x}, t)$.

(a) Use Maxwell's equations to prove the continuity equation $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$ and then derive the conservation of electric charge $Q = \int_V \rho \, d^3\mathbf{x}$. Which assumption do you need to make in order to establish this result?

(b) In empty space, with $\rho = |\mathbf{J}| = 0$, show that each component of $\mathbf{E}$ and $\mathbf{B}$ satisfies the wave equation. Compute the speed of the waves in terms of the permittivity $\epsilon_0 \simeq 8.85 \times 10^{-12}$ m^{-3} kg^{-1} s^4 A^2 and permeability $\mu_0 \simeq 1.25 \times 10^{-6}$ NA^{-2} of free space. Explain the importance of this result.

(c) Using Maxwell's equations and the expression for the energy stored in electric and magnetic fields inside a volume V:

$$U = \frac{1}{2} \int_V \left(\epsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0} \mathbf{B}^2 \right) d^3\mathbf{x},$$

write down an equation for the variation of the energy in terms of the Poynting vector, which you should define, and provide an interpretation. [*The identity* $\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B})$ *may be useful.*]

(d) For a linearly polarised monochromatic electromagnetic wave of frequency ω and wave vector $\mathbf{k}$ the electric field can be written as $\mathbf{E} = \mathbf{E}_0 \sin(\mathbf{k} \cdot \mathbf{x} - \omega t)$. Show that the Poynting vector is parallel to the wave vector $\mathbf{k}$ and compute its magnitude. Consider the time average of the Poynting vector and relate it to the average energy stored in the electric and magnetic fields.

(e) If a mobile phone transmits electromagnetic waves with a power of 1 watt, compute the average amplitude of the Poynting vector and the amplitude of the electric field at 10 cm from the handset. You may assume that the radiation is isotropic.

16C Fluid Dynamics

An incompressible viscous fluid of constant uniform viscosity μ and density ρ undergoes unidirectional flow of the form $\mathbf{u} = u(y,t)\mathbf{e}_x$ in two dimensions. Gravity is negligible.

(a) Use a small control fluid volume of size $\delta x \times \delta y$,

(i) to show that this flow satisfies mass conservation;

(ii) to derive the momentum conservation equation satisfied by $u(y,t)$ and the pressure $p(x)$.

(b) The flow is steady, is subject to a uniform pressure gradient $G = \mathrm{d}p/\mathrm{d}x$ and occurs between two rigid surfaces at $y = 0$ and $y = h$. The surface at $y = 0$ is stationary while the surface at $y = h$ translates with velocity $U\mathbf{e}_x$, where U is a constant parameter.

(i) Solve for the flow $u(y)$ in terms of G and U.

(ii) Compute the value G_0 of the applied pressure gradient G for which the shear stress at $y = 0$ is zero.

(iii) For $G = G_0$, deduce the volume flux in the x direction.

(iv) For $G = G_0$, use $u(y)$ to compute the shear stress exerted by the flow on the top plate. Show that it can also be obtained by using a force balance on a small control fluid volume of size $\delta x \times h$.

17B Numerical Analysis

Consider the ODE

$$y' = f(y), \quad y(0) = y_0 > 0, \tag{$*$}$$

where $f(y) = -\text{sign}(y)$, $y(t) \in \mathbb{R}$ and $t \in [0, T]$, with $T > y_0$. The sign function is defined as

$$\text{sign}(y) = \begin{cases} 1 & \text{for } y > 0 \\ 0 & \text{for } y = 0 \\ -1 & \text{for } y < 0. \end{cases}$$

(a) Does the function f satisfy a Lipschitz condition for $y \in \mathbb{R}$? Justify your answer.

(b) Show that there is a unique continuous function $y : [0, T] \to \mathbb{R}$ that is differentiable for all $t \in [0, T]$ except for some $\tilde{t} \in (0, T]$ and satisfies the ODE $(*)$ for all $t \in [0, T] \setminus \tilde{t}$.

(c) The Euler method for $(*)$ produces a sequence $\{y_n\}_{n \leqslant N}$, where $N = \lfloor \frac{T}{h} \rfloor$ and $h > 0$ is the step-size. Is

$$|y_n - y(nh)| \leqslant \mathcal{O}(h), \qquad \text{for } 0 \leqslant n \leqslant N,$$

where $y(t)$ is the solution described in part (b)? Justify your answer.

18H Statistics

Suppose X_1 and X_2 are i.i.d. $\mathcal{N}(\mu, 1)$ random variables.

(a) Write down the joint probability density function of (X_1, X_2).

(b) Prove that $T = X_1 + X_2$ is a sufficient statistic for μ. Is it a minimal sufficient statistic? Justify your answer.

(c) Suppose we wish to estimate $\theta := \mu^2$. Prove that $S = X_1^2 - 1$ is an unbiased estimator of θ. Find the mean square error of S. You may use the fact that $\mathbb{E}[Z^4] = 3$ for $Z \sim \mathcal{N}(0, 1)$.

(d) What is the probability density function of X_1 conditioned on T?

(e) Use the Rao–Blackwell theorem to derive an estimator with strictly smaller mean square error than S for estimating θ. Calculate the mean square error for the new estimator you derive and compare it with the mean square error of S calculated in part (c).

19H Markov Chains

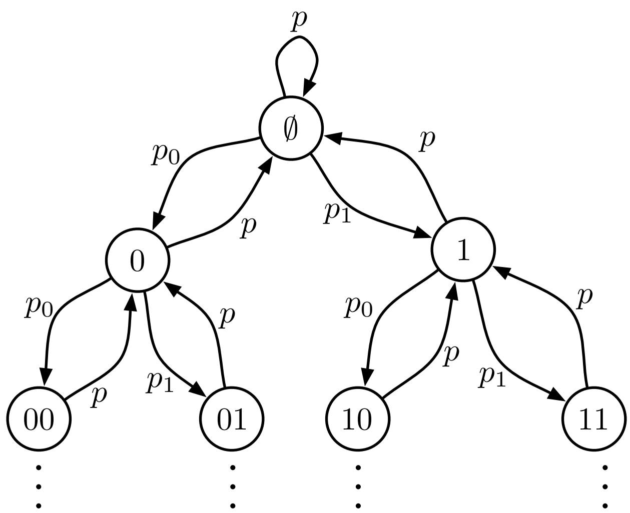
Label the vertices of a binary tree by all binary vectors, with the exception of the "root" node, which is labeled $\emptyset$. Let $p_0, p_1 > 0$ such that $p_0 + p_1 < 1$, and let $p = 1 - p_0 - p_1$. Consider a Markov chain X_n on the binary tree with transition probabilities as follows:

$$\mathbb{P}(X_{n+1} = (b_1, b_2, \ldots, b_k, i) | X_n = (b_1, b_2, \ldots, b_k)) = p_i \quad \text{for } i = 0, 1,$$
$$\mathbb{P}(X_{n+1} = (b_1, b_2, \ldots, b_{k-1}) | X_n = (b_1, b_2, \ldots, b_k)) = p$$

for any non-root vertex $(b_1, b_2, \ldots, b_k) \in \{0,1\}^k$, and

$$\mathbb{P}(X_{n+1} = i | X_n = \emptyset) = p_i \quad \text{for } i = 0, 1,$$
$$\mathbb{P}(X_{n+1} = \emptyset | X_n = \emptyset) = p$$

for the root vertex. The figure below shows the states and the transition probabilities for the first two levels of the tree.



(a) Prove that the Markov chain is irreducible and find its period. Justify your answers.

(b) What are the conditions on p_0, p_1 so that the chain is transient/null recurrent/positive recurrent? Justify your answer.

(c) Assume that the p_0, p_1 are chosen such that the chain is positive recurrent. Let $\ell(X_n)$ denote the length of the string representing state X_n. For example, $\ell(\emptyset) = 0$ and $\ell(0010) = 4$. Prove that the following limit exists

$$\lim_{n \to \infty} \mathbb{P}(\ell(X_n) = k | X_0 = \emptyset),$$

and determine its value.

END OF PAPER