UNIVERSITY OF CAMBRIDGE

MAT0

MATHEMATICAL TRIPOS Part IA

Wednesday, 07 June, 2023 1:30pm to 4:30pm

PAPER 4

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **all four** questions from Section I and **at most five** questions from Section II. Of the Section II questions, no more than three may be on the same course.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Numbers and Sets

A *permutation* of the integers $\{1, \ldots, n\}$ is a bijection from this set to itself. The permutation σ is said to be *up-down* if $\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \ldots$; it is said to be *down-up* if, instead, $\sigma(1) > \sigma(2) < \sigma(3) > \sigma(4) < \ldots$.

(a) Define a bijection between the set of up-down and the set of down-up permutations of $\{1, \ldots, n\}$.

(b) Let A_n be the number of up-down permutations of $\{1, \ldots, n\}$ for $n \geqslant 1$, and define $A_0 = 1$. Show that these numbers satisfy the equation

$$2A_{n+1} = \sum_{k=0}^{n} \binom{n}{k} A_k A_{n-k} \quad \text{for } n \geqslant 1\,.$$

[*Hint: Consider the possible up-down or down-up permutations for which a given element of* $\{1, \ldots, n+1\}$ *maps to* $n+1$.]

2E Numbers and Sets

State and prove the Chinese remainder theorem.

Find all solutions x of the simultaneous congruences

$$\begin{cases} x \equiv 4 \mod 6\,, \\ x \equiv 2 \mod 8\,. \end{cases}$$

Prove that for every positive integer d there exist integers a and b such that $4a^2 + 9b^2 - 1$ is divisible by d.

3C Dynamics and Relativity

A rocket moves vertically upwards in a uniform gravitational field and emits exhaust gas downwards with time-dependent speed $U(t)$ relative to the rocket. Derive the rocket equation

$$m(t)\frac{dv}{dt} + U(t)\frac{dm}{dt} = -m(t)g\,,$$

where $m(t)$ and $v(t)$ are respectively the rocket's mass and upward speed at time t.

Suppose now that $m(t) = m_0 - \alpha t$ and $U(t) = U_0 m_0 / m(t)$, where m_0, U_0 and α are constants. What is the condition for the rocket to lift off from rest at $t = 0$? Assuming that this condition is satisfied, find $v(t)$.

State the dimensions of all the quantities involved in your expression for $v(t)$, and verify that the expression is dimensionally consistent.

[*You may neglect any relativistic effects.*]

4C Dynamics and Relativity

In two-dimensional space-time an inertial frame S' has velocity v relative to another inertial frame S. State the Lorentz transformation that relates coordinates (x', t') in S' to coordinates (x, t) in S, assuming that the frames coincide when $t = t' = 0$.

Show that if $x_{\pm} = x \pm ct$ and $x'_{\pm} = x' \pm ct'$ then the Lorentz transformation can be expressed in the form

$$x'_{+} = \lambda(v) x_{+} \quad \text{and} \quad x'_{-} = \lambda(-v) x_{-}\,, \quad \text{where} \quad \lambda(v) = \left(\frac{c-v}{c+v}\right)^{1/2}. \qquad (*)$$

Deduce that $x^2 - c^2t^2 = x'^2 - c^2t'^2$.

Use $(*)$ to verify that successive Lorentz transformations with velocities v_1 and v_2 result in another Lorentz transformation with velocity v_3, to be determined in terms of v_1 and v_2.

SECTION II

5F Numbers and Sets

The Chebyshev polynomials are defined for $x \in \mathbb{R}$ by the recurrence relation

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x) \qquad \text{for } n \geqslant 1. \end{aligned}$$

(a) Prove that $T_n\big(\cos(y)\big) = \cos(ny)$ for all integers $n \geqslant 0$.

(b) Prove that $\cos(\pi/n)$ is algebraic for all integers $n \geqslant 1$.

(c) For each integer $n \geqslant 1$, determine whether $\cos(\pi/n)$ is rational or not. [*Hint: After stating the known answers for small n, it is useful to consider the form of $T_n(x)$ for odd n.*]

(d) In each of the following cases, prove that the sequence (a_n) is bounded and determine whether it has a limit:

(i) $a_n = \sum_{k=1}^{n} \big(1 - \cos(\pi/k)\big)$,

(ii) $a_n = \sum_{k=0}^{n} \cos(ky)$, with $\cos(y) \neq 1$.

6E Numbers and Sets

If p is a prime number, prove that $(p-1)! \equiv -1 \mod p$.

If $n > 4$ is a composite number, prove that $(n-1)! \equiv 0 \mod n$.

State the Fermat–Euler theorem and deduce from it Fermat's little theorem.

If p is any prime, prove that if $a \equiv b \mod p$, then $a^{p^n} \equiv b^{p^n} \mod p^{n+1}$ for all integers $n \geqslant 1$.

Let $a > 1$ be an integer. A *pseudo-prime of base a* is a composite number $n > 1$ satisfying $a^{n-1} \equiv 1 \mod n$. By considering the numbers $\dfrac{a^{2p}-1}{a^2-1}$, where p is prime, or otherwise, prove that for each a there are infinitely many pseudo-primes of base a.

7D Numbers & Sets

(a) Let X be a set and let $f : X \to X$ be an injective function. Show that $f^n : X \to X$ is injective, where f^n denotes the n-fold composite of f with itself.

The image of f is given by $\{f(x) : x \in X\}$ and denoted $f(X)$. Show that

$$X \supseteq f(X) \supseteq f^2(X) \supseteq f^3(X) \supseteq \cdots$$

Suppose there exists $k \in \mathbb{N}$ such that $f^k(X) = f^{k+1}(X)$. Show that $f^k(X) = f^{k+m}(X)$ for all $m \in \mathbb{N}$. Hence, or otherwise, find a subset A of X such that $f : A \to A$ is bijective.

(b) Let $X = \{x_1, x_2, \ldots, x_n\}$ and let W_k be the set of words in elements of X of length k, that is $W_k = \{w_1 \ldots w_k : w_i \in X \text{ for } 1 \leqslant i \leqslant k\}$. Let P_n be the set of bijections $f : X \to X$. We define a relation $\sim$ on W_k as follows. Suppose $w, z \in W_k$, then $w \sim z$ if and only if there exists $f \in P_n$ such that $w_1 \ldots w_k = f(z_1) \ldots f(z_k)$, where $w = w_1 \ldots w_k$ and $z = z_1 \ldots z_k$. Show that $\sim$ defines an equivalence relation on W_k.

List the equivalence classes of W_3 for each $n \in \mathbb{N}$.

List the equivalence classes of W_4 when $n = 3$.

Let $n = 4$ and $g \in P_4$ be such that

$$g : x_1 \mapsto x_2,\ x_2 \mapsto x_3,\ x_3 \mapsto x_4 \ \text{ and } \ x_4 \mapsto x_1.$$

Let $F = \{g, g^2, g^3, g^4\}$. We define a new equivalence relation $\underset{F}{\sim}$ on W_k. Suppose $w, z \in W_k$, then $w \underset{F}{\sim} z$ if and only if there exists $f \in F$ such that $w_1 \ldots w_k = f(z_1) \ldots f(z_k)$. Are the equivalence classes of W_3 under $\underset{F}{\sim}$ the same as the equivalence classes under $\sim$? Justify your answer. [You may assume that $\underset{F}{\sim}$ is an equivalence relation.]

8D Numbers & Sets

Prove that a countable union of countable sets is countable.

Infinite binary sequences are sequences of the form $a_1 a_2 a_3 \ldots$, where $a_i \in \{0, 1\}$ for $i \in \mathbb{N}$. Are the sets consisting of the following countable? Justify your answers.

(i) All infinite binary sequences.

(ii) Infinite binary sequences with either a finite number of $1s$ or a finite number of $0s$.

(iii) Infinite binary sequences with infinitely many $1s$ and infinitely many $0s$.

A function $f : \mathbb{Z} \to \mathbb{N}$ is called *periodic* if there exists a positive integer k such that $f(x + k) = f(x)$ for every $x \in \mathbb{Z}$. Is the set of periodic functions $f : \mathbb{Z} \to \mathbb{N}$ countable? Justify your answer.

Is the set of bijections from $\mathbb{N}$ to $\mathbb{N}$ countable? Justify your answer.

9C Dynamics and Relativity

Find the moment of inertia of a uniform-density sphere with mass M and radius a with respect to an axis passing through its centre.

Such a sphere is released from rest on a plane inclined at an angle α to the horizontal. Let t_s and t_r be the times taken for the sphere to travel a distance l along the plane assuming either sliding without friction or rolling without slipping, respectively. Discuss whether energy is conserved in each of the two cases. Show that $t_s/t_r = \sqrt{5/7}$.

The uniform-density sphere is replaced by a sphere of the same mass whose density varies radially such that its moment of inertia is γMa^2 for some constant γ. Determine the new value for t_s/t_r.

10C Dynamics and Relativity

(a) Write down the 4-momentum of a particle of rest mass m and 3-velocity $\mathbf{v}$, and the 4-momentum of a photon of frequency ω (having zero rest mass) moving in the direction of the unit 3-vector $\mathbf{e}$.

Show that if P_1 and P_2 are timelike future-pointing 4-vectors then $P_1 \cdot P_2 \geqslant 0$ (where the dot denotes the Lorentz-invariant scalar product). Hence or otherwise show that the law of conservation of 4-momentum forbids a photon to spontaneously decay into an electron–positron pair. [Electrons and positrons have equal and non-zero rest masses.]

(b) In the laboratory frame an electron travelling with 3-velocity $\mathbf{u}$ collides with a positron at rest. They annihilate, producing two photons of frequencies ω_1 and ω_2 that move off at angles θ_1 and θ_2 to $\mathbf{u}$, respectively. Explain why the 3-momenta of the photons and $\mathbf{u}$ lie in a plane.

By considering energy and two components of 3-momentum in the laboratory frame, or otherwise, show that

$$\frac{1+\cos(\theta_1+\theta_2)}{\cos\theta_1+\cos\theta_2} = \sqrt{\frac{\gamma-1}{\gamma+1}}$$

where $\gamma = 1/\sqrt{1-u^2/c^2}$.

11C Dynamics and Relativity

Consider a system of N particles with position vectors $\mathbf{r}_i(t)$ and masses m_i, where $i = 1, 2, \ldots, N$. Particle i experiences an external force $\mathbf{F}_i$ and an internal force $\mathbf{F}_{ij}$ from particle j, for each $j \neq i$. Stating clearly any assumptions you need, show that

$$\frac{d\mathbf{P}}{dt} = \mathbf{F} \quad \text{and} \quad \frac{d\mathbf{L}}{dt} = \mathbf{G}\,,$$

where $\mathbf{P}$ is the total momentum, $\mathbf{F}$ is the total external force, $\mathbf{L}$ is the total angular momentum about a fixed point $\mathbf{a}$, and $\mathbf{G}$ is the total external torque about $\mathbf{a}$.

Does the result $\dfrac{d\mathbf{L}}{dt} = \mathbf{G}$ still hold if the fixed point $\mathbf{a}$ is replaced by the moving centre of mass of the system? Justify your answer.

Suppose now that all the particles have the same mass m and that the external force on particle i is $-k\dfrac{d\mathbf{r}_i}{dt}$, where k is a constant. Show that

$$\mathbf{L}(t) = \mathbf{L}(0)e^{-kt/m}\,.$$

12C Dynamics and Relativity

A particle of mass m moves in a plane under an attractive force of magnitude $mf(r)$ towards the origin O. You may assume that the acceleration $\mathbf{a}$ in polar coordinates (r,θ) is given by

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\boldsymbol{\theta}}\,,$$

where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are the unit vectors in the directions of increasing r and θ respectively, and the dot denotes d/dt.

(a) Show that $l = r^2\dot{\theta}$ is a constant of the motion. Introducing $u = 1/r$, show that

$$\dot{r} = -l\frac{du}{d\theta}$$

and derive the geometric orbit equation

$$l^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = f\left(\frac{1}{u}\right).$$

(b) Suppose now that

$$f(r) = \frac{3r+9}{r^3}\,,$$

and that initially the particle is at distance $r_0 = 1$ from O, and moving with speed $v_0 = 4$ in the direction of decreasing r and increasing θ that makes an angle $\pi/3$ with the radial vector pointing towards O.

Show that $l = 2\sqrt{3}$ and find u as a function of θ. Hence, or otherwise, show that the particle returns to its original position after one revolution about O and then flies off to infinity.

END OF PAPER