UNIVERSITY OF CAMBRIDGE

MAT0

MATHEMATICAL TRIPOS Part IA

Monday, 5 June, 2023 9:00am to 12:00pm

PAPER 3

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **all four** questions from Section I and **at most five** questions from Section II. Of the Section II questions, no more than three may be on the same course.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1D Groups

Let G be a finite group and N a normal subgroup of G. Let C_n denote the cyclic group of order n.

Are the following statements true or false? Justify your answers.

(i) If $G/N \cong C_2$ and $N \cong C_2$ then $G \cong C_4$.

(ii) If $G/N \cong C_3$ and $N \cong C_2$ then $G \cong C_6$.

(iii) Let H be a finite group and M a normal subgroup of H. If $G/N \cong H/M$ and $N \cong M$ then $G \cong H$.

2D Groups

Prove that a Möbius map is determined by the image of just 3 points.

3B Vector Calculus

What does it mean for a vector field $\mathbf{F}$ in $\mathbb{R}^3$ to be *irrotational*?

Given a field $\mathbf{F}$ that is irrotational everywhere, and given a fixed point $\mathbf{x}_0$, write down the definition of a scalar potential $V(\mathbf{x})$ that satisfies $\mathbf{F} = -\boldsymbol{\nabla} V$ and $V(\mathbf{x}_0) = 0$. Show that this potential is well-defined.

Given vector fields $\mathbf{A}_0$ and $\mathbf{B}$ with $\boldsymbol{\nabla} \times \mathbf{A}_0 = \mathbf{B}$, write down the form of the general solution $\mathbf{A}$ to $\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$. State a necessary condition on $\mathbf{B}$ for such an $\mathbf{A}_0$ to exist.

4B Vector Calculus

Cartesian coordinates x, y, z and cylindrical polar coordinates ρ, ϕ, z are related by

$$x = \rho \cos \phi, \quad y = \rho \sin \phi \,.$$

Find scalars h_ρ, h_ϕ and unit vectors $\mathbf{e}_\rho, \mathbf{e}_\phi$ such that $\mathrm{d}\mathbf{x} = h_\rho \mathbf{e}_\rho \, \mathrm{d}\rho + h_\phi \mathbf{e}_\phi \, \mathrm{d}\phi + \mathbf{e}_z \mathrm{d}z \,.$

A region V is defined by

$$\rho_0 \leqslant \rho \leqslant \rho_0 + \Delta\rho \,, \qquad \phi_0 \leqslant \phi \leqslant \phi_0 + \Delta\phi \,, \qquad z_0 \leqslant z \leqslant z_0 + \Delta z \,,$$

where $\rho_0, \phi_0, z_0, \Delta\rho, \Delta\phi$ and Δz are positive constants. Write down, or calculate, the scalar areas of its six faces and its volume ΔV.

For a vector field $\mathbf{F}(\mathbf{x}) = F(\rho)\mathbf{e}_\rho$, calculate the value of

$$\lim_{\Delta\rho \to 0} \frac{1}{\Delta V} \int_{\partial V} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S \,,$$

where ∂V and $\mathbf{n}$ are the surface and outward normal of the region V.

SECTION II

5D Groups

State and prove Lagrange's theorem.

Let H and K be subgroups of a finite group G. Show that $H \cap K$ is a subgroup of G. What can be said about $H \cap K$ if H and K have co-prime orders? Justify your answer.

Let G be a finite group and x an element of G. Define the *order* of x in G and denote it by $o(x)$. Let k be a positive integer. Prove that $x^k = e$ if and only if $o(x)$ divides k. (Here e denotes the identity element of G.)

Now suppose x and y are elements of G with co-prime orders. Further suppose $xy = yx$. Prove that $o(xy) = o(x)o(y)$.

Let x and y be two non-identity elements of G.

(i) If $o(x)$ and $o(y)$ are co-prime is it always true that $o(xy) = o(x)o(y)$?

(ii) If $xy = yx$ is it always true that $o(xy) = o(x)o(y)$?

State Cauchy's theorem. Hence, or otherwise, show that there are exactly two groups of order 26 up to isomorphism.

6D Groups

Let N be a normal subgroup of a group G and let G/N denote the set of left cosets of N in G. Explain how G/N is given a well-defined group structure.

Let $x, y \in G$. The *commutator* of x and y is defined by $[x, y] = x^{-1}y^{-1}xy$. Let $\overline{G}$ be the set of finite products of commutators of G, that is elements of $\overline{G}$ are of the form $[x_1, y_1][x_2, y_2] \ldots [x_k, y_k]$, where $x_i, y_i \in G$ for $1 \leqslant i \leqslant k$. Prove that $\overline{G}$ is a normal subgroup of G.

Show that $G/\overline{G}$ is an abelian group. Further, show that if N is a normal subgroup of G and G/N is abelian, then $\overline{G}$ is a subgroup of N.

Determine $\overline{G}$ when G is each of the following groups. Justify your answers.

(i) D_8 the dihedral group of order 8.

(ii) A_5 the alternating group of degree 5.

(iii) S_5 the symmetric group of degree 5.

7D Groups

Let H and K be subgroups of a finite group G. Show that

$$|HK| = \frac{|H||K|}{|H \cap K|},$$

where $HK = \{hk : h \in H, k \in K\}$.

Let a and b be co-prime. If $|G : H| = a$ and $|G : K| = b$, show that $HK = G$.

Let $x \in G$. Define the *conjugacy class* of x in G and denote it by $\mathrm{Conj}_G(x)$. Define the *centraliser* of x in G and denote it by $C_G(x)$.

Let $x, y \in G$. Suppose $|\mathrm{Conj}_G(x)| = a$ and $|\mathrm{Conj}_G(y)| = b$ with a and b co-prime. Show that $C_G(x)C_G(y) = G$. Prove that

$$\mathrm{Conj}_G(xy) = \mathrm{Conj}_G(x)\mathrm{Conj}_G(y),$$

where $\mathrm{Conj}_G(x)\mathrm{Conj}_G(y) = \{uv : u \in \mathrm{Conj}_G(x), v \in \mathrm{Conj}_G(y)\}$. [*Hint: Observe that* $g^{-1}xgh^{-1}yh$ *may be written as* $h^{-1}(hg^{-1}xgh^{-1}y)h$, *where* $g, h \in G$.]

8D Groups

State and prove the first isomorphism theorem. [You may assume that images of homomorphisms are subgroups and that kernels of homomorphisms are normal subgroups.]

Define the groups $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{R})$. Prove that $\mathrm{SL}_n(\mathbb{R})$ is a normal subgroup of $\mathrm{GL}_n(\mathbb{R})$ and identify $\mathrm{GL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{R})$.

Let $\mathrm{M}_2(\mathbb{Z})$ denote the set of 2×2 matrices with entries in $\mathbb{Z}$. Let

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}) : ad - bc \neq 0 \text{ and } \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \in \mathrm{M}_2(\mathbb{Z}) \right\}.$$

Check that G is a group and that it is infinite. Show that

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}) : ad - bc = \pm 1 \right\}.$$

Consider the following subset of G,

$$H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G : a, d \equiv 1 \bmod 2, \;\; b, c \equiv 0 \bmod 2 \right\}.$$

By considering a suitable homomorphism, or otherwise, show that H is a normal subgroup of finite index in G.

9B Vector Calculus

The vector fields $\mathbf{u}(\mathbf{x}, t)$ and $\mathbf{w}(\mathbf{x}, t)$ obey the evolution equations

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} - \boldsymbol{\nabla} P\,,$$
$$\frac{\partial \mathbf{w}}{\partial t} = (\mathbf{w} \cdot \boldsymbol{\nabla})\mathbf{u} - (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{w}\,,$$

where P is a given scalar field. Show that the scalar field $h = \mathbf{u} \cdot \mathbf{w}$ obeys an evolution equation of the form

$$\frac{\partial h}{\partial t} = (\mathbf{w} \cdot \boldsymbol{\nabla})f + (\mathbf{u} \cdot \boldsymbol{\nabla})g\,,$$

where the scalar fields f and g should be identified.

Suppose that $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ and $\mathbf{w} = \boldsymbol{\nabla} \times \mathbf{u}$. Show that, if $\mathbf{u} \cdot \mathbf{n} = \mathbf{w} \cdot \mathbf{n} = 0$ on the surface S of a fixed volume V with outward normal $\mathbf{n}$, then

$$\frac{dH}{dt} = 0, \text{ where } H = \int_V h\, dV\,.$$

Suppose that $\mathbf{u} = (a^2 - \rho^2)\rho \sin z\, \mathbf{e}_\phi + a\rho^2 \sin z\, \mathbf{e}_z$ in cylindrical polar coordinates ρ, ϕ, z, where a is a constant, and that $\mathbf{w} = \boldsymbol{\nabla} \times \mathbf{u}$. Show that $h = -2a\rho^4 \sin^2 z$, and calculate the value of H when V is the cylinder $0 \leqslant \rho \leqslant a$, $0 \leqslant z \leqslant \pi$.

$$\left[\textit{In cylindrical polar coordinates } \boldsymbol{\nabla} \times \mathbf{F} = \frac{1}{\rho}\begin{vmatrix} \mathbf{e}_\rho & \rho\mathbf{e}_\phi & \mathbf{e}_z \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}.\right]$$

10B Vector Calculus

Show that

$$\boldsymbol{\nabla} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}\, \boldsymbol{\nabla} \cdot \mathbf{b} - \mathbf{b}\, \boldsymbol{\nabla} \cdot \mathbf{a} + (\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{a} - (\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{b}\,.$$

State Stokes' theorem for a vector field in $\mathbb{R}^3$, specifiying the orientation of the integrals.

The vector fields $\mathbf{m}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ satisfy the conditions $\mathbf{m} = \mathbf{n}$ and $\mathbf{v} \cdot \mathbf{n} = 0$ on an open surface S with unit normal $\mathbf{n}(\mathbf{x})$. By applying Stokes' theorem to the vector field $\mathbf{m} \times \mathbf{v}$, show that

$$\int_S (\delta_{ij} - n_i n_j) \frac{\partial v_i}{\partial x_j} dS = \oint_C \left[\mathbf{v} \cdot (d\mathbf{x} \times \mathbf{n})\right], \qquad (*)$$

where C is the boundary of S. Describe the orientation of $d\mathbf{x} \times \mathbf{n}$ relative to S and C.

Verify $(*)$ when S is the hemisphere $r = R$, $z \geqslant 0$ and $\mathbf{v} = r \sin\theta\, \mathbf{e}_\theta$ in spherical polar coordinates r, θ, ϕ.

[*You may use the formulae* $(\mathbf{e}_r \cdot \boldsymbol{\nabla})\mathbf{e}_\theta = \mathbf{0}$ *and*

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{1}{r^2}\frac{\partial(r^2 F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, F_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}\,,$$

and you may quote formulae for dS *and* $d\mathbf{x}$ *in these coordinates without derivation.*]

11B Vector Calculus

(a) Verify the identity

$$\boldsymbol{\nabla} \cdot (\kappa\psi\boldsymbol{\nabla}\phi) = \psi\boldsymbol{\nabla} \cdot (\kappa\boldsymbol{\nabla}\phi) + \kappa\boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\phi\,,$$

where $\kappa(\mathbf{x})$, $\phi(\mathbf{x})$ and $\psi(\mathbf{x})$ are differentiable scalar functions.

Let V be a region in $\mathbb{R}^3$ that is bounded by a closed surface S. The function $\phi(\mathbf{x})$ satisfies

$$\boldsymbol{\nabla} \cdot (\kappa\boldsymbol{\nabla}\phi) = 0 \text{ in } V \text{ and } \phi = f(\mathbf{x}) \text{ on } S\,,$$

where κ and f are given functions and $\kappa > 0$. Show that ϕ is unique.

The function $w(\mathbf{x})$ also satisfies $w = f(\mathbf{x})$ on S. By writing $w = \phi + \psi$, show that

$$\int_V \kappa|\boldsymbol{\nabla}w|^2\,dV \geqslant \int_V \kappa|\boldsymbol{\nabla}\phi|^2\,dV\,.$$

(b) A steady temperature field $T(\mathbf{x})$ due to a distribution of heat sources $H(\mathbf{x})$ in a medium with spatially varying thermal diffusivity $\kappa(\mathbf{x})$ satisfies

$$\boldsymbol{\nabla} \cdot (\kappa\boldsymbol{\nabla}T) + H = 0\,.$$

Show that the heat flux $\int_S \mathbf{q} \cdot d\mathbf{S}$ across a closed surface S, where $\mathbf{q} = -\kappa\boldsymbol{\nabla}T$, can be expressed as an integral of the heat sources within S.

By using this version of Gauss's law, or otherwise, find the temperature field $T(r)$ for the spherically symmetric case when

$$\kappa(r) = r^\alpha, \quad -1 < \alpha < 2, \qquad\qquad H(r) = \begin{cases} H_0 & \text{if } r \leqslant 1 \\ 0 & \text{if } r > 1 \end{cases}$$

subject to the condition that $T \to 0$ as $r \to \infty$. What goes wrong if $\alpha \leqslant -1$?

Deduce that if $w(r)$ satisfies $w(1) = 1$ and $w(r) \to 0$ as $r \to \infty$ (sufficiently rapidly for the integral to converge) then

$$\int_1^\infty r^{\alpha+2}\left(\frac{dw}{dr}\right)^2 dr \geqslant \alpha + 1\,.$$

12B Vector Calculus

(a) State the transformation law for the components of an nth-rank tensor $T_{ij\ldots k}$ under a rotation of the basis vectors, being careful to specify how any rotation matrix relates the new basis $\{\mathbf{e}'_i\}$ to the original basis $\{\mathbf{e}_j\}$, $i, j = 1, 2, 3$.

If $\phi(\mathbf{x})$ is a scalar field, show that $\partial^2\phi/\partial x_i \partial x_j$ transforms as a second-rank tensor.

Define what it means for a tensor to be *isotropic*. Write down the most general isotropic tensors of rank k for $k = 0, 1, 2, 3$.

(b) Explain briefly why T_{ijkl}, defined by

$$T_{ijkl} = \int_{\mathbb{R}^3} x_i x_j e^{-r^2} \frac{\partial^2}{\partial x_k \partial x_l}\left(\frac{1}{r}\right) dV\,, \quad \text{where } r = |\mathbf{x}|,$$

is an isotropic fourth-rank tensor.

Assuming that

$$T_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}\,,$$

use symmetry, contractions and a scalar integral to determine the constants α, β and γ.

[*Hint:* $\nabla^2(1/r) = 0$ for $r \neq 0$.]

END OF PAPER