UNIVERSITY OF CAMBRIDGE

MAT0

MATHEMATICAL TRIPOS Part IA

Friday, 2 June, 2023 1.30pm to 4:30pm

PAPER 2

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **all four** questions from Section I and **at most five** questions from Section II. Of the Section II questions, no more than three may be on the same course.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1A Differential Equations

Find the general solution $y(x)$ of the differential equation

$$y''' - 4y'' + 4y' = xe^{2x}.$$

2A Differential Equations

(a) Find the solution $y(x)$ of

$$x^2 y' - \cos(2y) = 1$$

subject to $y \to 9\pi/4$ as $x \to \infty$. [If your answer involves inverse trigonometric functions, then you should specify their range.]

(b) Find the general solution $u(x)$ of the equation

$$xu' = x + u.$$

3F Probability

(a) State and prove Markov's inequality.

(b) Let X be a standard normal random variable. Compute the moment generating function $M_X(t) = \mathbb{E}(e^{tX})$.

(c) Prove that, for all $v > 0$,

$$\int_v^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \leqslant e^{-v^2/2}.$$

4F Probability

A $2k$-*spalindrome* is a sequence of $2k$ digits that contains k distinct digits and reads the same backwards as forwards.

(a) What is the probability that a sequence of $2k$ digits, chosen independently and uniformly at random from $\{0, 1, \ldots, 9\}$, is a $2k$-spalindrome?

(b) Suppose now a sequence of $3k$ digits is chosen independently and uniformly at random from $\{0, 1, \ldots, 9\}$. What is the probability that this longer sequence contains a $2k$-spalindrome? [*Hint: Consider the event that the subsequence starting in position ℓ is a $2k$-spalindrome.*]

SECTION II

5A Differential Equations

(a) Consider the linear differential equation

$$y' + p(x)y = f(x), \tag{$*$}$$

where $p(x)$ and $f(x)$ are given nonzero functions. Show how to express the general solution $y(x)$ in terms of two integrals involving $p(x)$ and $f(x)$, to be specified.

If $y_1(x)$ and $y_2(x)$ are distinct solutions of $(*)$, express the general solution of $(*)$ in terms of $y_1(x)$ and $y_2(x)$.

(b) Find the general solution $y(x)$ of the differential equation

$$xy' - (2x^2 + 1)y = x^2.$$

Show that there is only one solution of this equation with $y(x)$ bounded as $x \to \infty$, and determine its limiting value. Sketch this solution.

6A Differential Equations

The function $y(x, \mu)$ satisfies

$$\frac{\partial y}{\partial x} = y + \mu(x + y^2)\,, \quad y(0, \mu) = 1\,, \tag{$*$}$$

and the function $u(x, \mu)$ is defined by $u = \partial y / \partial \mu$. Show that

$$\frac{\partial u}{\partial x} = u + x + y^2 + 2\mu y u\,, \quad u(0, \mu) = 0\,.$$

Determine $y(x, 0)$ and then $u(x, 0)$.

For small μ, the solution of $(*)$ can be approximated by a series

$$y(x, \mu) = y_0(x) + \mu y_1(x) + \mu^2 y_2(x) + \cdots.$$

Specify the functions $y_0(x), y_1(x)$ and $y_2(x)$.

7A Differential Equations

The Dirac δ-function can be defined by the properties $\delta(t) = 0$ for $t \neq 0$ and $\int_a^b f(t)\delta(t)\,dt = f(0)$ for any $a < 0 < b$ and function $f(t)$ that is continuous at $t = 0$. The function $H(t)$ is defined by

$$H(t) = \begin{cases} 1 & \text{for} \quad t \geqslant 0 \\ 0 & \text{for} \quad t < 0\,. \end{cases}$$

(a) Prove that

(i) $\delta(pt) = \delta(t)/|p|$ for any nonzero real constant p;

(ii) for any differentiable function $f(t)$

$$\int_{-\infty}^{\infty} f(t)\delta'(t)\,dt = -f'(0)\,;$$

(iii) $H'(t) = \delta(t)$.

(b) An electronic system has two time-dependent variables $x(t)$ and $y(t)$, and two inputs to which a constant unit signal is applied, each starting at a particular time. The differential equations governing the system take the form

$$\begin{aligned} \dot{x} + 2y &= H(t)\,, \\ \dot{y} - 2x &= H(t - \pi)\,. \end{aligned}$$

At $t = -\pi$, the system has $x = 1$ and $y = 0$. Find $x(t)$ for $t < 0$. Show that $x(t)$ can be written for $t > 0$ as

$$x(t) = a\sin 2t + b + q(t)\sin^2 t\,,$$

where the constants a and b and the function $q(t)$ are to be specified. Sketch $q(t)$ for $0 < t < 2\pi$.

8A Differential Equations

(a) Classify the equilibrium point of the system

$$\frac{dx}{dt} = 4x + 2y\,, \quad \frac{dy}{dt} = -x + y\,.$$

Sketch the phase portrait showing both the direction of any straight-line trajectories and the shapes of a representative selection of non-straight trajectories to indicate the direction of motion in each part of phase space.

(b) Consider the second-order differential equation for $x(t)$

$$\ddot{x} + 3\dot{x} - 4\log\frac{x^2+1}{2} = 0\,.$$

(i) Rewrite the equation as a system of two first-order equations for $x(t)$ and $y(t)$, where $y = \dot{x}$, and find the equilibrium points of that system.

(ii) Use linearisation to classify the equilibrium points.

(iii) On a sketch of the (x, y)-plane, show the regions where $\dot{x}$ and $\dot{y}$ are both positive, both negative, or one positive and one negative.

(iv) Using the information obtained in parts (i)–(iii), sketch the trajectories of the system, including the trajectories through $(1, 0)$.

9F Probability

In a group of people, each pair are friends with probability $1/2$, and friendships between different pairs of people are independent. Each person's birthday is distributed independently and uniformly among the 365 days of the year. Birthdays are independent of friendships.

The number of people in the group, N, has a Poisson distribution with mean 365.

(a) What is the expectation of the number of pairs of friends with the same birthday?

(b) Let Z_i be the number of people born on the ith day of the year. Find the joint probability mass function of $(Z_1, \ldots, Z_{365})$.

(c) What is the probability that no pair of friends have the same birthday? [You may express your answer in terms of the constant

$$C = \sum_{n=2}^{\infty} \frac{2^{-n(n-1)/2}}{n!} \approx 0.27\,. \;]$$

10F Probability

Let X be a random variable with probability density function

$$f(x) = \frac{x^{n-1}e^{-x}}{(n-1)!} \quad \text{for } x \geqslant 0,$$

where n is a positive integer.

(a) Find the moment generating function $M_X(t)$ for $t < 1$.

(b) Find the mean and variance of X.

(c) Prove that, for every $q \geqslant 0$,

$$\int_0^{n+q\sqrt{n}} \frac{x^{n-1}e^{-x}}{(n-1)!}\, dx \to \Phi(q) \quad \text{as } n \to \infty\,,$$

where Φ is the distribution function of a standard normal random variable. [You may cite any result from the course, provided that it is clearly stated.]

11F Probability

Let T_1 and T_2 be independent exponential random variables with means λ_1^{-1} and λ_2^{-1}, respectively. Let $V = \min(T_1, T_2)$ and $W = \max(T_1, T_2)$.

(a) Find the distribution of V. What is the probability that $V = T_1$?

(b) Find $\Pr(V \leqslant t \mid V > s)$ for $t > s > 0$.

From now on, suppose that $\lambda_1 = \lambda_2 = \lambda$.

(c) Prove that V and $W - V$ are independent. What is the distribution of $W - V$?

(d) Hence, find the distribution of $2V/(W + V)$.

12F Probability

Let X be a random variable taking values in $\{0, 1, 2, \dots\}$, with $\Pr(X \geqslant 2) > 0$.

(a) Define the *probability generating function* G_X of X. Show that the first and second derivatives of G_X are positive and non-decreasing on $(0, 1]$.

Now consider a branching process which starts with a population of 1. For each $n \geqslant 1$, each individual in generation n gives rise to an independent number of offspring, distributed as X, which together form generation $n + 1$.

(b) Let d be the probability that the population eventually becomes extinct. Prove that d is the smallest non-negative solution to $t = G_X(t)$.

(c) Let $\mathbb{E}(X) = \mu$. Show that if $\mu > 1$ then $d < 1$.

(d) Suppose that $\mu > 1$ and that X has variance σ^2. Show that for $t \in [0, 1]$,

$$G_X(t) \leqslant 1 - \mu(1 - t) + \frac{1}{2}(\sigma^2 + \mu^2 - \mu)(1 - t)^2.$$

Hence find an upper bound $d^* < 1$ for the extinction probability d, where d^* is given in terms of μ and σ^2.

END OF PAPER