UNIVERSITY OF CAMBRIDGE

MAT0

MATHEMATICAL TRIPOS Part IA

Thursday, 1 June, 2023 9:00am to 12:00pm

PAPER 1

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **all four** questions from Section I and **at most five** questions from Section II. Of the Section II questions, no more than three may be on the same course.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1A Vectors and Matrices

The principal value of the logarithm of a complex variable is defined to have its argument in the range $(-\pi, \pi]$.

(a) Evaluate $\log(-i)$, stating both the principal value and the other possible values.

(b) Show that i^{-2i} represents an infinite set of real numbers, which should be specified.

(c) By writing $z = \tan w$ in terms of exponentials, show that

$$\tan^{-1} z = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right).$$

Use this result to evaluate the principal value of

$$\tan^{-1}\left(\frac{2\sqrt{3}-3i}{7}\right).$$

2C Vectors and Matrices

For an $n \times n$ complex matrix A, define the *Hermitian conjugate* $A^\dagger$. State the conditions (i) for A to be unitary (ii) for A to be Hermitian.

Let A, B, C and D be $n \times n$ complex matrices and $\mathbf{x}$ a complex n-vector. A matrix N is defined to be normal if $N^\dagger N = NN^\dagger$.

(a) For A nonsingular, show that $B = A^{-1}A^\dagger$ is unitary if and only if A is normal.

(b) Let C be normal. Show that $|C\mathbf{x}| = 0$ if and only if $|C^\dagger \mathbf{x}| = 0$.

(c) Let D be normal. Deduce from part (b) that if $\mathbf{e}$ is an eigenvector of D with eigenvalue λ then $\mathbf{e}$ is also an eigenvector of $D^\dagger$ and find the corresponding eigenvalue.

3E Analysis

Let $a \in \mathbb{R}$ and let f and g be real-valued functions defined on $\mathbb{R}$. State and prove the chain rule for $F(x) = g\big(f(x)\big)$.

Now assume that f and g are non-constant on any interval. Must the function $F(x) = g\big(f(x)\big)$ be non-differentiable at $x = a$ if

(i) f is differentiable at a and g is not differentiable at $f(a)$?

(ii) f is not differentiable at a and g is differentiable at $f(a)$?

(iii) f is not differentiable at a and g is not differentiable at $f(a)$?

Justify your answers.

4E Analysis

State the comparison test. Prove that if $\sum_{n=0}^{\infty} a_n z_0^n$ converges and $|z_1| < |z_0|$, then $\sum_{n=0}^{\infty} a_n z_1^n$ converges absolutely.

Define the *radius of convergence* of a complex power series. [You do not need to show that the radius of convergence is well-defined.]

If $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R_1 and $\sum_{n=0}^{\infty} b_n z^n$ has radius of convergence R_2, show that the radius of convergence R of the series $\sum_{n=0}^{\infty} a_n b_n z^n$ satisfies $R \geqslant R_1 R_2$.

SECTION II

5A Vectors and Matrices

(a) The position vector $\mathbf{r}$ of a general point on a surface in $\mathbb{R}^3$ is given by the equations below
(i) $(\mathbf{r}-\mathbf{a})\cdot\mathbf{n}=0$,
(ii) $\mathbf{r}=\mathbf{b}+\lambda(\mathbf{d}-\mathbf{b})+\mu(\mathbf{f}-\mathbf{b})$,
(iii) $|\mathbf{r}-\mathbf{c}|=\rho$.

Identify each surface and describe the meaning of the constant vectors $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{f}$ and $\mathbf{n}$, together with the scalars λ,μ and $\rho>0$.

(b) Find the equation for the line of intersection of the planes $2x+3y-z=3$ and $x-3y+4z=3$ in the form $\mathbf{r}\times\mathbf{m}=\mathbf{u}\times\mathbf{m}$, where $\mathbf{m}$ is a unit vector and $\mathbf{u}\cdot\mathbf{m}=0$.

Find the minimum distance from this line to the line that is inclined at equal angles to the positive x-, y-, and z- axes and passes through the origin.

(c) The intersection of the surface $\mathbf{r}\cdot\mathbf{n}=p$, where $\mathbf{n}$ is a unit vector and p is a real number, and the sphere of radius A centred on the point with position vector $\mathbf{g}$ is a circle of radius R.

Find the position vector $\mathbf{h}$ of the centre of the circle and determine R as a function of $A,p,\mathbf{g}$ and $\mathbf{n}$. Discuss geometrically the condition on R to be real.

6C Vectors and Matrices

(a) Consider the matrix

$$M=\begin{pmatrix}2&1&0\\0&1&-1\\0&2&4\end{pmatrix}.$$

Determine whether or not M is diagonalisable.

(b) Prove that if A and B are similar matrices then A and B have the same eigenvalues with the same corresponding algebraic multiplicities. Is the converse true? Give either a proof (if true) or a counterexample with a brief reason (if false).

(c) State the Cayley–Hamilton theorem for an $n\times n$ matrix A and prove it for the case that A is a 2×2 diagonalisable matrix.

Suppose B is an $n\times n$ matrix satisfying $B^k=0$ for some $k>n$ (where 0 denotes the zero matrix). Show that $B^n=0$.

7B Vectors and Matrices

(a) Let A be an $n \times n$ non-singular matrix and let $G = A^\dagger A$ and $H = AA^\dagger$, where dagger denotes the Hermitian conjugate.

By considering $|A\mathbf{x}|$ for a vector $\mathbf{x}$, or otherwise, show that the eigenvalues of G are positive real numbers.

Show that if $\mathbf{e}_i$ is an eigenvector of G with eigenvalue λ_i then $\mathbf{f}_i = A\mathbf{e}_i$ is an eigenvector of H. What is the value of $|\mathbf{f}_i|/|\mathbf{e}_i|$?

(b) Using part (a), explain how to construct unitary matrices U and V from the eigenvectors of G and H such that $V^\dagger AU = D$, where D is a diagonal matrix to be specified in terms of the eigenvalues of G. [You may assume that, for any $n \times n$ Hermitian matrix, it is possible to find n orthogonal eigenvectors.]

(c) Find U, V and D for the case

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & -2 \\ -1 & -1 \end{pmatrix} .$$

8B Vectors and Matrices

Define what it means for an $n \times n$ real matrix to be *orthogonal*.

Show that the eigenvalues of an orthogonal matrix have unit modulus, and show that eigenvectors with distinct eigenvalues are orthogonal.

Let Q be a 3×3 orthogonal matrix with $\det Q = -1$. Show that -1 is an eigenvalue of Q.

Let $\mathbf{n}$ be a nonzero vector satisfying $Q\mathbf{n} = -\mathbf{n}$ and consider the plane Π through the origin that is perpendicular to $\mathbf{n}$. Show that Q maps Π to itself.

Show that Q acts on Π as a rotation through some angle θ, and show that $\cos\theta = \frac{1}{2}(\operatorname{tr} Q + 1)$.

Show also that $\det(Q - I) = 4(\cos\theta - 1)$.

[*You may quote the form of relationship between two matrix representations A and A' of a linear map α with respect to different bases, but should explain results derived from it.*]

9E Analysis

(a) Let $x_1 > 0$ and define a sequence (x_n) by

$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{1}{x_{n-1}}\right) \text{ for } n > 1.$$

Prove that $\lim_{n\to\infty} x_n = 1$.

Show that if a real sequence (x_n) satisfies

$$0 \leqslant x_{m+n} \leqslant x_m + x_n \quad \text{for all } m, n = 1, 2, \ldots,$$

then the sequence (x_n/n) is (i) bounded and (ii) convergent.

(b) Suppose that a series $\sum_{n=1}^{\infty} a_n$ of real numbers converges but not absolutely. Let

$$P_n = \sum_{i=1}^{n} \big(|a_i| + a_i\big), \qquad N_n = \sum_{i=1}^{n} \big(|a_i| - a_i\big).$$

Show that $\lim_{n\to\infty} P_n/N_n = 1$.

State the alternating series test. Let (b_n) be a sequence of positive real numbers such that

$$\lim_{n\to\infty} n\left(\frac{b_n}{b_{n+1}} - 1\right) = p,$$

where p is a positive real number. Show that the series $\sum_{n=1}^{\infty}(-1)^n b_n$ converges.

10E Analysis

State and prove the intermediate value theorem.

Give, with justification, an example of a function $\phi : [a,\infty) \to \mathbb{R}$ such that, for any $b > a$, ϕ takes on $[a,b]$ every value between $\phi(a)$ and $\phi(b)$ but ϕ is not continuous on $[a,b]$.

If a function $f : [a,b] \to \mathbb{R}$ is monotone on $[a,b]$ and takes every value between $f(a)$ and $f(b)$, show that f is continuous on $[a,b]$.

Let $g : (a,b) \to \mathbb{R}$ be a continuous function and suppose that there are sequences $x_n \to a$ and $y_n \to a$ as $n \to \infty$ such that $g(x_n) \to l$ and $g(y_n) \to L$ with $l < L$. Show that for each $\lambda \in [l,L]$ there is a sequence $z_n \to a$ such that $g(z_n) \to \lambda$.

11E Analysis

(a) State the mean value theorem. Deduce that

$$\frac{a-b}{a} < \log\frac{a}{b} < \frac{a-b}{b} \quad \text{for } 0 < b < a.$$

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be an n-times differentiable function, where $n > 0$. Show that for each $a \in \mathbb{R}$ and $h > 0$ there exists $b \in (a, a + nh)$ such that

$$\frac{1}{h^n}\Delta_h^n f(a) = f^{(n)}(b),$$

where $\Delta_h^{k+1} f(x) = \Delta_h^1\big(\Delta_h^k f(x)\big)$ and $\Delta_h^1 f(x) = f(x+h) - f(x)$.

(c) Let $I \subset \mathbb{R}$ be an open (non-empty) interval and $a \in I$. Suppose that a function $\varphi : I \to \mathbb{R}$ has a finite limit at a and $\lim_{x\to a}\varphi(x) = \varphi(a) + 1$. Can φ be the derivative of some differentiable function f on I? Justify your answer.

12E Analysis

Define the *upper* and *lower integral* of a function on $[a, b]$ and what it means for a function to be *(Riemann) integrable* on $[a, b]$.

(a) Let $\lfloor y \rfloor = \max\{i \in \mathbb{Z} : i \leqslant y\}$. Show that the function

$$u(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \quad \text{if } x \neq 0, \qquad u(0) = 0,$$

is integrable on $[0, 1]$. [You may assume that every continuous function on a closed bounded interval is integrable.]

(b) Let $f : [A, B] \to \mathbb{R}$ be a continuous function and $A < a < x < B$. Prove that

$$\lim_{h\to 0}\frac{1}{h}\int_a^x \big(f(t+h) - f(t)\big)dt = f(x) - f(a).$$

[Any version of the fundamental theorem of calculus from the course can be assumed if accurately stated.]

(c) Show that if a function $g : [a, b] \to \mathbb{R}$ is integrable, then there exists a sequence of continuous functions $\varphi_n : [a, b] \to \mathbb{R}$ such that $\int_\alpha^\beta g(x)dx = \lim_{n\to\infty}\int_\alpha^\beta \varphi_n(x)dx$ for any subinterval $[\alpha, \beta] \subseteq [a, b]$.

END OF PAPER