UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II 2022

List of Courses

Algebraic Geometry
Algebraic Topology
Analysis of Functions
Applications of Quantum Mechanics
Applied Probability
Asymptotic Methods
Automata and Formal Languages
Classical Dynamics
Coding and Cryptography
Cosmology
Differential Geometry
Dynamical Systems
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Further Complex Methods
Galois Theory
General Relativity
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Integrable Systems
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Mathematical Biology
Mathematics of Machine Learning
Number Fields
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Numerical Analysis
Principles of Quantum Mechanics
Principles of Statistics
Probability and Measure
Quantum Information and Computation
Representation Theory

Riemann Surfaces
Statistical Modelling
Statistical Physics
Stochastic Financial Models
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Waves

Paper 1, Section II

25G Algebraic Geometry

Let X be an irreducible affine variety. Define the tangent space to X at a point $p \in X$. What does it mean for X to be *smooth*?

Prove that any irreducible affine cubic has at most one singular point.

Prove that the set of smooth points of any irreducible variety is dense in the Zariski topology.

Let $X \subset \mathbb{P}^n$ be a smooth irreducible hypersurface. Recall there is a natural map

$$\pi : \mathbb{A}^{n+1} \setminus \{\underline{0}\} \to \mathbb{P}^n.$$

Let $Y \subset \mathbb{A}^{n+1}$ be the closure of $\pi^{-1}(X)$. Prove that Y contains at most one singular point. Give examples to show that Y can be smooth and that Y can be singular.

Paper 2, Section II

25G Algebraic Geometry

State the Riemann–Roch theorem for a smooth projective curve X. Using this theorem, calculate the degree of the canonical divisor of X in terms of the genus of X.

Prove that every smooth projective curve of genus g can be embedded in a fixed projective space $\mathbb{P}^n$, where n may depend on g.

State the Riemann–Hurwitz formula. Using this formula or otherwise, construct a smooth projective variety of dimension 2 that contains no curves of genus less than 3.

Let X be a smooth curve of degree d in $\mathbb{P}^2$ and let p be a point not lying on X. Prove that projection away from p gives rise to a morphism

$$\pi : X \to \mathbb{P}^1.$$

Give an upper bound on the number of points of X at which π can be ramified. [You do not need to show that your bound is sharp.]

Paper 3, Section II

24G Algebraic Geometry

[In this question all algebraic varieties are over $\mathbb{C}$.]

State Hilbert's Nullstellensatz for affine varieties. Suppose that I is a homogeneous ideal such that $\mathbb{V}(I) \subset \mathbb{P}^n$ is empty. What are the possibilities for I?

Let V be a smooth quadric hypersurface in $\mathbb{P}^3$. Construct a pair of disjoint, smooth and projective curves lying on V. Deduce that V is not isomorphic to $\mathbb{P}^2$.

Let W be a smooth projective curve. Prove that every rational map from W to a projective variety is a morphism. Give an example showing that if W is singular, this statement can fail.

Construct an algebraic variety $Z \subset \mathbb{P}^2 \times \mathbb{P}^1$ and a surjective morphism $\pi : Z \to \mathbb{P}^1$ such that there exists a point $p \in \mathbb{P}^1$ whose preimage $\pi^{-1}(p)$ is a smooth projective curve of genus 1, and another point $q \in \mathbb{P}^1$ such that $\pi^{-1}(q)$ has exactly 3 irreducible components.

Paper 4, Section II

24G Algebraic Geometry

What does it mean for two irreducible varieties to be *birational*? Prove that birational varieties have the same dimension.

Let K be a finitely generated field extension of $\mathbb{C}$. Prove that there exists a projective variety X over $\mathbb{C}$ whose function field is K.

Let X be the affine plane curve $\mathbb{V}(f) \subset \mathbb{A}^2$, where

$$f(x, y) = y^2 - x(x-1)^2.$$

For what values of d is X birational to a smooth projective plane curve of degree d?

Construct an affine variety X of dimension 2 that is birational to $\mathbb{A}^2$, and whose set of singular points is an irreducible subvariety of dimension 1.

Paper 1, Section II

21G Algebraic Topology

State the universal property which characterizes an amalgamated free product of groups. State the Seifert-van Kampen theorem.

Suppose that $\{U_1, U_2\}$ is an open cover of a topological space X, that $U_1 \cap U_2$ is path connected and that $x_0 \in U_1 \cap U_2$. If $i_k : U_k \to X$ is the inclusion, prove that $\pi_1(X, x_0)$ is generated by $i_{1*}(\pi_1(U_1, x_0))$ and $i_{2*}(\pi_1(U_2, x_0))$. [You may use the Lebesgue covering lemma if you state it clearly.]

Consider the Mobius band $M = I^2/\sim$, where $(0, x) \sim (1, 1-x)$. Identify its boundary $\partial M = (I \times \{0,1\})/\sim$ with S^1. Note that if $f : \partial M \to X$, the space obtained by *attaching a Mobius band to X using f* is $X \cup_f M = (X \amalg M)/\sim$, where now $\sim$ is the smallest equivalence relation containing $x \sim f(x)$ for all $x \in \partial M$. Now let Y be the space obtained by attaching two Mobius bands to $T^2 = S^1 \times S^1$ using the maps $f_1, f_2 : S^1 \to T^2$ given by $f_1(z) = (z, z)$ and $f_2(z) = (z^2, z^3)$. Give a two-generator one-relator presentation of $\pi_1(Y, y_0)$ for some $y_0 \in Y$. Show that this group is non-abelian.

Paper 2, Section II

21G Algebraic Topology

Let $p : \widehat{X} \to X$ be a covering map, and suppose that X and $\widehat{X}$ are path connected and locally path connected topological spaces. If $x_0, x_1 \in X$, show that $p^{-1}(x_0)$ and $p^{-1}(x_1)$ have the same cardinality. [You may use any theorems from the course, as long as you state them clearly.]

Define what it means for p to be a *normal covering map.* State an appropriate lifting theorem and use it to prove that if $p : \widehat{X} \to X$ is a *universal* covering map, then it is normal.

Let Σ_g be a surface of genus g and suppose that $p : \widehat{\Sigma}_g \to \Sigma_g$ is a connected covering map of degree $n \in \mathbb{N}$. For which values of g and n must p be normal? Justify your answer. For those values of g and n for which p need not be normal, give an explicit example of a non-normal covering map p.

Paper 3, Section II

20G Algebraic Topology

Consider the set $X \subset S^3$ given by $X = \{(x_1, x_2, x_3, x_4) \in S^3 : |x_4| \leqslant \frac{1}{2}\}$ and its boundary $\partial X = \{(x_1, x_2, x_3, x_4) \in S^3 : |x_4| = \frac{1}{2}\}$. Define Y and ∂Y to be the image of X and ∂X in $\mathbb{RP}^3 = S^3/\sim$, where $x \sim -x$. Show that Y is homotopy equivalent to $\mathbb{RP}^2$. Compute $H_*(\mathbb{RP}^3)$. [You may assume $\mathbb{RP}^3$ admits a triangulation containing Y and ∂Y as subcomplexes, and may use $H_*(\mathbb{RP}^2)$ if you state it precisely.]

Let $f : \partial Y \to \partial Y$ be the identity map, and define Z to be the space obtained by identifying two copies of Y along their boundary: $Z = Y \cup_f Y$. Compute $H_*(Z)$ and $\pi_1(Z, z_0)$, where $z_0 \in Z$. The universal covering space of Z is homeomorphic to a familiar space. What is it?

Paper 4, Section II

21G Algebraic Topology

Suppose that (C, d) and (C', d') are chain complexes, and that $f, g : C \to C'$ are chain maps. Show that f induces a map $f_* : H_*(C) \to H_*(C')$. Define what it means for f and g to be *chain homotopic.* Show that if f and g are chain homotopic, they induce the same map on homology.

Define a chain complex $(M(f), d_f)$ as follows: $M(f)_i = C_{i-1} \oplus C'_i$ and the map $(d_f)_i : M(f)_i \to M(f)_{i-1}$ is given by the matrix

$$\begin{pmatrix} d_{i-1} & 0 \\ (-1)^i f_{i-1} & d'_i \end{pmatrix}.$$

Verify that $(M(f), d_f)$ is a chain complex. Show that there is a long exact sequence

$$\ldots \to H_i(C) \xrightarrow{(-1)^{i+1} f_*} H_i(C') \to H_i(M(f)) \to H_{i-1}(C) \xrightarrow{(-1)^i f_*} H_{i-1}(C') \to \ldots$$

If f is chain homotopic to g, show that $(M(f), d_f)$ and $(M(g), d_g)$ are isomorphic as chain complexes.

Paper 1, Section II

23F Analysis of Functions

(a) State and prove the Sobolev trace theorem that maps $H^s(\mathbb{R}^n)$ into a suitable Sobolev space over $\mathbb{R}^{n-1}$.

(b) Show that there is no bounded linear operator $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^{n-1})$ satisfying $Tu = u|_{\mathbb{R}^{n-1}\times\{0\}}$ for all $u \in C(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$.

(c) For $u \in C_c^\infty(\mathbb{R}^2)$, prove that

$$\int_{\mathbb{R}^2} |u|^4 \, dx \leqslant C \int_{\mathbb{R}^2} |u|^2 \, dx \int_{\mathbb{R}^2} |\nabla u|^2 \, dx,$$

for a constant C independent of u. [*Hint: First show that, for all* $(x, y) \in \mathbb{R}^2$, $|u(x, y)|^2 \leqslant 2 \int_{\mathbb{R}} |u(x, t)||\nabla u(x, t)| \, dt$.]

Paper 2, Section II

23F Analysis of Functions

(a) Let $U \subset \mathbb{R}^n$ be open with finite Lebesgue measure. Let $p \in (1, \infty)$ and let $\Lambda \in L^p(U)'$ be positive. Prove there is $\omega \in L^q(U)$ where $1/p + 1/q = 1$ such that

$$\Lambda(f) = \int_U f\omega \, dx \qquad \text{for all } f \in L^p(U).$$

[You may use without proof that $\|g\|_{L^q(U)} = \sup\{\int_U |fg| \, dx : \|f\|_{L^p(U)} \leqslant 1\}$.]

(b) (i) Define the Fourier transform of $f \in L^1(\mathbb{R}^n)$.

(ii) Let $p, q \in (1, \infty)$, and assume $\|\hat{f}\|_{L^q(\mathbb{R}^n)} \leqslant C_{p,q}\|f\|_{L^p(\mathbb{R}^n)}$ for all $f \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Show that q is uniquely determined by p.

(iii) Compute the Fourier transform of $f(x) = |x|^{-1}$ on $\mathbb{R}^3$ up to a multiplicative constant which you do not need to determine. [You may use the Fourier transform of a Gaussian without proof.]

Paper 3, Section II

22F Analysis of Functions

(a) Let $U \subset \mathbb{R}^n$ be bounded and open, and let $m^2 > 0$. Given $f \in L^2(U)$, define what it means for u to be a *weak solution* to

$$\begin{aligned} -\Delta u + m^2 u &= f && \text{in } U \\ u &= 0 && \text{on } \partial U. \end{aligned}$$

Show that for any $f \in L^2(U)$ there is a unique weak solution u and let $Tf = u$. Show that $T : L^2(U) \to L^2(U)$ defines a compact operator. [You may use any theorems from the course if you state them carefully.]

(b) Let $U \subset \mathbb{R}^n$ be bounded and open, and let $(u_k) \subset L^2(U)$ be a sequence such that $u_k \rightharpoonup u$ weakly in $L^2(U)$. Assume that $\sup_k \int_{\{|p| \geqslant t\}} (|\hat{u}_k(p)|^2 + |\hat{u}(p)|^2)\, dp \to 0$ as $t \to \infty$. Show that then $u_k \to u$ in $L^2(U)$.

(c) Given $f \in H^r(\mathbb{R}^n)$, assume that $u \in L^2(\mathbb{R}^n)$ satisfies

$$\Delta^{2022} u + u = f \qquad \text{on } \mathbb{R}^n$$

in distributional sense. For which n is u a function that solves the equation in the classical sense? [You may cite any theorems from the course.]

Paper 4, Section II

23F Analysis of Functions

(a) Prove that the embedding $H^1(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ is not compact.

(b) Construct a bounded linear functional on $L^\infty(\mathbb{R}^n)$ that cannot be expressed as $f \in L^\infty(\mathbb{R}^n) \mapsto \int f(x) g(x)\, dx$ for any $g \in L^1(\mathbb{R}^n)$. [You may use theorems from the course if you state them carefully.]

(c) Prove that $H^n(\mathbb{R}^n)$ embeds continuously into $C^{0,\alpha}(\mathbb{R}^n)$, for some $\alpha \in (0,1)$.

(d) Let θ be the Heaviside function, defined by $\theta(x) = 1_{x \geqslant 0}$, $x \in \mathbb{R}$. Find the Hardy–Littlewood maximal function $M\theta$.

Paper 1, Section II

35D Applications of Quantum Mechanics

(a) A beam of particles of mass m and energy E, moving in one dimension, scatters off a potential barrier $V(x)$ which is localised near the origin $x = 0$ and is reflection invariant, $V(x) = V(-x)$ for all x. With reference to the asymptotic form of the wave function as $x \to \pm\infty$, define the corresponding *reflection* and *transmission coefficients*, denoted r and t respectively, and write down the *S-matrix* $\mathcal{S}$.

For the case $V(x) = V_0\,\delta(x)$, where $\delta(x)$ denotes the Dirac δ-function, determine r and t as functions of the energy E, and show explicitly that $\mathcal{S}$ is a unitary matrix.

(b) A particle of mass m and energy E moves in one dimension subject to a potential $\tilde{V}(x)$ obeying $\tilde{V}(x+a) = \tilde{V}(x)$ for all x. Define the corresponding *Floquet matrix* $\mathcal{M}$. Explain briefly how the Floquet matrix determines the resulting energy spectrum of continuous bands separated by forbidden regions. [You may state without proof any results from the course you might need.]

Determine $\mathcal{M}$ as a function of E for the case $\tilde{V}(x) = V_0 \sum_{n=-\infty}^{+\infty} \delta(x - na)$. Find algebraic equations which determine all the edges of the allowed energy bands. For each edge express $\exp(-ika)$ at the edge in terms of r and t. Here $r = r(E)$ and $t = t(E)$ are the reflection and transmission coefficients determined in part (a), and $E = \hbar^2 k^2/2m$ with $k > 0$.

Paper 2, Section II

36D Applications of Quantum Mechanics

Consider a quantum system with Hamiltonian $\hat{H}$ having a discrete spectrum with a unique groundstate $|\psi_0\rangle$ of energy E_0. For any state $|\psi\rangle$, define the *Rayleigh-Ritz quotient*, $R[\psi]$, and show that it attains its minimum value when $|\psi\rangle = |\psi_0\rangle$.

A particle of mass m moves in one dimension subject to the potential,

$$V(x) \quad = \quad \frac{\hbar^2}{2m}\left(x^6 - 3x^2 + 2\right).$$

Show that the system has an energy eigenstate with (unnormalised) wavefunction,

$$\tilde{\psi}(x) := \exp(-\beta x^n),$$

for a value of β, a positive integer value of n and an energy each of which you should determine.

Estimate the groundstate energy of this system using the variational principle with a Gaussian trial wavefunction of the form

$$\psi_\alpha(x) \quad := \quad \exp\left(-\frac{\alpha}{2}x^2\right)$$

with parameter $\alpha > 0$. Show that the best estimate of the ground-state energy is obtained for the unique value

$$\alpha = \alpha_* = \sqrt{\frac{(p + \sqrt{q})}{2}}\,,$$

where p and q are integers that you should determine. Give the corresponding approximate ground-state energy, E_0^*, in terms of α_*. You should not attempt to evaluate this function numerically. [*Hint: You may use without proof the following definite integral,*

$$\int_{-\infty}^{\infty} x^{2n} \exp\left(-\alpha x^2\right)\, dx \quad = \quad \frac{(2n)!}{n!(4\alpha)^n}\sqrt{\frac{\pi}{\alpha}}. \quad \Big]$$

Is your result consistent with the hypothesis that the exact eigenstate $\tilde{\psi}(x)$ found above is the true groundstate? Explain your reasoning carefully.

Paper 3, Section II

34D Applications of Quantum Mechanics

Let Λ be a Bravais lattice in three dimensions with primitive vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. Define the *reciprocal lattice* Λ^* and show that it is a Bravais lattice.

An incident particle of mass m and wavevector $\mathbf{k}$ scatters off a crystal which consists of identical atoms located at the vertices of a finite subset $\mathcal{S}$ of the lattice Λ,

$$\mathcal{S} \quad = \quad \{\mathbf{l} = l_1\mathbf{a}_1 + l_2\mathbf{a}_2 + l_3\mathbf{a}_3 : l_i \in \mathbb{Z},\ -L_i/2 \leqslant l_i \leqslant +L_i/2 \ \text{ for } \ i = 1, 2, 3\},$$

where L_1, L_2 and L_3 are positive even integers. After scattering the particle has wavevector $\mathbf{k}'$ with $|\mathbf{k}| = |\mathbf{k}'| = k$ and the scattering angle θ, with $0 \leqslant \theta \leqslant \pi$, is defined by $\mathbf{k} \cdot \mathbf{k}' = k^2 \cos\theta$. Show that the resulting scattering amplitude is proportional to

$$\Delta\left(\mathbf{k} - \mathbf{k}'\right) \quad := \quad \sum_{\mathbf{l} \in \mathcal{S}} \exp\left(i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{l}\right) .$$

For L_1, L_2, $L_3 \gg 1$, show that this quantity is strongly peaked for wavevectors $\mathbf{k}$ and $\mathbf{k}'$ obeying $\mathbf{k} - \mathbf{k}' = \mathbf{q}$ for some $\mathbf{q} \in \Lambda^*$.

Consider the case where Λ is a body centered cubic lattice with primitive vectors

$$\mathbf{a}_1 = \frac{a}{2}\left(\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z\right), \qquad \mathbf{a}_2 = \frac{a}{2}\left(\mathbf{e}_x - \mathbf{e}_y + \mathbf{e}_z\right), \qquad \mathbf{a}_3 = a\,\mathbf{e}_z\,,$$

where $a > 0$ and $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ are, respectively, unit vectors in the x-, y- and z-directions. For scattering at fixed energy $E = \hbar^2 k^2/2m$ with $ka \gg 1$, find the smallest non-zero value of the scattering angle θ for which the scattering amplitude has a strong peak (i.e. a peak such as you found in the previous part of the question).

Paper 4, Section II

34D Applications of Quantum Mechanics

(a) A scalar particle of mass m and charge e is moving in three dimensions in a background electromagnetic field with vector potential $\mathbf{A}(\mathbf{x},t)$ and zero scalar potential. The Hamiltonian is given as

$$\hat{H} \;=\; \frac{1}{2m}\left(-i\hbar\nabla + e\mathbf{A}\right)\cdot\left(-i\hbar\nabla + e\mathbf{A}\right)\,.$$

Specialise to the case of a constant, homogeneous magnetic field $\mathbf{B} = \nabla\times\mathbf{A} = (0,0,B)$ in the z-direction. Suppose further that the x and y coordinates of the particle are constrained to lie in a rectangular region R of the x-y plane with sides of length R_x and R_y, and that the particle has vanishing momentum in the z-direction. By solving the Schrödinger equation in a suitable gauge with periodic boundary conditions in the x- and y-directions, find the energy levels of the system and give the degeneracy of each level. [You may use without proof any results about the spectrum of the quantum harmonic oscillator you may need, and you may assume that R_x and R_y are large compared to other length scales in the problem.]

(b) An electron is a particle of mass m, charge e and spin $1/2$. It is described by a two-component wave function $\vec{\Psi}\in\mathbb{C}^2$ with energy eigenstates obeying a matrix Schrödinger equation

$$\hat{\mathbb{H}}\,\vec{\Psi} \;=\; E\vec{\Psi}\,,$$

where

$$\hat{\mathbb{H}} \;=\; \hat{H}\,\mathbb{I}_2 \;+\; \frac{e\hbar}{2m}\,\mathbf{B}\cdot\boldsymbol{\sigma}\,,$$

where $\hat{H}$ is the Hamiltonian for the spinless particle given above, $\mathbb{I}_2$ is the (2×2)-unit matrix and $\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$ is a three-component vector whose entries are the Pauli matrices σ_i, for $i=1,2,3$.

Find the energy levels of a single electron in a constant, homogeneous magnetic field $\mathbf{B}=(0,0,B)$ under the same conditions as in part (a). Give the degeneracy of each energy level.

Now consider N non-interacting electrons occupying these energy levels. Find the ground-state energy $E_{\rm gs}$ of the system as a function of N, identifying any thresholds which occur. Sketch the graph of $E_{\rm gs}$ against N. [*Hint: Recall that electrons are identical fermionic particles obeying the Pauli exclusion principle.*]

Paper 1, Section II

28J Applied Probability

(a) Arrivals of the Number 1 bus form a Poisson process of rate 1 bus per hour, and arrivals of the Number 8 bus form an independent Poisson process of rate 8 buses per hour. What is the probability that exactly three Number 8 buses pass by while I am waiting for a Number 1?

(b) Let $(N_t, t \geqslant 0)$ be a Poisson process of constant intensity λ on $\mathbb{R}_+$. Conditional on the event $N_t = n$, show that the jump times $(J_1, J_2, \ldots, J_n)$ are distributed as the order statistics of n i.i.d. $U[0,t]$ random variables.

(c) As above, let $N = (N_t, t \geqslant 0)$ be a Poisson process with intensity $\lambda > 0$ and let $(X_i)_{i\geqslant 1}$ be a sequence of i.i.d. random variables, independent of N. Show that if $g(s,x) : \mathbb{R}^2 \mapsto \mathbb{R}$ is a measurable function and J_i are the jump times of N, then for any $\theta \in \mathbb{R}$,

$$\mathbb{E}\left[\exp\left\{\theta \sum_{i=1}^{N_t} g(J_i, X_i)\right\}\right] = \exp\left\{\lambda \int_0^t (\mathbb{E}(e^{\theta g(s,X_1)}) - 1) ds\right\}.$$

(d) Define the age process (the time since the last renewal) $A(t) = t - J_{N_t}$, where J_n is the n-th jump time of the Poisson process N. Show that

$$\mathbb{E}A(t) = (1 - e^{-\lambda t})/\lambda.$$

[You may use without proof that $\mathbb{E}U_{(i)} = \frac{i}{n+1}$, where $U_{(i)}$ is the i-th order statistic of a sample of n i.i.d. $U[0,1]$ random variables.]

Paper 2, Section II

28J Applied Probability

(a) Define a simple birth process with parameter λ starting with one individual. If X_t denotes the number of individuals at time t in a simple birth process (with $X_0 = 1$), find $\mu(t) := \mathbb{E}X_t$. [You may assume that $\mu(t)$ is a continuous function of t.]

(b) Let $\lambda \in [1,2]$ and consider the continuous-time Markov chain on $\mathbb{N} = \{0,1,2,\ldots\}$ with rates

$$q_{i,i-1} = 2^i, \quad q_{ii} = -(\lambda+1)\cdot 2^i, \quad q_{i,i+1} = \lambda 2^i \qquad \text{for } i \geqslant 1,$$

and $q_{0,1} = \lambda, q_{0,0} = -\lambda$.

For what values of $\lambda \in [1,2]$ is X recurrent? For what values of $\lambda \in [1,2]$ does X have an invariant distribution? For what values of $\lambda \in [1,2]$ is X explosive? Justify your answers.

[You may assume the recurrence and transience properties of simple random walks on $\mathbb{N}$. You may also assume without proof that for a transient simple random walk on $\mathbb{N}$, $\sup_i \mathbb{E}_i V_i < \infty$, where V_i is the number of visits to the state i.]

(c) Let X be an irreducible continuous-time Markov chain with jump chain Y. Prove or provide a counterexample (with proper justification) to the following: if a state is positive recurrent for X, it is positive recurrent for Y.

[*You may quote any result from the lectures that you need, without proof, provided it is clearly stated.*]

Paper 3, Section II

27J Applied Probability

(a) Define $M/M/1$ and $M/M/\infty$ queues and state (without proofs) their stationary distributions, as well as all the necessary conditions for their existence. State Burke's theorem for an $M/M/\infty$ queue.

(b) Calls arrive at a telephone exchange as a Poisson process of constant rate λ, and the lengths of calls are independent exponential random variables of parameter μ. Assuming that infinitely many telephone lines are available, set up a Markov chain model for this process.

Show that for large t the distribution of the number of lines in use at time t is approximately Poisson with mean λ/μ.

Let X_t denote the number of lines in use at time t, given that n are in use at time 0. Find $\mathbb{E}s^{X_t}$ for any $s \in [-1,1]$. Hence or otherwise, identify the distribution of X_t.

[You may use without proof that the probability generating function of a Poisson(λ) random variable is $e^{\lambda(s-1)}$.]

(c) Compute the expected length of the busy period for an $M/M/1$ and an $M/M/\infty$ queue. (The busy period B is the length of time between the arrival of the first customer and the first time afterwards that all servers are free).

[*You may quote any result from the lectures that you need, without proof, provided it is clearly stated.*]

Paper 4, Section II

27J Applied Probability

(a) Let $d \geqslant 1$ and let $\lambda : \mathbb{R}^d \mapsto \mathbb{R}$ be a non-negative measurable function such that $\int_A \lambda(x)dx < \infty$ for all bounded Borel sets A. Define a *non-homogeneous spatial Poisson process* on $\mathbb{R}^d$ with *intensity function* λ.

(b) Assume that the positions $(x,y,z) \in \mathbb{R}^3$ of stars in space are distributed according to a Poisson process on $\mathbb{R}^3$ with constant intensity λ. Show that their distances from the origin $g(x,y,z) = \sqrt{x^2+y^2+z^2}$ form another (non-homogeneous) Poisson process on $\mathbb{R}_+$. Find its intensity function. Find the density function for the distribution of the distance of the closest star from the origin.

(c) An art gallery has ten rooms, and visitors are required to view them all in sequence. Visitors arrive at the instants of a non-homogeneous Poisson process on $\mathbb{R}_+$ with intensity function $\lambda(x)$. The ith visitor spends time $Z_{i,j}$ in the jth room, where the random variables $Z_{i,j}$ for $i \geqslant 1, 1 \leqslant j \leqslant 10$ are i.i.d. random variables, independent of the arrival process. Let $t \geqslant 0$ and let $V_j(t)$ be the number of visitors in room j at time t. Show for fixed t that $V_j(t)$ for $1 \leqslant j \leqslant 10$ are independent random variables. Find their distributions.

[*You may quote any result from the lectures that you need, without proof, provided it is clearly stated.*]

Paper 2, Section II

32E Asymptotic Methods

(a) Let $\phi_n(x) > 0$, for $n = 0, 1, 2, \ldots$, be a sequence of real functions defined on $\{x \in \mathbb{R} : 0 < |x - x_0| < a\}$ which is an asymptotic sequence as $x \to x_0$.

(i) Let $\psi_0(x) = \phi_0(x)$ and

$$\psi_n(x) = \frac{\phi_{n-1}(x)\phi_n(x)}{\phi_{n-1}(x) + \phi_n(x)}\,, \qquad n = 1, 2, 3, \ldots$$

Show that $(\psi_n(x))_{n=0}^{\infty}$ is an asymptotic sequence as $x \to x_0$.

Is it true that $\phi_n(x) \sim \psi_n(x)$ as $x \to x_0$ for every $n = 0, 1, 2, \ldots$? You should either give a proof or a counterexample.

(ii) Let $\chi_0(x) = \phi_0(x)$ and

$$\chi_n(x) = \sqrt{\phi_{n-1}(x)\phi_n(x)}\,, \qquad n = 1, 2, 3, \ldots$$

Show that $(\chi_n(x))_{n=0}^{\infty}$ is an asymptotic sequence as $x \to x_0$.

Is it true that $\phi_n(x) \sim \chi_n(x)$ as $x \to x_0$ for every $n = 0, 1, 2, \ldots$? You should either give a proof or a counterexample.

(b) Let $(\phi_n(x))_{n=0}^{\infty}$ and $(\psi_n(x))_{n=0}^{\infty}$ be two sequences of real functions defined on $\{x \in \mathbb{R} : 0 < |x - x_0| < a\}$ which are asymptotic sequences as $x \to x_0$. Suppose that

$$\phi_n(x) \sim \psi_n(x) \quad \text{as} \quad x \to x_0\,,$$

for $n = 0, 1, 2, \ldots$, and that for some sequence of real numbers $(a_n)_{n=0}^{\infty}$ we have

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x) \quad \text{as} \quad x \to x_0\,.$$

Does there necessarily exist a sequence of real numbers $(b_n)_{n=0}^{\infty}$ such that

$$f(x) \sim \sum_{n=0}^{\infty} b_n \psi_n(x) \quad \text{as} \quad x \to x_0\,?$$

You should either give a proof or a counterexample.

Paper 3, Section II

30E Asymptotic Methods

A stationary Schrödinger equation in one dimension has the form

$$\varepsilon^2 \frac{d^2\psi}{dx^2} = -(E - V(x))\psi\,, \quad \text{for} \quad x \in \mathbb{R}\,, \tag{$*$}$$

where $\varepsilon > 0$ is assumed to be very small and the potential $V(x)$ is given by

$$V(x) = \begin{cases} \frac{1}{4}|x| & \text{for} \quad |x| \leqslant 4 \\ \sqrt{|x|} - 1 & \text{for} \quad |x| \geqslant 4 \end{cases}.$$

The connection formula for the approximate energies E of bound states ψ in $(*)$ is

$$\frac{1}{\varepsilon}\int_a^b (E - V(x))^{1/2}\,dx = (n + \frac{1}{2})\pi\,. \tag{$**$}$$

(a) State the appropriate values of a, b and n.

(b) For $E \geqslant 0$ define

$$f(E) = \int_a^b (E - V(x))^{1/2}\,dx,$$

with a, b as in (a). Find and sketch f, and deduce that for each n and ε, $(**)$ has a unique solution $E = E_n$.

(c) Show that for n fixed and ε sufficiently small, E_n can be determined explicitly and give an expression for it.

(d) Show that as $n \to \infty$ with ε fixed, E_n satisfies

$$E_n \sim cn^\alpha\,,$$

and determine the values of c and α.

Paper 4, Section II

31E Asymptotic Methods

Justifying your steps carefully, use the method of steepest descent to find the first term in the asymptotic approximation of the function:

$$I(x) = \int_C \frac{1}{z^2 + 16}\, e^{x\cosh z}\,dz\,, \quad \text{as} \quad x \to \infty\,,$$

where $x \in \mathbb{R}$ and the integral is over the contour

$$C = \{z \in \mathbb{C} :\ z = p + iq\,, q = 2\arctan p\,,\ p \in \mathbb{R}\},$$

taken in the direction of increasing p.

Paper 1, Section I

4I Automata & Formal Languages

(a) Define what it means for a grammar to be in *Chomsky normal form.*

(b) Suppose G is a grammar in Chomsky normal form. If $w \in \mathcal{L}(G)$ has $|w| = n$, what is the length of a G-derivation of w? [No justification is required.]

(c) Let $\Sigma = \{a, b\}$, $V = \{S, A, B, C\}$. Consider the grammar $G = (\Sigma, V, P, S)$ in Chomsky normal form given by $P = \{S \to AC, C \to BA, A \to AB, B \to BA, A \to a, B \to b\}$. Show that the word *abbabba* is in $\mathcal{L}(G)$ by providing a G-parse tree for it.

A grammar G is said to be in *weak Chomsky normal form* if all production rules are either of the form $A \to a$, $A \to BC$, or $A \to BCD$, for variables A, B, C, D and letters a.

(d) Give a grammar G' in weak Chomsky normal form that is

(i) equivalent to the grammar G from part (c) and

(ii) there is a G'-derivation for the word *abbabba* of length strictly shorter than the number given in part (b).

Justify your answer.

Paper 2, Section I

4I Automata & Formal Languages

Let Σ be an alphabet and $\mathbb{W} := \Sigma^*$ be the set of words over Σ.

(a) Define what it means for $A \subseteq \mathbb{W}$ to be *computably enumerable.*

[You do not need to define what it means for a partial function to be computable.]

(b) Prove that for $\varnothing \neq A \subseteq \mathbb{W}$ the following statements are equivalent:

(i) the set A is computably enumerable;

(ii) the set A is the domain of a partial computable function;

(iii) the set A is the range of a partial computable function;

(iv) the set A is the range of a total computable function.

[*You may assume that the truncated computation function is computable, and that the map* $w \mapsto ((w)_0, (w)_1)$ *is a bijection from* $\mathbb{W}$ *to* $\mathbb{W}^2$ *that can be performed by a register machine.*]

Paper 3, Section I
4I Automata & Formal Languages

(a) Let $G = (\Sigma, V, P, S)$ be a formal grammar and let $\Omega = \Sigma \cup V$. Define $\mathcal{L}(G)$.

[You do not need to define the binary relation $\xrightarrow{G}$ on Ω^*.]

(b) Define what it means for two grammars to be *equivalent*.

(c) Define what it means for two grammars to be *isomorphic*.

(d) Fix $\Sigma = \{a, b, c\}$ and consider the following pairs of grammars with start symbol S and given by their respective sets of productions P_0 and P_1; for each pair, determine whether they are equivalent or non-equivalent. Justify your answers.

(i) $P_0 = \{S \to Aa,\ S \to Sb,\ A \to Ab,\ A \to a,\ B \to Aa,\ B \to b\}$,
$P_1 = \{S \to Sb,\ C \to Da,\ C \to b,\ D \to Db,\ D \to a,\ S \to Da\}$.

(ii) $P_0 = \{S \to AB,\ A \to Aa,\ A \to a,\ B \to Bb,\ B \to b,\ AB \to c\}$,
$P_1 = \{S \to XabY,\ X \to Xa,\ X \to a,\ Y \to Yb,\ Y \to b,\ XY \to c\}$.

(iii) $P_0 = \{S \to aAa,\ A \to bAb,\ A \to b\}$,
$P_1 = \{S \to aYa, Y \to ZZ, Z \to aZa, Z \to bZY, Z \to YZ, Y \to bYb, Y \to b\}$.

[You may assume that isomorphic grammars are equivalent.]

Paper 4, Section I
4I Automata & Formal Languages

(i) Define what it means for a grammar to be *regular*.

(ii) Let $G = (\Sigma, V, P, S)$ be a regular grammar and $\Omega = \Sigma \cup V$. Prove that if $\alpha \in \Omega^*$ and $S \xrightarrow{G} \alpha$, then there are $w \in \mathbb{W}$ and $A \in V$ such that $\alpha = wA$ or $\alpha = w$.

(iii) Let $G = (\Sigma, V, P, S)$ be a regular grammar, $A, B \in V$, and $w, v \in \mathbb{W}$. Prove that if $wA \xrightarrow{G} vB$, then there is some word $u \in \mathbb{W}$ such that $A \xrightarrow{G} uB$.

If $G = (\Sigma, V, P, S)$ is a regular grammar and A is a variable, we call A *accessible in* G if there is a word $w_1 \in \Sigma^*$ such that $S \xrightarrow{G} w_1 A$; we call A *looping in* G if there is a word $w_2 \in \Sigma^*$ such that $A \xrightarrow{G} w_2 A$; we call A *terminable in* G if there is a word $w_3 \in \Sigma^*$ such that $A \xrightarrow{G} w_3$.

(iv) Let G be a regular grammar. Prove that if $\mathcal{L}(G)$ is infinite then there is a variable that is accessible, looping, and terminable in G.

Paper 1, Section II

12I Automata & Formal Languages

Let Σ be an alphabet, $\mathbb{W}$ the set of words over Σ, $A, B \subseteq \mathbb{W}$, and $\mathcal{C}$ any set of subsets of $\mathbb{W}$.

(i) Define what $A \leqslant_{\mathrm{m}} B$ means.

(ii) Define what it means for a set A to be $\mathcal{C}$-*complete.*

(iii) Define what it means for A to be in Σ_1.

(iv) Define the *halting problem* $\mathbf{K}$.

(v) Prove that the halting problem $\mathbf{K}$ is Σ_1-complete.

A set $P \subseteq \mathbb{W}$ is in Π_2 if and only if there is a computable partial function $f : \mathbb{W} \times \mathbb{W} \dashrightarrow \mathbb{W}$ such that for all $w \in \mathbb{W}$, we have that $w \in P$ if and only if for all $v \in \mathbb{W}$, $f(w, v)\downarrow$.

(vi) We define $\mathbf{Tot} \subset \mathbb{W}$ to be the set $\{v\,;\, \mathrm{W}_v = \mathbb{W}\}$. Show that $\mathbf{Tot}$ is Π_2-complete.

[*In this entire question, you are allowed to use the fact that truncated computation functions are computable, provided that you give a precise and correct definition of the function used. You may use the partial function $f_{w,1}$ without providing a definition.*]

Paper 3, Section II

12I Automata & Formal Languages

Let Σ be an alphabet and $\mathbb{W}$ the set of words over Σ. Let $D = (\Sigma, Q, \delta, q_0, F)$ be a deterministic automaton.

(i) Define $\mathcal{L}(D)$, the set of words accepted by the automaton D, precisely defining all auxiliary functions needed for your definition.

(ii) State the *pumping lemma* for the language $\mathcal{L}(D)$. Specify the pumping number precisely in terms of D.

[No proof is required.]

(iii) Let $\Sigma = \{a, b\}$. Consider the regular language

$$L := \{wa^k \,;\, w \in \Sigma^* \text{ with } |w| \leqslant 10 \text{ and } k > 0\}.$$

Show that the minimal deterministic automaton for L has at least ten states.

Let $A \subseteq \mathbb{W}$. Define an equivalence relation on $\mathbb{W}$ by

$$v \sim_A w :\Longleftrightarrow \text{ for all } u, \text{ we have } vu \in A \text{ if and only if } wu \in A.$$

(iv) Let $A \subseteq \mathbb{W}\backslash\{\varepsilon\}$. Show that A is a regular language if and only if the relation $\sim_A$ has finitely many equivalence classes.

Paper 1, Section I

8D Classical Dynamics

The Lagrangian for a particle of charge q and mass m in an electromagnetic field is

$$L = \frac{1}{2} m \dot{\mathbf{r}}^2 - q(\phi - \dot{\mathbf{r}} \cdot \mathbf{A}) \, ,$$

where $\mathbf{A}(\mathbf{r}, t)$ is the vector potential and $\phi(\mathbf{r}, t)$ is the scalar potential associated with the electromagnetic field.

(a) Determine how L changes under the gauge transformation

$$\phi \mapsto \phi - \frac{\partial f}{\partial t} \, , \qquad \mathbf{A} \mapsto \mathbf{A} + \boldsymbol{\nabla} f \, ,$$

where $f(\mathbf{r}, t)$ is a smooth function. Why does this change in L not affect the Euler-Lagrange equations?

(b) Show that the Euler-Lagrange equations imply the Lorentz force law.

(c) Now suppose that the electric field vanishes and the magnetic field is constant and uniform. Show that the component of the particle's canonical momentum along the direction of the magnetic field is conserved.

Paper 2, Section I

8D Classical Dynamics

A rigid body rotates with angular velocity $\boldsymbol{\omega}(t)$ around its centre of mass. Define what is meant by the *fixed space frame* and the *principal body frame*.

Write down an expression for how the body axes change in time. Hence derive Euler's equations for the torque-free motion of a rigid body.

Consider an axisymmetric body with principal moments of inertia $I_1 = I_2 \neq I_3$. Show that Euler's equations imply the angular momentum $\mathbf{L}$, the angular velocity $\boldsymbol{\omega}$ and the body's symmetry axis are always coplanar.

Paper 3, Section I

8D Classical Dynamics

Consider a 3-dimensional system with phase space coordinates $(\mathbf{q}, \mathbf{p})$.

(a) Define the *Poisson bracket* $\{f, g\}$ of two smooth functions on phase space.

(b) Show that $f(\mathbf{q}, \mathbf{p})$ is conserved along a particle's trajectory if and only if $\{f(\mathbf{q}, \mathbf{p}), H\} = 0$, where H is the Hamiltonian.

(c) Derive a constraint satisfied by a function $f(\mathbf{q}, \mathbf{p})$ given that $\{f(\mathbf{q}, \mathbf{p}), \mathbf{q} \cdot \mathbf{p}\} = 0$. Show that any smooth function obeying $f(\lambda \mathbf{q}, \lambda^{-1} \mathbf{p}) = f(\mathbf{q}, \mathbf{p})$, where λ is a real constant, satisfies this constraint.

Paper 4, Section I

8D Classical Dynamics

What is meant by an *adiabatic invariant* of a mechanical system?

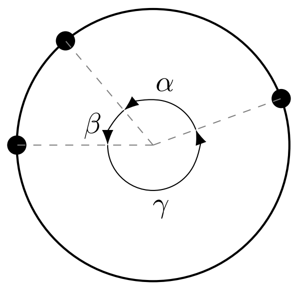
A particle of mass m and energy E moves between two fixed, parallel walls that are a distance L apart. The particle travels freely in a direction perpendicular to the walls except when it collides elastically with a wall at which point its velocity changes instantaneously. Compute the action $I = \oint p\, dq$ and verify that $T = dI/dE$ is the period of oscillation.

Suppose that the distance between the walls is varied very slowly so that $L(t)$ depends on time. How does the energy of the particle depend on time? Give a brief physical explanation for why the particle's energy changes.

Paper 2, Section II

14D Classical Dynamics

Three identical particles, each of mass m, are constrained to move around a fixed circle of radius r that lies in a horizontal plane. You may assume that the particles do not collide. The angles between the locations of the particles are α, β, γ as in the figure, which shows the view from above.



(a) Write down a constraint obeyed by α, β, γ. What degree of freedom is not described by these three angles?

(b) The particles feel the influence of a potential

$$V(\alpha, \beta, \gamma) = V_0(e^{-2\alpha} + e^{-2\beta} + e^{-2\gamma}),$$

where V_0 is a positive constant. Solving your constraint to find $\gamma = \gamma(\alpha, \beta)$, obtain a Lagrangian governing the dynamics of the particles' relative separations as a function of α, β, $\dot{\alpha}$ and $\dot{\beta}$.

(c) Find an equilibrium configuration of the system and show that it is stable. Find three linearly independent normal modes, together with their frequencies, that describe small perturbations about this equilibrium.

(d) The physical system is unchanged by permutations of (α, β, γ). Explain how this is consistent with your answer to part (c).

Paper 4, Section II

15D Classical Dynamics

What does it mean for a phase space coordinate transformation to be *canonical*? Consider a coordinate transformation $(q, p) \mapsto (Q, P)$ on the phase space of a system with one degree of freedom. Show that if this transformation is defined in terms of a generating function $F(q, P)$ via

$$Q = \left.\frac{\partial F}{\partial P}\right|_q \qquad \text{and} \qquad p = \left.\frac{\partial F}{\partial q}\right|_P$$

then it is canonical.

Find the phase space coordinate transformation associated to the generating function

$$F(q, P) = \int_0^q \sqrt{2P - u^2}\, du\,.$$

Obtain Hamilton's equations for Q and P in the case $H(q, p) = \frac{1}{2}(p^2 + q^2)$. Hence find $Q(t)$ and $P(t)$ and check that these agree with the usual solution for a simple harmonic oscillator.

A particle of energy E has Hamiltonian $H(q, p) = \frac{1}{2}(p^2 + q^2) + \epsilon q^4$, where $2q^2\epsilon \ll 1$ for all q in the range $-\sqrt{2E} \leqslant q \leqslant \sqrt{2E}$. By choosing an appropriately modified generating function $F_\epsilon(q, P)$, show that

$$\frac{q(t)}{p(t)} = \tan(t - t_0) - \epsilon\, I(q_0(t), E)\,(1 + \tan^2(t - t_0)) + \epsilon q_0^2(t) \tan^3(t - t_0) + \mathcal{O}(\epsilon^2),$$

where $q_0(t) = \sqrt{2E}\sin(t - t_0)$ and $I(x, y)$ is defined by

$$I(x, y) = \int_0^x \frac{u^4}{(2y - u^2)^{3/2}}\, du\,.$$

Paper 1, Section I

3I Coding and Cryptography

(a) Let $\mathcal{A}$ be an alphabet of (finite) cardinality m. What does it mean to say that a code $c : \mathcal{A} \to \{0,1\}^*$ is (i) *prefix-free* or (ii) *optimal*?

Suppose that letters $\mu_1, \ldots, \mu_m$ are sent with probabilities $p_1, \ldots, p_m$. Let c be an optimal prefix-free binary code with word lengths $\ell_1, \ldots, \ell_m$.

Show that if $p_i > p_j$ then $\ell_i \leqslant \ell_j$. Show also that among the codewords of maximal length there must exist two that differ only in the last digit.

(b) Letters $\mu_1, \ldots, \mu_5$ are transmitted with probabilities $0.4, 0.2, 0.2, 0.1, 0.1$. Determine whether there are optimal binary codings with either (i) all but one codeword of the same length, or (ii) each codeword a different length. Justify your answers.

Paper 2, Section I

3I Coding and Cryptography

(a) (i) Consider a source $(X_n)_{n \geqslant 1}$ of random variables taking values in some finite alphabet $\mathcal{A}$. What does it mean for a source to be *Bernouilli*? What does it mean for a source to be *reliably encodable at rate r*? What is the *information rate* of a source?

(ii) Show that the information rate of a Bernouilli source $(X_n)_{n \geqslant 1}$ is at most the expected word length of an optimal code $c : \mathcal{A} \to \{0,1\}^*$ for X_1.

(b) Let $(X_n)_{n \geqslant 1}$ be a source with letters drawn from a finite alphabet $\mathcal{A}$. This source is not necessarily assumed to be Bernouilli. Let $N \geqslant 1$ be an integer and let $Y_i = (X_{(i-1)N+1}, X_{(i-1)N+2}, \ldots, X_{iN})$. Show that the information rate of the source $(Y_n)_{n \geqslant 1}$ is N times that for $(X_n)_{n \geqslant 1}$.

Paper 3, Section I

3I Coding and Cryptography

Let C be a binary linear $[n, m, d]$-code.

Define (i) the *parity check extension* C^+ of C and (ii) the *punctured code* C^- (assuming $n \geqslant 2$). Show that C^+ and C^- are both linear.

What is the *shortening* C' of C (assuming $n \geqslant 2$)? When is C' a linear code?

For the changes to C defined in (i) and (ii), describe the effect of both these changes on the generator and parity check matrices. For the case of (ii) you may assume that $d \geqslant 2$ and you puncture in the last place.

Paper 4, Section I

3I Coding and Cryptography

(a) If $C \subseteq \mathbb{F}_2^n$ is a linear code, define the *dual code* $C^{\perp}$ and explain why it is also linear. If C is cyclic, show directly that $C^{\perp}$ is cyclic. Explain briefly how the generator polynomials of C and $C^{\perp}$ are related.

(b) Factorise $X^7 - 1$ over the field $\mathbb{F}_2$ and hence list all the binary cyclic codes of length 7. Identify versions of Hamming's original code and its dual in your list. What are the other cyclic codes of length 7? You should relate them to codes defined explicitly in the course.

Paper 1, Section II

11I Coding and Cryptography

(a) What is a *binary symmetric channel* (BSC) with error probability p? Write down its channel matrix. Why can we assume that $p < \frac{1}{2}$? State Shannon's second coding theorem and use it to compute the capacity of this channel.

(b) Codewords 00 and 11 are sent with equal probability through a BSC with error probability p. Compute the mutual information between the codeword sent and the first digit received as output. Show that the extra mutual information gained on receipt of the second digit is $H(2p(1-p)) - H(p)$ bits. [Here $H(p)$ denotes the entropy of a random variable which takes the value 1 with probability p and 0 with probability $1-p$.]

(c) Consider a ternary alphabet and a channel that has channel matrix

$$\begin{pmatrix} 1-2\alpha & \alpha & \alpha \\ \alpha & 1-2\alpha & \alpha \\ \alpha & \alpha & 1-2\alpha \end{pmatrix}.$$

Calculate the capacity of the channel.

Paper 2, Section II

12I Coding and Cryptography

(a) What is a *one-time pad*?

Suppose that X and Y are independent random variables taking values in $\mathbb{Z}_n$, the integers modulo n. Using Gibbs' inequality, or otherwise, show that

$$H(X+Y) \geqslant \max\{H(X), H(Y)\}.$$

Why is this result of interest in the context of one-time pads? Does this result remain true if X and Y are not independent? Give reasons for your answer.

(b) The notorious spymaster Stan uses a one-time pad to communicate with the even more notorious spy Ollie. The messages are coded in the obvious way, namely, if the pad has C, the third letter of the alphabet and the message has I, the ninth, then the encrypted message has L as the $(3+9)$th. We will work modulo 26. Unknown to Stan and Ollie, the person whom they employ to carry the messages is actually the police agent Eve in disguise. The police are close to arresting Ollie when Eve is given the message

$$LRPFOJQLCUD.$$

Eve knows that the actual message is

$$FLYXATXONCE,$$

and wants to change things so that Ollie deciphers the message as

$$REMAINXHERE.$$

What message should Eve deliver?

(c) Let K be the field with 2^d elements. Recall that the multiplicative group $K^\times$ is a cyclic group; let α be a generator. Let $T : K \to \mathbb{F}_2$ be any non-zero $\mathbb{F}_2$-linear map. You are given that the $\mathbb{F}_2$-bilinear form $K \times K \to \mathbb{F}_2$ such that $(x, y) \mapsto T(xy)$ is non-degenerate (i.e. $T(xy) = 0$ for all $y \in K$ implies $x = 0$).

(i) Show that the sequence $x_n = T(\alpha^n)$ is the output from a linear feedback shift register of length at most d.

(ii) The *period* of $(x_n)_{n \geqslant 0}$ is the least integer $r \geqslant 1$ such that $x_{n+r} = x_n$ for all sufficiently large n. Show that the sequence in (i) has period $2^d - 1$.

Paper 1, Section I

9B Cosmology

(a) A homogeneous and isotropic fluid has an energy density $\rho(t)$ and pressure $P(t)$. Use the relation $\mathrm{d}E = -P\,\mathrm{d}V$ for the energy E to derive the continuity equation in a universe with scale factor $a(t)$,

$$\dot{\rho} + 3\,\frac{\dot{a}}{a}\,(\rho + P) = 0\,,$$

where the overdot indicates differentiation with respect to time t. *[Hint: recall that the physical volume $V(t) = a(t)^3 V_0$, where V_0 is the co-moving volume.]*

(b) Given that the scale factor $a(t)$ satisfies the Raychaudhuri equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho + 3P)\,,$$

where G is Newton's constant and c is the speed of light, show that the quantity

$$Q = \frac{8\pi G}{3c^2}\rho\, a^2 - \dot{a}^2\,,$$

is time independent.

(c) The pressure P is related to ρ by the equation of state

$$P = \omega\,\rho, \quad |\omega| < 1\,.$$

Given that $a(t_0) = 1$, find $a(t)$ for an *expanding* universe with $Q = 0$, and hence show that $a(t_\star) = 0$ for some $t_\star < t_0$.

Paper 2, Section I

9B Cosmology

The number density n of photons in thermal equilibrium at temperature T takes the form

$$n = \frac{8\pi}{c^3}\int_0^\infty \frac{\nu^2\mathrm{d}\nu}{\exp(h\nu/k_B T) - 1}\,, \qquad (\star)$$

where h is Planck's constant, k_B is the Boltzmann constant and c is the speed of light.

Using $(\star)$, show that the photon number density n and energy density ρ can be expressed in the form

$$n = \alpha\, T^3 \qquad \text{and} \qquad \rho = \xi\, T^4\,,$$

where the constants α and ξ need not be evaluated explicitly.

At time $t = t_{\mathrm{dec}}$ and temperature $T = T_{\mathrm{dec}}$, photons decouple from thermal equilibrium. By considering how the photon frequency redshifts as a flat universe expands, show that the form of the equilibrium frequency distribution is preserved if the temperature for $t > t_{\mathrm{dec}}$ is defined by

$$T = \frac{a(t_{\mathrm{dec}})}{a(t)}T_{\mathrm{dec}}\,.$$

Paper 3, Section I

9B Cosmology

The equilibrium number density of fermions of mass m at temperature T and chemical potential μ is

$$n = \frac{4\pi g_s}{h^3}\int_0^\infty \frac{p^2 \mathrm{d}p}{\exp\left[\frac{E(p)-\mu}{k_B T}\right]+1},$$

where g_s is the degeneracy factor, $E(p) = c\sqrt{p^2+m^2c^2}$, c is the speed of light, k_B is the Boltzmann constant, p is the magnitude of the particle momentum and h is Planck's constant. For a non-relativistic gas with $pc \ll mc^2$ and $k_BT \ll mc^2 - \mu$, show that the number density becomes

$$n = g_s\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}\exp\left[\frac{\mu - m\,c^2}{k_B T}\right]. \qquad (\star)$$

[You may assume that $\int_0^\infty \mathrm{d}x\, x^2\, e^{-x^2/\alpha} = \sqrt{\pi}\alpha^{3/2}/4$ for $\alpha > 0$.]

Before recombination, equilibrium is maintained between neutral hydrogen, free electrons, protons and photons through the interaction

$$p + e^- \leftrightarrow H + \gamma\,.$$

Using the non-relativistic number density $(\star)$, deduce Saha's equation relating the electron and hydrogen number densities,

$$\frac{n_e^2}{n_H} \approx \left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2}\exp\left[-\frac{E_{\mathrm{bind}}}{k_B T}\right],$$

where $E_{\mathrm{bind}} = (m_p + m_e - m_H)c^2$ is the hydrogen binding energy. State clearly any assumptions made.

Paper 4, Section I

9B Cosmology

What is the *flatness problem*? By using the Friedmann and continuity equations, show that a period of accelerated expansion of the scale factor $a(t)$ in the early stages of the universe can solve the flatness problem if $\rho + 3P < 0$, where ρ is the energy density and P is the pressure. *[Hint: it may be useful to compute* $\mathrm{d}(\rho a^2)/\mathrm{d}t$.*]*

In the very early universe one can neglect the spatial curvature and the cosmological constant. Suppose that in addition there is a homogenous scalar field ϕ with potential

$$V(\phi) = m^2\phi^2\,,$$

and the Friedmann equation is

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,,$$

where $H = \dot{a}/a$ is the Hubble parameter. The field ϕ obeys the evolution equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0\,.$$

During inflation, ϕ evolves slowly after starting from a large initial value ϕ_i at $t = 0$. State what is meant by the *slow-roll approximation*. Show that in this approximation

$$\begin{aligned}\phi(t) &= \phi_i - \frac{2}{\sqrt{3}}mt \\ a(t) &= a_i \exp\left[\frac{m\phi_i}{\sqrt{3}}t - \frac{1}{3}m^2t^2\right] = a_i \exp\left[\frac{\phi_i^2 - \phi(t)^2}{4}\right]\,,\end{aligned}$$

where a_i is the initial value of a.

Paper 1, Section II

15B Cosmology

In a homogeneous and isotropic universe, the scale factor $a(t)$ obeys the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K\,c^2}{a^2} = \frac{8\,\pi\,G}{3c^2}\rho\,,$$

where K is a constant curvature parameter and ρ is the energy density which, together with the pressure P, satisfies the continuity equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0\,. \qquad (\star)$$

(a) Use the equations to show that the rate of change of the Hubble parameter $H = \dot{a}/a$ satisfies

$$\dot{H} + H^2 = -\frac{4\pi G}{3c^2}(\rho + 3P)\,.$$

(b) Suppose that an *expanding* universe is filled with radiation (with energy density ρ_r and pressure $P_r = \rho_r/3$) as well as a cosmological constant component (with density ρ_Λ and pressure $P_\Lambda = -\rho_\Lambda$). Both radiation and cosmological constant components satisfy the continuity equation $(\star)$ separately.

Given that the energy densities of these two components are measured today ($t = t_0$) to be

$$\rho_{r0} = \beta\frac{3c^2 H_0^2}{8\pi G} \quad \text{and} \quad \rho_{\Lambda 0} = \frac{3c^2 H_0^2}{8\pi G} \quad \text{with constant} \quad \beta > 0 \quad \text{and} \quad a(t_0) = 1\,,$$

show that the curvature parameter must satisfy $Kc^2 = \beta\,H_0^2$. Hence, derive the following relations for the Hubble parameter H and its time derivative:

$$H^2 = \frac{H_0^2}{a^4}(\beta - \beta\,a^2 + a^4)\,,$$
$$\dot{H} = -\beta\frac{H_0^2}{a^4}(2 - a^2)\,.$$

(c) Show qualitatively that universes with $a(0) = 0$ and $\beta > 4$ will recollapse to a Big Crunch in the future. *[Hint: you may find it useful to sketch $a^4 H^2$ and $a^4 \dot{H}$ versus a^2 for representative values of β.]*

(d) For $\beta = 4$, find an explicit solution for the scale factor $a(t)$ satisfying $a(0) = 0$. Find the limiting behaviours of this solution for large and small t.

Paper 3, Section II

14B Cosmology

Small density perturbations $\delta_{\mathbf{k}}(t)$ in pressureless matter inside the cosmological horizon obey the following Fourier evolution equation

$$\ddot{\delta}_{\mathbf{k}} + 2\frac{\dot{a}}{a}\dot{\delta}_{\mathbf{k}} - \frac{4\pi G\bar{\rho}_c}{c^2}\delta_{\mathbf{k}} = 0\,,$$

where the overdot indicates differentiation with respect to time t, $a(t)$ the scale factor of the universe, G is Newton's constant, c the speed of light, $\mathbf{k}$ is the co-moving wavevector and $\bar{\rho}_c$ is the background density of the pressureless gravitating matter.

(a) Let $t_{\rm eq}$ be the time of matter-radiation equality. Show that during the matter-dominated epoch, $\delta_{\mathbf{k}}$ behaves as

$$\delta_{\mathbf{k}}(t) = A(\mathbf{k})\left(\frac{t}{t_{\rm eq}}\right)^{2/3} + B(\mathbf{k})\left(\frac{t}{t_{\rm eq}}\right)^{-1}\,,$$

where $A(\mathbf{k})$ and $B(\mathbf{k})$ are functions of $\mathbf{k}$ only.

(b) For a given wavenumber $k \equiv |\mathbf{k}|$, show that the time t_H at which this mode crosses inside the horizon, *i.e.* $c\,t_H \approx 2\pi a(t_H)/k$, is given by

$$\frac{t_H}{t_0} \approx \begin{cases} \left(\frac{k_0}{k}\right)^3\,, & t_H > t_{eq} \\ \dfrac{1}{\sqrt{1+z_{\rm eq}}}\left(\dfrac{k_0}{k}\right)^2\,, & t_H < t_{eq} \end{cases}$$

where t_0 is the age of this universe, $k_0 \equiv 2\pi/(c\,t_0)$, and the matter-radiation equality redshift is given by $1 + z_{\rm eq} = (t_0/t_{\rm eq})^{2/3}$.

(c) Assume that early in the radiation era there is no significant perturbation growth in $\delta_{\mathbf{k}}$ and that primordial perturbations from inflation are scale-invariant with a constant amplitude at the time of horizon crossing given by $\langle \delta_{\mathbf{k}}(t_H)^2 \rangle \approx V^{-1}C/k^3$, where C is a constant and V is a volume. Use the results in parts (a) and (b) to project these perturbations forward to $t_0 \gg t_H$, and show that the power spectrum of perturbations today (at $t = t_0$) is given by

$$P(k) \equiv V\langle \delta_{\mathbf{k}}(t_0)^2 \rangle = \begin{cases} \frac{Ck}{k_0^4}\,, & k < k_{eq} \\ \frac{Ck_{\rm eq}}{k_0^4}\left(\frac{k_{\rm eq}}{k}\right)^3\,, & k > k_{\rm eq} \end{cases}$$

where $k_{\rm eq}$ is the wavenumber of modes that entered the horizon at matter-radiation equality.

Paper 1, Section II
26G Differential Geometry

(a) Let X, Y be smooth manifolds and $f : X \to Y$ be a smooth map. What does it mean for y_0 to be a *regular value* of f? Give an example of a smooth map f that has a regular value, together with a regular value of f, justifying your answer. State Sard's theorem.

(b) Let X and Y be compact manifolds of dimension n and $f : X \to Y$ be a smooth map. Define the degree mod 2 of f, quoting carefully (but without proof) the results from the course necessary to make this well defined.

(c) Let $S \subset \mathbb{R}^3$ be a surface and $p \in S$. Define the exponential map $\exp_p$, explaining carefully its domain. Explain also briefly why the exponential map is smooth. Give an explicit example where the domain of $\exp_p$ is not T_pS, and an example where $\exp_p$ is not surjective, justifying carefully your answers.

(d) Let $S \subset \mathbb{R}^3$ be a compact surface, and let V be a smooth vector field on S. Consider the map $\phi : S \to S$ defined by $\phi(p) = \exp_p(V(p))$. Explain why this map is well-defined and smooth. What is its degree mod 2?

Paper 2, Section II
26G Differential Geometry

(a) For regular curves in $\mathbb{R}^3$, parametrised by arc length s, define *curvature* k and *torsion* τ and derive the Frénet formulas. Indicate carefully all additional assumptions for these to be well defined.

(b) Suppose two regular curves in $\mathbb{R}^3$ both have curvature identically zero and the same arc length. Are they related by a proper Euclidean motion? Justify your answer. Does the answer change if we replace curvature identically zero with curvature identically one?

(c) We say that a quantity $Q(\gamma, s)$, defined for all regular curves γ parametrised by arc length s, is a *pointwise Euclidean invariant of curves* if

$$Q(\gamma, s) = Q(E \circ \gamma, s) \text{ for all proper Euclidean motions } E, \text{and}$$

$$Q(\gamma_{s_0}, s) = Q(\gamma, s - s_0) \text{ for all } s_0 \in \mathbb{R}, \text{ where } \gamma_{s_0}(s) := \gamma(s - s_0).$$

Show that $Q(\gamma, s) := k(s)$ and $Q(\gamma, s) := \tau(s)$, where k and τ refer to the curvature and torsion of the curve γ respectively, are both examples of such pointwise Euclidean invariants of curves.

(d) One can trivially construct other such pointwise Euclidean invariants by applying functions of curvature and torsion, e.g. defining $Q(\gamma, s) := k^2(s)$ or $Q(\gamma, s) := k(s) + \tau(s)$. Are these the only examples, i.e. is it true that if $Q(\gamma, s)$ is any pointwise Euclidean invariant, then $Q(\gamma, s) = f(k(s), \tau(s))$ for some function f (independent of the curve γ)? Justify your answer.

Paper 3, Section II

25G Differential Geometry

(a) Let $S \subset \mathbb{R}^3$ be an oriented surface. Define the Gaussian curvature $K(p)$ and mean curvature $H(p)$ of S at p. Prove that these are Euclidean invariants, i.e. if $E : \mathbb{R}^3 \to \mathbb{R}^3$ is a proper Euclidean motion and $\widetilde{S} = E(S)$ and $\widetilde{K}$, $\widetilde{H}$ denote the Gaussian and mean curvature of $\widetilde{S}$ (with a choice of orientation that you should describe), respectively, then $\widetilde{K}(E(p)) = K(p)$, $\widetilde{H}(E(p)) = H(p)$. Do the Gaussian and mean curvatures depend on the orientation?

(b) Show that there is no Euclidean motion taking a piece of the cylinder to a piece of the plane, and infer that for a general surface S, the property $K = 0$ identically does not imply that there is a Euclidean motion taking S to a piece of the plane. Exhibit similarly two surfaces each with $K = 1$ identically, no respective pieces of which are related by a Euclidean motion, and similarly two surfaces each with $K = -1$ identically.

(c) Let $\mathcal{R} \subset \mathbb{R}^3$ be a compact submanifold of dimension 3 with connected boundary $S = \partial\mathcal{R}$. Note that $S \subset \mathbb{R}^3$ is an orientable surface and can be oriented by the unique normal vector N pointing towards $\mathcal{R}$. Now let $\widetilde{S} \subset \mathcal{R}$ be a surface (without boundary). Suppose $p \in S \cap \widetilde{S}$. Show that $\widetilde{H}(p) \geqslant H(p)$, where H and $\widetilde{H}$ denote the mean curvature of S and $\widetilde{S}$, respectively, where both surfaces are (locally) oriented at p by the N described above. Is it necessarily the case that $\widetilde{K}(p) \geqslant K(p)$? Justify your answer.

Paper 4, Section II

25G Differential Geometry

(a) Given a compact orientable surface with smooth boundary, define the *area element* dA, *Euler characteristic* χ, and *geodesic curvature* k_g of the boundary, explaining briefly why the first two are well defined. State the Gauss–Bonnet theorem for the surface. [You need not consider the case of corners.]

(b) Let S be a compact orientable surface without boundary, and let $\gamma : I \to S$ be a smooth closed curve on S, parametrised by arc length, which separates S into two surfaces with boundary, S_1 and S_2, such that S is the union $S = S_1 \cup S_2$ where $\partial S_1 = \partial S_2 = S_1 \cap S_2 = \gamma(I)$. Suppose there exists an isometry $\phi : S_1 \to S_2$, and moreover, for each $x, y \in \gamma(I)$, an isometry $\phi_{x,y} : S \to S$ such that $\phi_{x,y}(\gamma(I)) = \gamma(I)$ and such that $\phi_{x,y}(x) = y$. Show that γ is a geodesic.

(c) In the above problem, suppose we drop the assumption of the existence of the isometry ϕ. Is γ still necessarily a geodesic?

(d) Alternatively, suppose we drop the assumption of the existence of the isometries $\phi_{x,y}$. Is γ still necessarily a geodesic?

Paper 1, Section II

32A Dynamical Systems

(a) State and prove Dulac's theorem. State the Poincaré-Bendixson theorem.

(b) Consider the system

$$\dot{x} = r - x(1+s) + x^2 y \quad (1)$$
$$\dot{y} = sx - x^2 y, \quad (2)$$

where r and s are positive numbers. Show that there is a unique fixed point. Show that for a suitable choice of α to be determined, with $0 < \alpha < r$, trajectories enter the closed region bounded by $x = \alpha$, $y = s/\alpha$, $x + y = r + s/\alpha$ and $y = 0$. Deduce that when $s - 1 > r^2$ the system has a periodic orbit.

Paper 2, Section II

33A Dynamical Systems

(a) Define a Lyapunov function for a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ on $\mathbb{R}^n$ with a fixed point $\mathbf{x}^*$. Explain what it means for a fixed point of the flow to be Lyapunov stable. State and prove Lyapunov's first stability theorem.

(b) Consider the second order differential equation

$$\ddot{x} = F(x) - \mu \dot{x},$$

where $\mu > 0$ and $F(x) = -2x(1-x^2)^2$. Show that there are three fixed points in the $(x, \dot{x})$ plane. Show that one of these is the origin and that it is Lyapunov stable. Show further that the origin is asymptotically stable, and that the ω-limit set of each point in the phase space is one of the three fixed points, justifying your answer carefully.

Paper 3, Section II

31A Dynamical Systems

Consider the dependence of the system

$$\dot{x} = (a^2 - x)(a - y^2) \tag{1}$$

$$\dot{y} = x - y \tag{2}$$

on the parameter a. Find the fixed points and plot their location on the (a, x)-plane. Hence, or deduce, that there are bifurcations at $a = 0$ and $a = a^* > 0$ which is to be determined.

Investigate the bifurcation at $a = 0$ by making the substitutions $X = x - a^2$ and $Y = y - a^2$. Find the extended centre manifold in the form $Y(X, a)$ correct to second order. Find the evolution on the extended centre manifold and hence determine the stability of the fixed points.

Use a plot to show which branches of the fixed points in the (a, x)-plane are stable and which are unstable and state, without calculation, the type of bifurcation at a^*. Hence sketch the structure of the (x, y) phase plane close to the bifurcation at a^* where $|a - a^*| \ll 1$ in the cases i) $a < a^*$ and ii) $a > a^*$.

Paper 4, Section II

32A Dynamical Systems

For the map $x_{n+1} = F(x_n, \lambda) := \lambda x_n(1 - x_n^2)$ with $\lambda > 0$ and $x_n \in [0, 1]$, show the following:

(i) There is an upper limit on λ if points are not to be mapped outside the domain $[0, 1]$. Find this value.

(ii) For $\lambda < 1$ the origin is the only fixed point and is stable.

(iii) If $\lambda > 1$, then the origin is unstable and a new fixed point x^* exists. This new fixed point x^* is stable for $1 < \lambda < 2$ and unstable for $\lambda > 2$.

(iv) For λ close to but larger than 2, and with $X_n = x_n - x^*$ and $0 < \mu = \lambda - 2 \ll 1$, the map can be locally represented as

$$X_{n+1} = -X_n + \alpha\mu X_n + \beta X_n^2 + \gamma X_n^3 + O(\mu^2), \tag{$*$}$$

where α, β and γ are constants that you should evaluate in terms of appropriate derivatives of F. Hence show that the 2-cycle born in the bifurcation at $\lambda = 2$ has points

$$x_\pm = x^* \pm \sqrt{\frac{-\alpha\mu}{\gamma + \beta^2}}.$$

[You do not need to substitute the expressions you found for α, β and γ into this formula.]

(v) The 2-cycle is stable for $\lambda > 2$, with $\lambda - 2$ small.

Paper 1, Section II

37A Electrodynamics

Consider spacetime with coordinates $x^\mu = (ct, \mathbf{x})$ and metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$, where $\mu, \nu = 0,1,2,3$ and c is the speed of light. An electromagnetic field described by the vector potential $A_\mu(x)$ fills spacetime, and a particle of mass m and charge q moves through it along the worldline $x^\mu(\lambda)$, where λ is a parameter along the worldline.

(a) Explain using the requirements of Lorentz invariance and gauge invariance why the action

$$S = -mc \int (-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu)^{\frac{1}{2}}\, d\lambda + q \int A_\mu(x)\dot{x}^\mu d\lambda$$

is suitable for describing the relativistic mechanics of the particle, where $\dot{x}^\mu = dx^\mu/d\lambda$.

(b) By varying the action with respect to a worldline with fixed end points, obtain the Euler-Lagrange equations of motion

$$m\frac{du^\mu}{d\tau} = qF^\mu{}_\nu u^\nu$$

where $u^\mu(\tau) = dx^\mu/d\tau$ is the four-velocity, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor and τ is the proper time.

(c) Show that the rate of change of the particle energy $\epsilon = \gamma mc^2$ satisfies

$$\frac{d\epsilon}{dt} = q\mathbf{E}\cdot\mathbf{v}$$

where $\mathbf{v} = d\mathbf{x}/dt$ is the particle velocity, $\mathbf{E}$ is the electric field and $\gamma = 1/\sqrt{1 - v^2/c^2}$.

(d) Hence, or otherwise, derive the following expression for the acceleration of the particle

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m\gamma}[\mathbf{E} + \mathbf{v}\times\mathbf{B} - \frac{1}{c^2}\mathbf{v}(\mathbf{v}\cdot\mathbf{E})],$$

where $\mathbf{B}$ is the magnetic field. Derive the non-relativistic limit of the above expression and comment on its relationship with the Lorentz force law.

Paper 3, Section II

36A Electrodynamics

The retarded four-potential $A^\mu(\mathbf{x},t) = (\phi/c, \mathbf{A})$ due to a charge density $J^\mu(\mathbf{x}',t')$ is

$$A^\mu(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int \frac{J^\mu(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|} d^3x',$$

where the integral is over all space.

(a) Explain briefly the physical meaning of the above expression and why causality requires $t' = t_{\text{ret}}$, where $t_{\text{ret}} = t - |\mathbf{x}-\mathbf{x}'|/c$.

(b) Consider a particle of charge q moving along the worldline $y^\mu = (ct, \mathbf{y}(t))$ and let $\mathbf{R}(t) = \mathbf{x} - \mathbf{y}(t)$ be the vector from the location of the charge at time t to the field point $\mathbf{x}$. Explain why the implicit equation

$$t_{\text{ret}} + \frac{R(t_{\text{ret}})}{c} = t,$$

determining the retarded potential, can have only one solution.

(c) Hence, or otherwise, obtain the Lienard-Wiechert potentials

$$\phi(\mathbf{x},t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{R - \frac{\mathbf{v}}{c}\cdot\mathbf{R}} \quad \text{and} \quad \mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\frac{q\mathbf{v}}{R - \frac{\mathbf{v}}{c}\cdot\mathbf{R}}$$

for the charge, where $\mathbf{v} = d\mathbf{y}/dt$ is the particle velocity. Clearly specify the time at which the right hand sides are to be evaluated.

(d) For a charge moving without acceleration, show by explicit computation that the resulting potentials satisfy the gauge-fixing condition

$$\frac{1}{c^2}\frac{\partial\phi}{\partial t} + \nabla\cdot\mathbf{A} = 0.$$

Paper 4, Section II

36A Electrodynamics

Consider a dielectric medium whose electromagnetic properties are described by the electric displacement $\mathbf{D}$, the magnetisation $\mathbf{H}$, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$.

(a) Write down the Maxwell equations for these four fields in the presence of a free charge density ρ and a free current density $\mathbf{J}$.

(b) Hence establish the identity

$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{E} \cdot \mathbf{J}.$$

(c) Consider a linear dielectric medium with the constitutive relations

$$D_i = \varepsilon_{ij} E_j, \quad B_i = \mu_{ij} H_j$$

where ε_{ij} and μ_{ij} are symmetric matrices, independent of t, representing the anisotropic dielectric response of the material, and the summation convention applies here and below. For a volume V enclosed by the surface S, derive the integral relation

$$\frac{\partial}{\partial t} \int_V \frac{1}{2} (\varepsilon_{ij} E_i E_j + \mu_{ij} H_i H_j) dV + \int_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = -\int_V \mathbf{E} \cdot \mathbf{J} \, dV.$$

In the absence of free currents, interpret the above relation in terms of an energy density ϵ and an energy flux $\mathbf{N}$, clearly identifying each.

(d) Consider a linear dielectric medium with

$$D_i = \varepsilon_{ij} E_j, \quad B_i = \mu \delta_{ij} H_j,$$

where μ and ε_{ij} are independent of space and time, and δ_{ij} is the Kronecker delta. Assuming plane waves

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{e} \sin(\mathbf{k} \cdot \mathbf{x} - \omega t), \quad \mathbf{B}(\mathbf{x}, t) = \mathbf{b} \sin(\mathbf{k} \cdot \mathbf{x} - \omega t)$$

in this medium, show that Maxwell's equations in the absence of free charges and currents imply that the wave vector $\mathbf{k}$, the frequency ω and polarisation $\mathbf{e}$ must satisfy

$$[\mathbf{k} \times (\mathbf{k} \times \mathbf{e})]_i + \omega^2 \mu \varepsilon_{ij} e_j = 0.$$

(e) Show that the energy flux $\mathbf{N}$ identified above, applied to the situation in part (d), points in the direction of wave propagation when the polarisation is an eigenvector of the matrix ε_{ij}.

Paper 1, Section II
39C Fluid Dynamics II

(a) In an incompressible Stokes flow, show that the Laplacian of the vorticity is zero. If the flow is also two-dimensional, deduce the equation satisfied by the streamfunction $\psi(x, y)$.

(b) A stationary two-dimensional rigid disk of radius a, centred at the origin, is subject to an external shear flow $\mathbf{u}_\infty = \gamma y \mathbf{e}_x$, where γ is a constant and $\mathbf{e}_x$ is the unit vector in the x-direction.

(i) What are the equations and boundary conditions satisfied by ψ for the flow outside the disk?

(ii) Solve for ψ, ensuring that your solution satisfies all the required boundary conditions.

(iii) Compute the hydrodynamic torque exerted by the shear flow on the disk.

[*Hint: in polar coordinates,*

$$(\nabla \times \mathbf{u}) \cdot \mathbf{e}_z = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta},$$

$$\nabla^2 f(r, \theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2},$$

$$\sigma_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right].]$$

Paper 2, Section II

39C Fluid Dynamics II

(a) Consider the incompressible flow of a Newtonian fluid with constant viscosity μ and constant density ρ subject to a body force per unit mass $\mathbf{f}$. Derive the equation for rate of change of kinetic energy in a finite volume Ω with boundary $\partial\Omega$ and give the physical interpretation for each term.

(b) Explain, justifying your arguments with appropriate order-of-magnitude estimates, how the energy balance can be used to estimate the drag on a steadily moving bubble of fixed shape in fluid at rest at infinity, when the Reynolds number is large, without having to solve for the details of the boundary layer around the bubble. [You may ignore all contributions from body forces.]

(c) A two-dimensional circular bubble of radius a, in fluid that is at rest far from the bubble, is moving steadily with velocity $U\mathbf{e}_x$, where $\mathbf{e}_x$ is the unit vector in the x-direction. The flow occurs at high Reynolds number and is assumed to be irrotational outside the boundary layer. [Again you may ignore the effect of any body forces.]

(i) Solve for the irrotational flow outside the boundary layer.

(ii) Using the method in part (b), or otherwise, estimate the drag force exerted on the bubble.

(iii) Give brief reasons why the same approach could not be applied to a rigid body.

[*Hint: In polar coordinates the rate-of-strain tensor has components*

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{r\theta} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right], \quad e_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}.]$$

Paper 3, Section II

38C Fluid Dynamics II

A two-dimensional lubrication flow occurs between two rigid surfaces in a fluid that has otherwise uniform pressure p_0. The bottom surface at $y = 0$ moves in the horizontal direction with velocity $\mathbf{u} = U\mathbf{e}_x$ while the top surface at $y = h(x)$ moves towards $y = 0$ with velocity $\mathbf{u} = -V\mathbf{e}_y$, with $\mathbf{e}_x$ and $\mathbf{e}_y$ being unit vectors in the x- and y-directions respectively. Both surfaces are of length L in the x direction. Consider the instant when both occupy the region $0 < x < L$.

(a) State *all* conditions involving h, U, V and L ensuring that the flow between the two surfaces is in the lubrication limit.

(b) Solve for the flow in the x direction between the two surfaces.

(c) Use conservation of mass to derive an expression for the pressure gradient between the two surfaces as a function of x.

[*Hint: You may find it convenient to introduce the notation* $\langle f \rangle$ *to denote the mean value of a function* f *over the range* $0 \leqslant x \leqslant L$.]

(d) In the particular case $U = 0$, show that the pressure gradient is necessarily zero somewhere between the two surfaces.

(e) Find the value of U such that the force in the x direction on the bottom surface is zero at the instant considered.

Paper 4, Section II

38C Fluid Dynamics II

Consider a two-dimensional wake of constant width $2h$ in an otherwise uniform horizontal flow of speed U. The unperturbed velocity $\mathbf{u} = u\mathbf{e}_x$, where $\mathbf{e}_x$ is a unit vector in the x-direction, is given by

$$u = \begin{cases} U, & y > h \\ 0, & -h < y < h \\ U, & y < -h. \end{cases}$$

The two shear layers at $y = \pm h$ are perturbed symmetrically so that at time t their location is $y = \pm[h + \eta(x,t)]$. The flow is assumed to be irrotational everywhere, the fluid is inviscid and the effects of gravity may be ignored.

(a) Sketch the unperturbed flow and the shape of the deformed shear layers.

(b) State the equation satisfied by the velocity potential ϕ and all the boundary conditions applicable to the three fluid domains ($y < -h - \eta(x,t)$, $-h - \eta(x,t) < y < h + \eta(x,t)$ and $y > h + \eta(x,t)$).

(c) What are the conditions on η and $\partial\eta/\partial x$ necessary in order to linearise the equations and boundary conditions? State the linearised versions of the boundary conditions on ϕ and its derivatives valid under those conditions.

(d) Justify why a full description of the linearised problem is provided by considering solutions of the form

$$\eta(x,t) = \mathrm{Re}\left\{\eta_0 \exp(ikx + \sigma t)\right\},$$

where Re is the real part.

(e) Solve for the dispersion relation linking σ and k with the parameters of the problem. [*Hint: The specified symmetry of the perturbation may allow simplification of the algebra.*] Deduce the conditions on k under which the wake flow is unstable.

(f) In the limit $hk \gg 1$, interpret the result for σ in light of what you know about the Kelvin-Helmholtz instability.

Paper 1, Section I

7E Further Complex Methods

Starting from the Euler product formula for the gamma function,

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\dots(z+n)},$$

show that

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z/k},$$

where Euler's constant is defined by $\gamma = \lim_{n\to\infty}(1 + \frac{1}{2} + \dots + \frac{1}{n} - \log n)$. You may assume that $\gamma = 0.5772\dots$.

The digamma function $\psi(z)$ is defined by $\psi(z) = d(\log\Gamma(z))/dz$. Show that

$$\psi(z) = -\gamma - \frac{1}{z} + z\sum_{k=1}^{\infty} \frac{1}{(z+k)k}.$$

Use this formula to deduce that, for z real and positive, $\psi'(z) > 0$ and hence that $\psi(z)$ has a single zero on the positive real axis which is located in the interval $(1, 2)$.

Paper 2, Section I

7E Further Complex Methods

The Riemann zeta function $\zeta(s)$ is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

which converges for $\mathrm{Re}(s) > 1$.

Show for $\mathrm{Re}(s) > 1$ that

$$(1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{1+e^t}\, dt.$$

Deduce, giving brief justification, an expression for the analytic continuation of $\zeta(s)$ into the region $\mathrm{Re}(s) > 0$.

Hence show that $\zeta(s)$ has a simple pole at $s = 1$ and evaluate the corresponding residue.

Paper 3, Section I

7E Further Complex Methods

Consider the differential equation

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0.$$

State the conditions on $p(z)$ and $q(z)$ for the point $z = z_0$, with z_0 finite, to be (i) an *ordinary point* and (ii) a *regular singular point*. Derive the corresponding conditions for $z_0 = \infty$.

Determine the most general forms of $p(z)$ and $q(z)$ for which $z = 0$ and $z = \infty$ are regular singular points and all other points are ordinary points. Give the corresponding general form of the solution.

Deduce the further restriction on the form of $p(z)$ and $q(z)$ if $z = 0$ is the only regular singular point and all other points are ordinary points.

Paper 4, Section I

7E Further Complex Methods

The hypergeometric function $F(a, b; c; z)$ is the solution of the hypergeometric equation, i.e. the Fuchsian equation determined by the Papperitz symbol

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{matrix}\right\}$$

with $F(a, b; c; z)$ analytic at $z = 0$ and satisfying $F(a, b; c; 0) = 1$.

Explain carefully the meaning of each of the elements appearing in the Papperitz symbol, including any aspects that are required for it to correspond to the hypergeometric equation.

Show that

$$F(a, c-b; c; \frac{z}{z-1}) = (1-z)^a F(a, b; c; z),$$

stating clearly any general results for transforming Fuchsian differential equations or manipulating Papperitz symbols that you use.

Paper 1, Section II

14E Further Complex Methods

The Laguerre differential equation is

$$zy'' + (1-z)y' + \lambda y = 0,$$

where λ is a real constant.

Show that $z=0$ is a regular singular point of the Laguerre equation. Briefly explain why in the neighbourhood of $z=0$ the equation has only one solution, $y_1(z)$, that takes the form of a power series, up to multiplication by a constant. A second, linearly independent, solution is $y_2(z)$. What do you expect to be the leading term in an expansion of $y_2(z)$ in the neighbourhood of $z=0$?

Seek solutions to the Laguerre equation of the form

$$y(z) = \int_\gamma e^{zt} f(t) dt,$$

determining the form required for the function $f(t)$ and the conditions required on the contour γ.

Assume that $\mathrm{Re}(z) > 0$. Consider separately each of the cases:

(i) $\lambda < 0$ and λ non-integer;

(ii) $\lambda > 0$ and λ non-integer;

(iii) λ equal to a negative integer;

(iv) λ equal to a non-negative integer.

Show that in each of these cases one possible choice of γ, say γ_1, is a finite closed contour, and another, say γ_2, is a contour starting at a finite value of t and extending to $-\infty$. Provide a sketch giving a clear specification of these contours in each of the cases (i)-(iv).

Show that the $y(z)$ obtained from the finite closed contour γ_1 is a constant multiple of the solution $y_1(z)$ and that in case (iv) this solution is a polynomial in z. What can you say about the form of this solution in case (iii)?

Paper 2, Section II

13E Further Complex Methods

The functions $g(z)$ and $h(z)$ are defined by

$$g(z) = \int_0^z \frac{1}{(1-t^2)^{1/2}}\, dt \quad \text{and} \quad h(z) = \int_0^z \frac{1}{(1-t^2)^{1/2}} \frac{1}{(1-k^2t^2)^{1/2}}\, dt,$$

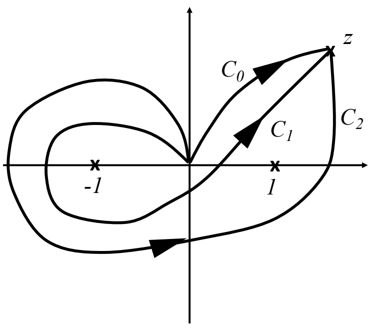
where each integral can be taken along any curve C in the complex t-plane that does not pass through a branch point of the integrand. In both cases the value of the integrand is chosen to be 1 at $t=0$.

(a) First consider $g(z)$. Let $G(z)$ be the value of $g(z)$ evaluated when C is forbidden from crossing the real axis except in the interval $(-1,1)$, with z allowed to lie anywhere in the complex plane except on the parts $(-\infty,-1]$ and $[1,\infty)$ of the real axis. C_0 in the diagram below is such a contour.

(i) Explain why $G(z)$ is a single valued function of z, but $g(z)$ may not be.

(ii) Evaluate $g(z)$ in terms of $G(z)$ when C is each of C_1 and C_2 shown in the diagram below.

(iii) Give, with brief reasoning, all possible values of $g(z)$ as the curve C is varied.



(b) Now consider $h(z)$. Let k be real with $0<k<1$. Let $H(z)$ be the value of $h(z)$ evaluated when C is forbidden from crossing the real axis except in the interval $(-1,1)$, with z allowed to lie anywhere in the complex plane except on the parts $(-\infty,-1]$ and $[1,\infty)$ of the real axis.

(i) Explain why $H(z)$ is a single valued function of z, but $h(z)$ may not be.

(ii) Show, by identifying suitable contours C, that possible values of $h(z)$ include $4K+H(z)$, $2K-H(z)$ and $2iL+H(z)$, where

$$K = \int_0^1 \frac{1}{(1-t^2)^{1/2}} \frac{1}{(1-k^2t^2)^{1/2}}\, dt \quad \text{and} \quad L = \int_1^{1/k} \frac{1}{(t^2-1)^{1/2}} \frac{1}{(1-k^2t^2)^{1/2}}\, dt.$$

(c) Deduce that the inverse function $\mathcal{H}(w)$ defined by $h(\mathcal{H}(w)) = w$ is a doubly periodic function and give expressions for the two periods.

(d) Assuming that $\mathcal{H}$ is a meromorphic function, explain briefly why it must have at least one pole.

Paper 1, Section II
18I Galois Theory

(a) What does it mean to say that a finite extension L/K is *normal*? Show that L/K is normal if and only if L is a splitting field of some polynomial over K.

(b) Let $M/L/K$ be finite extensions. Which of the following statements are true, and which are false? Give a proof or counterexample in each case.

i) If M/K is normal then M/L is normal.

ii) If M/K is normal then L/K is normal.

iii) If M/L and L/K are normal, then M/K is normal.

(c) Let L be a splitting field of $T^4 - 7$ over $\mathbb{Q}$. Show that $\mathrm{Gal}(L/\mathbb{Q})$ is the dihedral group of order 8. Determine all the subfields of L, and express each of them in the form $\mathbb{Q}(x)$ for some $x \in L$. Which of them are normal extensions of $\mathbb{Q}$?

Paper 2, Section II
18I Galois Theory

Let L/K be a finite extension of fields of characteristic $p > 0$.

(a) Let $x \in L$. What does it mean to say that x is *separable over* K? Show that x is separable over K if and only if its minimal polynomial is not of the form $g(T^p)$ for some $g \in K[T]$.

(b) We say that $x \in L$ is *purely inseparable* over K if for some $n \geqslant 0$, $x^{p^n} \in K$. Show that x is purely inseparable over K if and only if its minimal polynomial is of the form $T^{p^n} - y$, for some $n \geqslant 0$ and some $y \in K$.

(c) Let $g \in K[T]$ be a monic nonconstant polynomial, and $f(T) = g(T^p)$. Assume that L is a splitting field for g over K, and let M be a splitting field for f over L. Show that M is also a splitting field for f over K, and that every root of f in M is purely inseparable over L. Show also that for every $\sigma \in \mathrm{Aut}(L/K)$ there exists a unique automorphism τ of M whose restriction to L equals σ.

(d) Suppose that g is irreducible and separable. Show that $\mathrm{Aut}(M/K)$ acts transitively on the roots of f in M. Deduce that either every root of f lies in L, or every root has degree p over L.

Let $h \in K[T]$ be an irreducible monic factor of f. Show that either $h = f$, or that h is separable. Deduce that f is reducible if and only if every coefficient of g is a p-th power in K.

[*You may assume without proof the uniqueness of splitting fields, and that a nonconstant polynomial f is separable if and only if* $(f, f') = 1$.]

Paper 3, Section II

18I Galois Theory

(a) Show that a finite subgroup of the multiplicative group of a field is cyclic.

(b) What is a *primitive n-th root of unity*? Show that if K contains a primitive m-th root of unity and a primitive n-th root of unity, then it contains a primitive N-th root of unity, where N is the least common multiple of m and n.

(c) Define the *cyclotomic polynomials* Φ_n and show that they have integer coefficients. Show also that the reduction of Φ_n modulo a prime p is separable if p does not divide n.

(d) Let K be a field of characteristic zero, L a splitting field for Φ_n over K, and let $G = \mathrm{Gal}(L/K)$ be its Galois group. Write down an injective homomorphism from G into $(\mathbb{Z}/n\mathbb{Z})^\times$, and show that it is surjective if and only if Φ_n is irreducible over K.

(e) Let L be a splitting field for Φ_n over $\mathbb{Q}$. Show that the number of roots of unity in L is n if n is even, and $2n$ if n is odd. [You may assume that Φ_n is irreducible over $\mathbb{Q}$.]

Paper 4, Section II

18I Galois Theory

(a) Define the *discriminant* of a monic polynomial.

Let K be a field with $\mathrm{char}(K) \neq 2$, and let $f \in K[T]$ be a monic, separable polynomial of degree n. Show that the Galois group of f is contained in A_n if and only if the discriminant of f is a square in K.

Compute the Galois group of $T^3 - 2T + 2$ over $\mathbb{Q}$ and over $\mathbb{Q}(\sqrt{-19})$.

[*The discriminant of* $T^3 + aT + b$ *is* $-4a^3 - 27b^2$.]

(b) Let K be a field of characteristic 2, and $f = T^3 + aT + b \in K[T]$. Let L be a splitting field for f over K.

(i) Show that f is separable if and only if $b \neq 0$.

(ii) Assuming that f is separable, show that $g = T^2 + bT + a^3 + b^2$ splits into distinct linear factors in $L[T]$. By considering the action of the Galois group G of f on the roots of g, or otherwise, show that G is contained in A_3 if and only if g splits into linear factors in $K[T]$.

Paper 1, Section II

38B General Relativity

A Klein-Gordon scalar field ϕ satisfies the equation of motion $\nabla^\alpha \nabla_\alpha \phi = m^2 \phi$ where m is a constant. Its stress-energy tensor takes the form:

$$T_{\mu\nu} = \frac{1}{2}\Big[\nabla_\mu \phi \nabla_\nu \phi + g_{\mu\nu}\big(A \nabla_\rho \phi \nabla^\rho \phi + B\phi^2\big)\Big]. \qquad (*)$$

(a) Using the fact that the stress-energy tensor is covariantly conserved, determine the value of the parameters A and B.

(b) Using the Einstein equation, write an expression for the Ricci curvature $R_{\mu\nu}$ in terms of ϕ and its derivatives, in a $D > 2$ dimensional spacetime. Simplify your answer as much as possible.

(c) Now consider a general stress-energy tensor of the form $(*)$, with A and B not necessarily given by the values you have found above. The stress-energy tensor is said to satisfy the *weak energy condition* if

$$T_{\mu\nu} X^\mu X^\nu \geqslant 0$$

for all timelike vectors X^μ. Find the most general constraints on A and B such that $(*)$ satisfies the weak energy condition, and show that your answer to part (a) satisfies these constraints.

[Hint: you may find it useful to work in normal coordinates and furthermore to choose these coordinates such that $X^\mu = \delta_0{}^\mu$.]

Paper 2, Section II

38B General Relativity

Consider the geometry of 2-dimensional hyperbolic space:

$$ds^2 = a^2(dr^2 + \sinh^2 r\, d\phi^2)$$

where a is a constant. The coordinates have ranges $0 \leqslant r$ and $0 \leqslant \phi < 2\pi$.

(a) For a general metric with components $g_{\alpha\beta}$, give an expression for the Christoffel symbols, $\Gamma^\alpha_{\beta\gamma}$, in terms of the metric components and their derivatives. Use this formula to calculate the Christoffel symbols for the metric above.

(b) Using the geodesic equation, show that lines of constant ϕ are always geodesics, but circles of constant $r > 0$ never are.

(c) Calculate both of the nonzero components of the Riemann tensor $R^\alpha{}_{\beta\gamma\delta}$.

[*You may use:* $R^\alpha{}_{\beta\gamma\delta} := \partial_\gamma \Gamma^\alpha_{\beta\delta} - \partial_\delta \Gamma^\alpha_{\beta\gamma} + \Gamma^\mu_{\beta\delta}\Gamma^\alpha_{\mu\gamma} - \Gamma^\mu_{\beta\gamma}\Gamma^\alpha_{\mu\delta}$.]

(d) Show that the Ricci scalar R is constant.

Paper 3, Section II

37B General Relativity

(a) Let M be the mass of a star and consider a photon with impact parameter b which passes near the star. In this problem, by following the steps below, you will derive the general relativistic formula for the total angle $\delta\phi$ by which the photon bends.

The general relativistic formulae for equatorial null orbits in the Schwarzschild metric (in units where $c = G = 1$) are:

$$\frac{1}{2}\dot{r}^2 + V(r) = \frac{1}{2}E^2, \qquad V(r) = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2},$$

where dot is derivative with respect to proper time, and $L = r^2\dot{\phi}$ is the angular momentum.

(i) Write down the geodesic equation for the trajectory of the photon, parameterized by the ϕ coordinate. Switch to an inverse radial coordinate $y = 1/r$. By differentiating the geodesic equation by ϕ, show that $y'' + y = 3My^2$. Here $'$ denotes $d/d\phi$.

(ii) Solve this equation in the flat space regime ($M = 0$), for a trajectory for which $r \to \infty$ at $\phi = 0, \pi$.

(iii) Using perturbation theory in M identify a differential equation for Δy, the first order perturbation of y due to nonzero M.

(iv) Find the homogeneous and particular solutions for Δy.

(v) Taking $r \to \infty$ at $\phi = 0$, show that the leading order result for the bending of the light ray is:

$$|\delta\phi| \approx \frac{4M}{b}.$$

(b) In Nordström's theory of gravitation, the metric is required to take the form

$$g_{\mu\nu} = \phi^2\eta_{\mu\nu},$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $\phi > 0$ is a dynamical scalar field which approaches the value 1 far from any isolated gravitating system.

Write down the equation satisfied by an affinely parameterised geodesic of the metric $g_{\mu\nu}$. What can you deduce about the bending of light rays around a star of mass M in Nordström's theory? Is this result compatible with observations?

Paper 4, Section II

37B General Relativity

(a) Consider a linearized gravitational plane wave of the form

$$\bar{h}_{\mu\nu} = H_{\mu\nu} e^{ik_\rho x^\rho}$$

where $H_{\mu\nu}$ is independent of x^α, $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$ is the trace-reversed perturbation to the Minkowski metric $\eta_{\mu\nu}$, and we are using Lorentz gauge $\partial^\mu \bar{h}_{\mu\nu} = 0$.

(i) What restrictions are there on k^μ and $H_{\mu\nu}$? Justify your answers.

(ii) Derive the residual gauge symmetry remaining in $H_{\mu\nu}$, even after imposing Lorentz gauge.

[*You may use:* $G_{\mu\nu} = -\frac{1}{2}\partial^\rho \partial_\rho \bar{h}_{\mu\nu} + \partial^\rho \partial_{(\mu} \bar{h}_{\nu)\rho} - \frac{1}{2}\eta_{\mu\nu} \partial^\rho \partial^\sigma \bar{h}_{\rho\sigma}$.]

(b) Suppose that LIGO detects the merger of two black holes, each of which is about 30 solar masses, from an event which takes place approximately a few billion lightyears away.

(i) Estimate the frequency (in Hz) of the gravitational wave source, from the perspective of a hypothetical observer close to the binary system and at rest with respect to it, during the last orbit of the black holes before they merge. In solving this problem you may use the (Newtonian) Kepler's law:

$$T^2 = \frac{4\pi^2}{GM} r^3.$$

Here T is the period and for purposes of estimation you may take $r = 6MG/c^2$, the general relativistic formula for the inner-most stable circular orbit for a test particle in a Schwarzschild geometry. As these assumptions are inexact, do not keep more than one significant figure.

[*You may use:* $c \approx 3.0 \times 10^8$ m/s, $G \approx 6.7 \times 10^{-11}\, \mathrm{m}^3/(\mathrm{kg\, s}^2)$, and the solar mass $M_\odot \approx 2.0 \times 10^{30}$ kg.]

(ii) Write down a Big Bang metric suitable for calculations in our universe, which is spatially flat. You may leave the scale factor $a(t)$ as an undetermined function (where t is the proper time).

Let t_e be the time of emission, and t_o be the time of detection. Write down a formula for the frequency of the gravitational wave as it is observed by LIGO, from the perspective of Earth's local reference frame. [For purposes of solving this problem, you may treat the Earth and the binary black hole system as both being at rest relative to the cosmological frame of reference.]

Paper 1, Section II

17H Graph Theory

(a) By considering the random graph $G(n, p)$, with $p = (1/2)n^{-2/3}$, show that for every $k \geqslant 1$ there exists a graph G that contains no $K_{3,3}$ and has $\chi(G) > k$.

(b) (i) For $p = n^{-2/3} \log n$, let $G \sim G(n, p)$. Show that

$$\lim_{n\to\infty} \mathbb{P}(G \text{ contains at least 100 copies of } K_4) = 1.$$

(ii) Let $\{H_1, \ldots, H_r\}$ be vertex disjoint copies of K_4, in a graph G. (Recall that this means that $V(H_i) \cap V(H_j) = \emptyset$, for all $i \neq j$.)

Show that

$$\lim_{n\to\infty} \mathbb{P}(G \text{ contains at least 100 vertex disjoint copies of } K_4) = 1.$$

[*Hint: you may wish to consider the number of K_4's in the graph and also the number of pairs of K_4's that are not disjoint.*]

Paper 2, Section II

17H Graph Theory

(a) Define the *chromatic polynomial* $P_G(t)$ of a graph G and prove that it is a polynomial.

(b) Let $f(t) = t^4 - 4t^3 + 5t^2 - 2t$. Explain why f is not the chromatic polynomial of a bipartite graph. Is f the chromatic polynomial of some graph? Justify your answers.

(c) Let G be a connected graph, let $A = A(G)$ be its adjacency matrix and let $d(G)$ denote the diameter of G. Show that the matrices $I, A, A^2, \ldots, A^{d(G)}$ are linearly independent.

(d) Give an infinite family of connected graphs $\{G_n\}$ with adjacency matrices $\{A_n\}$ so that

$$I, A_n, A_n^2, \ldots, A_n^{d(G_n)+1}$$

are linearly dependent, for each n.

Paper 3, Section II

17H Graph Theory

(a) Let $r \geqslant 2$. Prove Turán's theorem in the form: if G is an n vertex graph that does not contain a K_{r+1} then

$$e(G) \leqslant \left(1 - \frac{1}{r}\right)\frac{n^2}{2}.$$

(b) For $t \leqslant n-1$, show that if G is a connected n vertex graph with $\delta(G) \geqslant t/2$ then G contains a path P_t of length t.

(c) For graphs G, H define the *Ramsey number* $r(G,H)$ to be the minimum n such that every red-blue colouring of the edges of K_n contains either a red copy of G or a blue copy of H. For $s \geqslant 2, t \geqslant 1$, show that

$$r(K_s, P_t) \geqslant (s-1)t + 1.$$

(d) Show further that for $s \geqslant 2, t \geqslant 1$ we have

$$r(K_s, P_t) = (s-1)t + 1.$$

Paper 4, Section II

17H Graph Theory

(a) State Menger's theorem relating the size of x-y separators in a graph to the number of independent x-y paths.

(b) State Hall's theorem and, assuming Menger's theorem, prove Hall's theorem.

(c) Let $k \geqslant 1$ and let $[0,1]^3 = A_1 \cup \cdots \cup A_k$ and $[0,1]^3 = B_1 \cup \cdots \cup B_k$ be two partitions of the unit cube into sets of equal volume. Show there is a permutation σ of $[k]$ so that $A_i \cap B_{\sigma(i)} \neq \emptyset$, for all $i \in [k]$.

(d) Let G be a $2k$-connected graph that contains a K_{2k} and let $x_1, \ldots, x_k, y_1, \ldots, y_k$ be distinct vertices in G. Show that there exist paths $P_1, \ldots, P_k$ for which

$$V(P_i) \cap V(P_j) = \emptyset,$$

for all $i \neq j$ where, for each i, P_i is an x_i-y_i path.

[*In parts (c) and (d) you may assume results from the course provided they are stated clearly.*]

Paper 1, Section II

33E Integrable Systems

Let $q = q(x,t)$ and $r = r(x,t)$ be complex valued functions and consider the matrices (U, V) defined by

$$U(\lambda) = \begin{pmatrix} i\lambda & iq \\ ir & -i\lambda \end{pmatrix}, \quad V(\lambda) = 2i\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2i\lambda \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} + \begin{pmatrix} 0 & q_x \\ -r_x & 0 \end{pmatrix} - i \begin{pmatrix} rq & 0 \\ 0 & -rq \end{pmatrix}.$$

Derive the zero curvature equation as the consistency condition for the system of equations

$$\Psi_x = U\Psi\,, \qquad \Psi_t = V\Psi$$

and show that it holds precisely when q, r satisfy a system of the form

$$ir_t + r_{xx} + aqr^2 = 0\,, \tag{1}$$

$$iq_t - q_{xx} - arq^2 = 0\,, \tag{2}$$

where a is a real number which you should determine. Show that if $r = \overline{q}$ this system reduces to the *nonlinear Schrödinger equation*

$$ir_t + r_{xx} + a|r|^2 r = 0\,, \tag{NLS1}$$

and find a similar reduction to the equation

$$ir_t + r_{xx} - a|r|^2 r = 0\,. \tag{NLS2}$$

Write these equations in Hamiltonian form. Search for solutions to (NLS1) and (NLS2) of the form $e^{-iEt}f(x)$ with real constant E and smooth, rapidly decreasing real-valued f. In each case either find such a solution explicitly, or explain briefly why it is not expected to exist.

[*Hint: you may use without derivation the indefinite integral*

$$\int \frac{dy}{\sqrt{\lambda^2 y^2 - y^4}} = -\frac{1}{\lambda}\mathrm{sech}^{-1}\frac{y}{\lambda}\,.]$$

Paper 2, Section II

34E Integrable Systems

Assume $\phi = \phi(x,t)$ is a solution of

$$-\phi_{xx} + u(x,t)\phi = \lambda(t)\phi, \qquad -\infty < x < \infty\,, \tag{S}$$

where $u = u(x,t)$ is smooth. Define $Q = Q(x,t)$ by $Q = \phi_t + u_x\phi - 2(u+2\lambda)\phi_x$ and show that there exists a number α, which you should find, such that

$$\partial_x(\phi_x Q - \phi Q_x) = \phi^2\Big(\dot{\lambda} + \alpha(u_t + u_{xxx} - 6uu_x)\Big) \tag{$*$}$$

where $\dot{\lambda} = \frac{d\lambda}{dt}$.

Now let $u = u(x,t)$ be a smooth solution of the KdV equation $u_t + u_{xxx} - 6uu_x = 0$, which is rapidly decreasing in x, and consider the case when $\phi = \varphi_n$ is the discrete eigenfunction of (S) corresponding to eigenvalue $\lambda_n = -\kappa_n^2 < 0$. Deduce from $(*)$ that $\lambda_n(t) = \lambda_n(0)$. [You may assume that $\kappa_n > 0$ and φ_n is normalized, i.e., $\int_{-\infty}^{\infty} \varphi_n(x,t)^2 dx = 1$ for all times t.]

Deduce further that in this case $Q(x,t) = h_n(t)\varphi_n(x,t)$ for some function $h_n = h_n(t)$ and, by multiplying by φ_n, making use of (S) and integrating, show that $h_n(t) = 0$ and $Q = 0$. Finally, derive from this the time evolution of the discrete normalization $c_n(t)$ which is defined by the asymptotic relation

$$\varphi_n(x,t) \approx c_n(t)e^{-\kappa_n x} \quad \text{as} \quad x \to +\infty\,.$$

[You may assume the differentiated version of this relation also holds.]

Paper 3, Section II

32E Integrable Systems

(a) Compute the group of transformations generated by the vector field

$$V = t\partial_t + x\partial_x\,,$$

and hence, or otherwise, calculate the second prolongation of the vector field V and show that V generates a group of Lie symmetries of the wave equation $u_{tt} - u_{xx} = 0$.

Use the group of symmetries you have just found for the equation $u_{tt} - u_{xx} = 0$ to obtain a group invariant solution for this equation.

(b) Compute the group of transformations generated by the vector field

$$4t^2\partial_t + 4tx\partial_x - (x^2 + 2t)\partial_u$$

and verify that they give rise to a group of Lie symmetries of the equation $u_t = u_{xx} + u_x^2$.

Paper 1, Section II

22F Linear Analysis

(a) State the open mapping theorem and the closed graph theorem, and prove that the former implies the latter.

(b) Let V be a Banach space. Give the definition of the *dual space* V^*, and prove that V^* is a Banach space.

(c) Let V be a Banach space over the real field, and let $T : V \to V^*, v \mapsto T_v$ be a linear map between these two Banach spaces that satisfies $T_v(v) \geqslant 0$ for all $v \in V$. Prove that T is continuous.

Paper 2, Section II

22F Linear Analysis

(a) Let $(V, \|\cdot\|)$ be a normed vector space over $\mathbb{R}$, and $v, w \in V$. Define

$$S_1^{vw} := \left\{ z \in V \ : \ \|z - v\| = \|z - w\| = \frac{1}{2}\|v - w\| \right\}$$

and then inductively, for $n \geqslant 2$,

$$S_n^{vw} := \left\{ z \in S_{n-1}^{vw} \ : \ \forall\, \tilde{z} \in S_{n-1}^{vw}, \ \|z - \tilde{z}\| \leqslant \frac{1}{2}\mathrm{diam}(S_{n-1}^{vw}) \right\},$$

with the definition $\mathrm{diam}(S) := \sup_{z,\tilde{z} \in S} \|z - \tilde{z}\|$. Prove that $\cap_{n \geqslant 1} S_n^{vw} = \{\frac{v+w}{2}\}$.

(b) Let $(V, \|\cdot\|_V)$ and $(\widetilde{V}, \|\cdot\|_{\widetilde{V}})$ be normed vector spaces over $\mathbb{R}$, and $u : V \to \widetilde{V}$ an isometry, i.e. a map with the property that $\|u(v) - u(w)\|_{\widetilde{V}} = \|v - w\|_V$. Using part (a), prove that $u(\frac{v+w}{2}) = \frac{u(v)+u(w)}{2}$ for all $v, w \in V$.

(c) Assume furthermore that the isometry $u : V \to \widetilde{V}$ satisfies $u(0) = 0$. Prove that u is linear.

Paper 3, Section II
21F Linear Analysis

Recall that a topological space X is called *normal* if for any pair of non-empty disjoint closed subsets $A, B \subset X$, there is a pair of disjoint open subsets $U_1, U_2 \subset X$ so that $A \subset U_1$ and $B \subset U_2$. Also recall that the Urysohn lemma states that in a normal topological space X, for any pair of non-empty disjoint closed subsets $A, B \subset X$, there is an $f : X \to [0, 1]$ continuous so that $f = 0$ on A and $f = 1$ on B.

(a) State and prove the Tietze extension theorem. [You may use the Urysohn lemma.]

(b) Consider a normal topological space X, and $A \subset X$ a non-empty closed subset that can be realised as a countable intersection of open sets. Show that there exists $f : X \to [0, 1]$ continuous so that f vanishes on A and on A only.

(c) Consider a normal topological space X, and $A, B \subset X$ a pair of non-empty disjoint closed subsets that can both be realised as countable intersections of open sets. Show that there exists $f : X \to [0, 1]$ continuous so that f vanishes on A and on A only, and is equal to 1 on B and on B only.

Paper 4, Section II
22F Linear Analysis

Below, H denotes a Hilbert space over $\mathbb{C}$.

(a) Consider a sequence (x_n) in H with the property that there exists an $x \in H$ such that for any $y \in H$, $\langle x_n, y\rangle$ converges to $\langle x, y\rangle$ in $\mathbb{C}$. Prove that the sequence (x_n) is bounded. [The uniform boundedness principle may be used without proof, provided it is properly stated.]

(b) With (x_n) and x as above, prove that there exists another sequence $(\tilde{x}_k)$ in H such that $\|\tilde{x}_k - x\|_H \to 0$ and such that each $\tilde{x}_k$ is a convex combination of terms in (x_n).

(c) Deduce that if $C \subset H$ is closed and convex, and (x_n) is a sequence in C as in part (a), i.e. with the property that there exists $x \in H$ such that for any $y \in H$, $\langle x_n, y\rangle \to \langle x, y\rangle$, then in fact $x \in C$.

(d) Is the statement in part (c) still true when C is closed but not necessarily convex? [You must either provide a proof if true or a detailed counterexample if untrue.]

Paper 1, Section II

16H Logic and Set Theory

(a) State and prove the Compactness Theorem for first-order logic. State and prove the Upward Lowenheim-Skolem Theorem. State the Downward Lowenheim-Skolem Theorem, and explain briefly why it is true.

(b) For a language L and an L-structure A, an *automorphism* of A is a bijection from A to itself that preserves all the functions and relations of L. An L-structure is *rigid* if it has no automorphism apart from the identity map.

(i) If a theory T in a language L has arbitrarily large finite non-rigid models, show that T also has an infinite non-rigid model.

(ii) If a theory T in a language L has arbitrarily large finite models, and every finite model of T is rigid, does it follow that every infinite model of T is rigid? Justify your answer.

[*You may assume the Completeness Theorem for first-order logic.*]

Paper 2, Section II

16H Logic and Set Theory

(a) Let α be a non-zero ordinal. Show that there is a greatest ordinal β such that $\omega^\beta \leqslant \alpha$. Deduce that there exist a non-zero natural number n and an ordinal $\gamma < \omega^\beta$ such that $\alpha = \omega^\beta.n + \gamma$.

(b) An ordinal δ is called *additively closed* if whenever $\beta < \delta$ and $\gamma < \delta$ then also $\beta + \gamma < \delta$. Show that a non-zero ordinal is additively closed if and only if it is of the form ω^α for some α.

(c) An ordinal δ is called *multiplicatively closed* if whenever $\beta < \delta$ and $\gamma < \delta$ then also $\beta\gamma < \delta$. Show that an ordinal greater than 2 is multiplicatively closed if and only if it is of the form $\omega^{(\omega^\alpha)}$ for some α.

[*You may assume standard properties of ordinal arithmetic.*]

Paper 3, Section II

16H Logic and Set Theory

In this question we work in a fixed model V of ZFC.

(a) Prove that every set has a transitive closure. [If you apply the Axiom of Replacement to a function-class F, you must explain clearly why F is indeed a function-class.]

(b) State the Axiom of Foundation and the Principle of ϵ-Induction, and show that they are equivalent (in the presence of the other axioms of ZFC).

(c) We say that a set x is *reasonable* if every member of $TC(\{x\})$ is countable. Which of the following are true and which are false? Justify your answers.

(i) A set is reasonable if and only if $TC(\{x\})$ is countable.

(ii) The reasonable sets are all members of V_α, for some α.

(iii) The reasonable sets form a model of ZFC.

[In (c) you may assume any results from the course.]

Paper 4, Section II

16H Logic and Set Theory

In this question we work in ZF, not ZFC. As usual, for cardinals κ and λ we write $\kappa \leqslant \lambda$ if there is an injection from K to L, where K and L are sets of cardinalities κ and λ, respectively.

(a) Show that the assertion that $\leqslant$ is a total ordering on cardinals (in other words, that for any κ and λ we have $\kappa \leqslant \lambda$ or $\lambda \leqslant \kappa$) is equivalent to the Axiom of Choice.

(b) Show that the Axiom of Choice implies that, for any infinite cardinal κ, we have $\kappa^2 = \kappa$.

(c) Suppose that κ and λ are non-zero cardinals such that $\kappa\lambda \leqslant \kappa + \lambda$. Prove that there must exist either an injection or a surjection from K to L.

(d) Show that the assertion that for any infinite cardinal κ we have $\kappa^2 = \kappa$ is equivalent to the Axiom of Choice. [*Hint: for a given set X, you may wish to consider the disjoint union of X with $\gamma(X)$.*]

[*You may assume Hartogs' Lemma, and you may use the equivalence of the Axiom of Choice with any of its equivalents from the course.*]

Paper 1, Section I

6C Mathematical Biology

Consider a birth-death process in which births always give rise to 3 offspring, with rate λ, while the death rate per individual is β. Draw a transition diagram and write down the master equation for this system.

Show that the population mean is given by

$$\langle n \rangle = \frac{3\lambda}{\beta}\left(1 - e^{-\beta t}\right) + n_0 e^{-\beta t},$$

where n_0 is the initial population mean, and that the population variance satisfies

$$\sigma^2 \to \frac{6\lambda}{\beta} \quad \text{as} \quad t \to \infty.$$

Paper 2, Section I

6C Mathematical Biology

In an SIR model for an infectious disease the population N is divided into susceptible $S(t)$, infected $I(t)$ and recovered (non-infectious) $R(t)$. The disease is assumed to be non-lethal, so the total population does not change in time.

Consider the following SIR model,

$$\frac{dS}{dt} = fR - \beta IS, \quad \frac{dI}{dt} = \beta IS - \nu I, \quad \frac{dR}{dt} = \nu I - fR,$$

and explain the meaning of each of the terms in the equations. Assume that at $t = 0$, $S \simeq N$, while $I, R \ll N$.

(a) Setting $f = 0$, show that if $\beta N < \nu$ no epidemic occurs.

(b) Now take $f > 0$ and suppose that there is an epidemic. Show that the system has a nontrivial fixed point and that it is stable for small disturbances. Show that the eigenvalues of the Jacobian matrix are complex for sufficiently small f but real for sufficiently large f. Give a qualitative sketch of $I(t)$ in the two cases.

Paper 3, Section I

6C Mathematical Biology

A gene product with concentration g is produced by a chemical S of concentration s, is autocatalysed and degrades linearly according to the kinetic equation

$$\frac{dg}{dt} = f(g,s) = s + k\frac{g^2}{1+g^2} - g,$$

where $k > 2$ is a constant.

First consider the case $s = 0$. Show that there are two positive steady states, and determine their stability. Sketch the reaction rate $f(g,0)$.

The system starts in the steady state $g = 0$ with $s = 0$. The value of s is then increased to the value s_1, held at this value for a long time, and then reduced to zero. Show that, if s_1 is greater than a value $s_c(k)$, a biochemical switch can be achieved to a state $g = g_* > 0$ whose value you should determine. Give a clear mathematical specification of the value $s_c(k)$. [An explicit formula is not needed.]

For the case $k \gg 1$, use a suitable approximate form of $f(g,s)$ to show that $s_c(k) \simeq Ck^{-1}$ where C is a constant that you should derive.

Paper 4, Section I

6C Mathematical Biology

The concentration $C(x,t)$ of a morphogen obeys the differential equation

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} + f(C),$$

in the domain $0 \leqslant x \leqslant L$, with boundary conditions $C(0,t) = 0$ and $\partial C(L,t)/\partial x = 0$, with D a positive constant and $f(C)$ a nonlinear function of C with $f(0) = 0$ and $f'(0) > 0$. Linearising the dynamics around $C = 0$, and representing $C(x,t)$ as a suitable Fourier expansion, find the condition on L such that the system is linearly stable. Express your answer in terms of D and $f'(0)$.

Paper 3, Section II

13C Mathematical Biology

Consider the reaction-diffusion system in one spatial dimension $-\infty < x < \infty$,

$$\begin{aligned} \frac{\partial u}{\partial t} &= D\frac{\partial^2 u}{\partial x^2} + f(u) + \rho(u-v), & (1) \\ \epsilon\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + u - v, & (2) \end{aligned}$$

where $D > 0$ is the activator diffusion constant, $\rho > 0$ is a constant, and $0 < \epsilon \ll 1$ so that the inhibitor v is a fast variable relative to the activator u. The nonlinear function $f(u)$ is taken to have the properties $f(0) = 0$ and $f'(0) = -r$ with $0 \leqslant r \leqslant 1$.

(a) Setting $\epsilon = 0$, show that the inhibitor dynamics can be solved to express the Fourier amplitude $\hat{v}(k,t)$ of the inhibitor in terms of the Fourier amplitude $\hat{u}(k,t)$ of the activator.

(b) Using the relation found in part (a), and linearising around the state $u = 0$, find the dynamics of perturbations around $u = 0$ and thus the growth rate $\sigma(k)$ as a function of the wavenumber k.

(c) From the result in (b), show that the threshold of a pattern-forming instability lies along a curve in the $r - \rho$ plane given by

$$\rho_c(r) = \left(\sqrt{r} + \sqrt{D}\right)^2, \qquad (3)$$

along which the critical wavenumber is

$$k_c = \left(\frac{r}{D}\right)^{1/4}. \qquad (4)$$

Paper 4, Section II

14C Mathematical Biology

Consider a population subject to the following birth-death process. When the number of individuals in the population is n, the probability of an increase from n to $n+1$ per unit time is $\gamma+\beta n$ and the probability of a decrease from n to $n-1$ is $\alpha n(n-1)$, where α, β, and γ are constants.

Draw a transition diagram and show that the master equation for $P(n,t)$, the probability that at time t the population has n members, is

$$\frac{\partial P}{\partial t} = \alpha n(n+1)P(n+1,t) - \left[\alpha n(n-1)+\gamma+\beta n\right]P(n,t) + \left[\gamma+\beta(n-1)\right]P(n-1,t). \quad (1)$$

Show that $\langle n\rangle$, the mean number of individuals in the population, satisfies

$$\frac{d\langle n\rangle}{dt} = -\alpha\langle n^2\rangle + (\alpha+\beta)\langle n\rangle + \gamma.$$

Deduce that, in a steady state,

$$\langle n\rangle = \frac{\alpha+\beta}{2\alpha} \pm \sqrt{\frac{(\alpha+\beta)^2}{4\alpha^2} + \frac{\gamma}{\alpha} - (\Delta n)^2},$$

where Δn is the standard deviation of n. Given the form of the expression above, when is the choice of the minus sign not admissible?

Show that, under conditions to be specified, the master equation (1) may be approximated by a Fokker-Planck equation of the form

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial n}\left[g(n)P(n,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial n^2}\left[h(n)P(n,t)\right].$$

Find the functions $g(n)$ and $h(n)$.

In the case $\alpha \ll \gamma$ and $\alpha \ll \beta$, find the leading-order approximation to n_* such that $g(n_*) = 0$. Defining the new variable $x = n - n_*$, explain how an approximate form of $P(x)$ may be obtained in the neighbourhood of $x = 0$ in the steady-state limit, showing clearly the dependence of $P(x)$ on the properties of the functions $g(n)$ and $h(n)$ at $n = n_*$. Deduce leading order estimates for $\langle n\rangle$ and $(\Delta n)^2$ in terms of α, β and γ.

Compare your results to those obtained from the master equation above and give justification of why the conditions for applicability of the Fokker-Planck equation hold in this case.

Paper 1, Section II

31J Mathematics of Machine Learning

(a) What does it mean for a set $C \subseteq \mathbb{R}^d$ to be *convex*?

(b) What does it mean for a function $f : C \to \mathbb{R}$ to be *strictly convex*? Show that any minimiser of f must be unique.

(c) Define the *projection* $\pi_C(x)$ of a point $x \in \mathbb{R}^d$ onto a closed convex set C. Briefly explain why this is unique. [Standard results about convex functions may be used without proof, and you need not show that $\pi_C(x)$ always exists.]

(d) Prove that $\pi \in C$ is the projection of x onto a closed convex set C if

$$(x - \pi)^T(z - \pi) \leqslant 0 \qquad \text{for all } z \in C.$$

(e) Let C be a closed convex set given by

$$C := \left\{ \begin{pmatrix} v \\ s \end{pmatrix} \in \mathbb{R}^p \times \mathbb{R} : \ \|v\|_2 \leqslant s \right\}.$$

Using part (d) or otherwise, show that if $(u, t) \in \mathbb{R}^p \times \mathbb{R}$ satisfy $\|u\|_2 \geqslant |t|$ then

$$\pi_C\left(\begin{pmatrix} u \\ t \end{pmatrix}\right) = \frac{1}{2}\left(1 + \frac{t}{\|u\|_2}\right)\begin{pmatrix} u \\ \|u\|_2 \end{pmatrix}.$$

What is $\pi_C\left(\begin{pmatrix} u \\ t \end{pmatrix}\right)$ when $\|u\|_2 \leqslant -t$?

(f) Let C be as in (e) and let $(X_i, Y_i) \in \mathbb{R}^{p+1} \times \mathbb{R}$ for $i = 1, \ldots, n$ be data formed of input–output pairs. Write down the projected gradient descent procedure for finding the empirical risk minimiser with squared error loss over the hypothesis class $\mathcal{H} = \{h : h(x) = \beta^T x, \text{ where } \beta \in C\}$, giving explicit forms for any gradients or projections involved.

Paper 2, Section II

31J Mathematics of Machine Learning

(a) Let $\mathcal{H}$ be a hypothesis class of functions $h : \mathcal{X} \to \{-1, 1\}$ with $|\mathcal{H}| > 2$ and $\mathcal{X} = \mathbb{R}^p$. Define the *shattering coefficient* $s(\mathcal{H}, n)$ and the *VC dimension* $\mathrm{VC}(\mathcal{H})$ of $\mathcal{H}$.

(b) Explain why if $\mathcal{H}_1, \mathcal{H}_2$ are hypothesis classes as above, then $s(\mathcal{H}_1 \cup \mathcal{H}_2, n) \leqslant s(\mathcal{H}_1, n) + s(\mathcal{H}_2, n)$.

Let us use the notation that, for a class $\mathcal{F}$ of functions $f : \mathbb{R}^p \to \mathbb{R}$, we write

$$\mathcal{H}_{\mathcal{F}} := \{h : h(x) = \operatorname{sgn} \circ f(x), \text{ where } f \in \mathcal{F}\}$$

for the class of functions derived through composition with the sgn function.

(c) Now let $\mathcal{F}_1 := \{f : f(x) = x^T \beta, \text{ where } \beta \in \mathbb{R}^p\}$. Stating any results from the course you need, show that

$$s(\mathcal{H}_{\mathcal{F}_1}, n) \leqslant (n+1)^p.$$

(d) Next for a class $\mathcal{G}$ of functions $g : \mathbb{R}^p \to \{-1, 1\}$, define for some fixed $m \in \mathbb{N}$,

$$\mathcal{F}_2 := \left\{ f : f(x) = \sum_{j=1}^{m} \alpha_j g_j(x), \text{ where } g_j \in \mathcal{G},\ \alpha \in \mathbb{R}^m \right\}.$$

Show that if $|\mathcal{G}| < \infty$,

$$s(\mathcal{H}_{\mathcal{F}_2}, n) \leqslant (n+1)^m |\mathcal{G}|^m.$$

Show furthermore that even if $|\mathcal{G}| = \infty$, we have

$$s(\mathcal{H}_{\mathcal{F}_2}, n) \leqslant (n+1)^m s(\mathcal{G}, n)^m.$$

[*Hint: Fix* $x_{1:n} \in \mathcal{X}^n$ *and consider* $\mathcal{G}'$ *with* $|\mathcal{G}'| \leqslant s(\mathcal{G}, n)$ *and* $\mathcal{G}'(x_{1:n}) = \mathcal{G}(x_{1:n})$*.*]

(e) Finally let $\mathcal{F}_3$ be the class of functions $f : \mathbb{R}^p \to \mathbb{R}$ given by a neural network with a single hidden layer of m nodes and activation function given by sgn. Show that

$$s(\mathcal{H}_{\mathcal{F}_3}, n) \leqslant (n+1)^{(p+1)m}.$$

Paper 4, Section II

30J Mathematics of Machine Learning

(a) Let $(X_1, Y_1), \ldots, (X_n, Y_n) \in \mathbb{R} \times \mathbb{R}$ be input–output pairs with $n \geqslant 4$. Describe the optimisation problem that a *regression tree algorithm* using a *squared error loss splitting criterion* would take to find the first split point.

(b) Assuming that the inputs are sorted so that $X_1 < \cdots < X_n$, show that the above may be solved in $O(n)$ computational operations.

(c) Now write down the squared error loss empirical risk minimiser $\hat{f}_m : \mathbb{R} \to \mathbb{R}$ over $\mathcal{F} : \{x \mapsto \alpha + x\beta \, : \alpha \in \mathbb{R}, \, \beta \in \mathbb{R}\}$, when trained only on data $(X_1, Y_1), \ldots, (X_m, Y_m)$ for $m \geqslant 2$. [You need not derive it.]

(d) Denote by $\hat{g}_m : \mathbb{R} \to \mathbb{R}$ the equivalent of $\hat{f}_m$ in part (c) when instead training only on $(X_{m+1}, Y_{m+1}), \ldots, (X_n, Y_n)$ for $m \leqslant n-2$. Show carefully how minimising

$$\sum_{i=1}^{m} (Y_i - \hat{f}_m(X_i))^2 + \sum_{i=m+1}^{n} (Y_i - \hat{g}_m(X_i))^2$$

over $m = 2, \ldots, n-2$ may be performed in $O(n)$ computations.

Paper 1, Section II
20H Number Fields

(a) (i) The zeta function $\zeta_K(s)$ of a number field K is the infinite sum $\zeta_K(s) = \sum_{\mathfrak{a} \leqslant \mathcal{O}_K} N(\mathfrak{a})^{-s}$. Show that it factors 'formally' as an infinite product, where the product has a term for each prime ideal of $\mathcal{O}_K$.

[You do not need to show ζ_K converges.]

(ii) Now let $K = \mathbb{Q}(\sqrt{d})$, where $d \neq 0, 1$ and d is square free. Show the zeta function factors 'formally', $\zeta_K(s) = \zeta_{\mathbb{Q}}(s) L(\chi, s)$ where

$$L(\chi, s) = \prod_{p \text{ prime}} (1 - \chi(p) p^{-s})^{-1}$$

for an explicit function χ, which you should determine in terms of how the ideal (p) factorises in $\mathcal{O}_K$.

['Formally' means the terms match up, i.e. you do not need to discuss convergence.]

(b) Let $p \equiv 11 \pmod{12}$ be a prime, and $K = \mathbb{Q}(\sqrt{-p})$.

Show that 3 splits completely in $\mathcal{O}_K$.

Let $\mathfrak{p}_1$ be a factor of the prime ideal (3) of $\mathcal{O}_K$. Define the *class group* Cl_K of K, and explain what it means for the ideal $\mathfrak{p}_1$ of $\mathcal{O}_K$ to have order n in Cl_K.

Suppose $n > 0$ is such that $p > 3^{n+2}$. Show the order of $\mathfrak{p}_1$ in Cl_K does not equal n if n is odd.

Paper 2, Section II
20H Number Fields

(a) The *trace form* for a number field K is the bilinear form $(x, y) := \mathrm{Tr}_{K/\mathbb{Q}}(xy)$.

(i) Prove that the trace form is non-degenerate.

(ii) Let $\alpha_1, \ldots, \alpha_n \in \mathcal{O}_K$ be a basis for $K/\mathbb{Q}$. Let

$$\Delta(\alpha_1, \ldots, \alpha_n) = \det(\mathrm{Tr}_{K/\mathbb{Q}}(\alpha_i \alpha_j)_{i,j}).$$

Show that the minimum absolute value for Δ is a positive integer, obtained precisely when $\alpha_1, \ldots, \alpha_n$ is a $\mathbb{Z}$-basis of $\mathcal{O}_K$.

(b) Let $K = \mathbb{Q}(\sqrt[3]{3})$. Compute the class group of K. You may assume the ring of integers is $\mathbb{Z}[\sqrt[3]{3}]$.

Paper 4, Section II

20H Number Fields

(a) State Minkowski's lemma.

Let $K = \mathbb{Q}(\sqrt{d})$, with $d \in \mathbb{Q}$, $d > 0$, d not a square. Prove that there are infinitely many $\alpha \in \mathcal{O}_K$ with $N(\alpha) < |D_K|^{1/2}$, where D_K is the discriminant of K.

(b) Determine the units in the ring of integers $\mathcal{O}_K$ in the cases (i) $K = \mathbb{Q}(\sqrt{10})$ and (ii) $K = \mathbb{Q}(\sqrt{-3})$. You must prove that your answers are correct.

(c) Let $K = \mathbb{Q}(\zeta)$, where $\zeta^5 = 1$. Determine $\mathcal{O}_K^*$ as an abelian group. [You do not have to describe explicit generators.]

Find explicit elements of $\mathcal{O}_K^*$ which generate a subgroup H of finite index (that is, for which $\mathcal{O}_K^*/H$ is finite).

Paper 1, Section I

1G Number Theory

Determine whether or not each of the following equations has a solution in integers x and y. Briefly describe the results and algorithms you use.

(i) $1007x + 2314y = 37$,

(ii) $2508x + 3211y = 55$,

(iii) $5x^2 + 16xy + 13y^2 = 365$.

[When there are solutions, you are not required to find any.]

Paper 2, Section I

1G Number Theory

State Lagrange's theorem concerning roots of polynomial congruences. Fix a prime number p. We say that d is *good* if the congruence $x^d \equiv 1 \pmod{p}$ has exactly d solutions modulo p. Prove that any divisor of a good number is again good.

Let $n = pq$ where p and q are distinct primes with $\mathrm{GCD}(p-1, q-1) = 10$. For how many bases $b \in (\mathbb{Z}/n\mathbb{Z})^\times$ is n a Fermat pseudoprime to the base b? For how many of these bases is n a strong pseudoprime?

[*The existence of primitive roots may not be assumed without proof.*]

Paper 3, Section I

1G Number Theory

(a) Prove that for $n \geqslant 1$ we have

$$\frac{2^{2n}}{2n+1} \leqslant \binom{2n}{n} \leqslant (2n)^{\sqrt{2n}} \prod_{p \leqslant 2n,\ p \text{ prime}} p.$$

[The formula for the exact power of p dividing $n!$ may be quoted without proof.]

(b) Deduce that $\sum_{p \leqslant x,\ p \text{ prime}} \log p \geqslant \frac{1}{2}x$ for all x sufficiently large.

(c) A positive integer n is called *decisive* if every integer $1 < a < n$ coprime to n is in fact prime. Prove that there are only finitely many decisive numbers.

Paper 4, Section I
1G Number Theory

Compute the continued fraction expansion of $\sqrt{11}$. Show that for all $n \geqslant 0$ the convergents p_n/q_n satisfy

$$p_{n+1} + q_{n+1}\sqrt{11} = \begin{cases} \alpha(p_n + q_n\sqrt{11}) & \text{if } n \text{ is odd,} \\ \beta(p_n + q_n\sqrt{11}) & \text{if } n \text{ is even,} \end{cases}$$

for real numbers α and β which you should determine.

Paper 3, Section II
11G Number Theory

Explain what it means for a positive definite integral binary quadratic form to be *reduced*. Let $d < 0$ be an integer with $d \equiv 0$ or $1 \pmod 4$. Define the class number $h(d)$ and prove that $1 \leqslant h(d) < \infty$.

Let q be a prime number with $q \equiv 3 \pmod 8$. Show that $h(-8q) \geqslant 2$. Further show that if $h(-8q) = 2$ then a prime number p greater than q is represented by $x^2 + 2qy^2$ if and only if $p \equiv \pm 1 \pmod 8$ and p is a quadratic residue mod q.

Paper 4, Section II
11G Number Theory

(a) Define the *Legendre symbol* $\left(\frac{a}{p}\right)$. State and prove Euler's criterion. Deduce a formula for $\left(\frac{-1}{p}\right)$.

(b) Let A be a 2×2 matrix with integer entries. Explain why if p is a prime number then

$$(I + A)^p \equiv I + A^p \pmod p.$$

Taking $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $p = 4k \pm 1$, show that $(-4)^k \equiv 1$ or $2 \pmod p$. Deduce a formula for $\left(\frac{2}{p}\right)$.

(c) Let p be an odd prime number and

$$T = \sum_{a=1}^{p-1} a\left(\frac{a}{p}\right).$$

(i) Show that if $p \equiv 1 \pmod 4$ then $T = 0$.

(ii) Show that if $p > 3$ then $T \equiv 0 \pmod p$.

(d) Show that if $p \equiv 7 \pmod 8$ then the sum of the quadratic residues modulo p in the interval $(0, p/2)$ is $(p^2 - 1)/16$.

Paper 1, Section II

41C Numerical Analysis

Consider the diffusion equation in 2D on a square domain $(x, y) \in [0, 1]^2$

$$\frac{\partial u}{\partial t}(x, y, t) = \nabla^2 u(x, y, t), \tag{1}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian. We assume zero Dirichlet boundary conditions $u(x, 0, t) = u(x, 1, t) = u(0, y, t) = u(1, y, t) = 0$ for all $t \geqslant 0$.

We discretize the domain $[0, 1]^2$ by a regular grid (ih, jh) where $0 \leqslant i, j \leqslant m + 1$ and $h = 1/(m + 1)$.

(a) Show that if we discretize the Laplacian operator ∇^2 by the five-point finite-difference scheme, we get an ordinary differential equation of the form

$$\frac{d\mathbf{u}(t)}{dt} = \frac{1}{h^2}(A_x + A_y)\mathbf{u}(t) \qquad \mathbf{u}(t) \in \mathbb{R}^{m^2}, \tag{2}$$

where $\mathbf{u}_{ij} \approx u(ih, jh)$, and A_x and A_y are two matrices of size $m^2 \times m^2$ that correspond respectively to discretizations of $\partial^2/\partial x^2$ and $\partial^2/\partial y^2$. You should verify that your matrices A_x and A_y commute, i.e., $A_x A_y = A_y A_x$.

(b) Consider the following time-stepping scheme for (2), where $k > 0$ is the time step and $\mu = k/h^2$:

$$\begin{cases} \mathbf{u}^{n+1/2} & = \mathbf{u}^n + \mu A_y \mathbf{u}^{n+1/2} \\ \mathbf{u}^{n+1} & = \mathbf{u}^{n+1/2} + \mu A_x \mathbf{u}^{n+1/2}. \end{cases}$$

(i) Explain why $\mathbf{u}^{n+1}$ can be computed from $\mathbf{u}^n$ using at most $\mathcal{O}(m^2)$ arithmetic operations.

(ii) Show that $\mathbf{u}^{n+1} = C\mathbf{u}^n$ for some matrix C that you should make explicit. Deduce conditions on μ for the method to be stable.

[*Hint: For (ii), you can use the fact that A_x and A_y are diagonalizable in the same orthogonal basis of eigenvectors $(\mathbf{v}^{(p,q)})_{1 \leqslant p,q \leqslant m}$ where $\mathbf{v}^{(p,q)} \in \mathbb{R}^{m^2}$, and that $A_x \mathbf{v}^{(p,q)} = \lambda_p \mathbf{v}^{(p,q)}$ and $A_y \mathbf{v}^{(p,q)} = \lambda_q \mathbf{v}^{(p,q)}$ and $\lambda_p = -4\sin^2(p\pi h/2)$.*]

(c) We consider the following modified discretization method to compute $\mathbf{u}^{n+1}$ from $\mathbf{u}^n$:

$$\begin{cases} \widetilde{\mathbf{u}}^{n+1/2} & = \mathbf{u}^n + \mu A_y \widetilde{\mathbf{u}}^{n+1/2} \\ \widetilde{\mathbf{u}}^{n+1} & = \widetilde{\mathbf{u}}^{n+1/2} + \mu A_x \widetilde{\mathbf{u}}^{n+1/2} \\ \mathbf{u}^{n+1} & = \widetilde{\mathbf{u}}^{n+1} + \mu A_x(\mathbf{u}^{n+1} - \mathbf{u}^n). \end{cases}$$

By writing the method as $\mathbf{u}^{n+1} = D\mathbf{u}^n$ for some matrix D, and analyzing the eigenvalues of D, show that this method is stable for any choice of $\mu > 0$.

Paper 2, Section II

41C Numerical Analysis

Consider the variable coefficient advection equation

$$\frac{\partial u}{\partial t}(x,t) + c(x)\frac{\partial u}{\partial x}(x,t) = 0 \qquad (1)$$

where $x \in (-\infty, \infty)$ and $t \geqslant 0$. Assume that $c(x) > 0$ is 2-periodic, i.e., $c(x+2) = c(x)$. We will seek a 2-periodic solution $u(x,t)$ that satisfies $u(x,t) = u(x+2,t)$ for all t.

(a) Assume $c(x)$ has a finite decomposition in a Fourier basis

$$c(x) = \sum_{n=-d}^{d} \widehat{c}_n e^{i\pi n x} \qquad (\widehat{c}_i = 0 \text{ for } |i| > d).$$

Give an expression for $\widehat{c}_n$ in terms of $c(x)$. Using the fact that $c(x) > 0$ for all x, show that the $(2d+1) \times (2d+1)$ matrix $[\widehat{c}_{n-m}]_{-d \leqslant n,m \leqslant d}$ is Hermitian positive definite.

(b) We seek a solution $u(x,t)$ of (1) of the form

$$u(x,t) = \sum_{n=-d}^{d} \widehat{u}_n(t) e^{i\pi n x}.$$

Let $\widehat{\mathbf{u}}(t) = (\widehat{u}_n(t))_{|n| \leqslant d} \in \mathbb{C}^{2d+1}$. Applying the spectral method to (1) derive an ODE of the form

$$\frac{d\widehat{\mathbf{u}}(t)}{dt} = i\pi B \widehat{\mathbf{u}}(t) \qquad (2)$$

for some matrix B of size $(2d+1) \times (2d+1)$ that you should specify.

(c) Explain why the eigenvalues of B are all real. Deduce that the explicit Euler discretization of (2) is unstable.

[*Hint: you can assume, without proof, that if P and Q are two Hermitian matrices and P is positive definite, then the eigenvalues of PQ are all real.*]

(d) Consider the case $c(x) = 2 + \cos(\pi x) - (1/2)\sin(\pi x)$ and $d = 1$. Form the matrix B and compute its eigenvalues.

Paper 3, Section II

40C Numerical Analysis

Let A be an $n \times n$ real symmetric positive definite matrix and consider the linear system of equations $A\mathbf{x} = \mathbf{b}$, with $\mathbf{b}, \mathbf{x} \in \mathbb{R}^n$. Let $F(\mathbf{x}) = (1/2)\mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x}$.

(a) Define the *steepest descent method with exact line search* to minimize F. Show that for the 2×2 linear system

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}, \quad \mathbf{b} = \mathbf{0} \in \mathbb{R}^2 \quad (\gamma > 1), \tag{1}$$

with the starting point $\mathbf{x}^{(0)} = (\gamma, 1)$, the k-th iterate of this method satisfies

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^*\|_2}{\|\mathbf{x}^{(0)} - \mathbf{x}^*\|_2} = \left(\frac{\kappa - 1}{\kappa + 1}\right)^k \tag{2}$$

where κ is the *condition number* of A that you should define.

Define the *conjugate gradient method.* If the conjugate gradient method is applied to this example, at most how many iterations will be needed to reach $\mathbf{x}^*$?

(b) Return to the case of general $n \times n$ A as specified at the beginning of the question. The *heavy-ball method* to minimize $F(\mathbf{x})$ is defined by the following iterations

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \alpha \nabla F(\mathbf{x}^{(k)}) + \beta(\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}), \tag{3}$$

for some constants $\alpha, \beta > 0$, with the initial point $\mathbf{x}^{(0)} = 0$. Show that $\mathbf{r}^{(k)} \in \mathcal{K}_k(A, \mathbf{b})$ where $\mathbf{r}^{(k)} = \mathbf{b} - A\mathbf{x}^{(k)}$ is the residual at the kth iterate, and $\mathcal{K}_k(A, \mathbf{b})$ is the kth Krylov subspace of A with respect to $\mathbf{b}$.

(c) Let $\mathbf{e}^{(k)} = \mathbf{x}^* - \mathbf{x}^{(k)}$ be the error for the iterates of the heavy-ball method. Show that we can find a matrix M of size $2n \times 2n$ such that

$$\begin{pmatrix} \mathbf{e}^{(k+1)} \\ \mathbf{e}^{(k)} \end{pmatrix} = M \begin{pmatrix} \mathbf{e}^{(k)} \\ \mathbf{e}^{(k-1)} \end{pmatrix}.$$

Your matrix M should be explicit, and depend only on A, α and β. Assuming A is diagonal, show that M can be made block diagonal with 2×2 blocks by an appropriate permutation of its rows and columns (i.e. there is a permutation matrix P such that PMP^T is block diagonal).

(d) Compute the spectral radius of M for the particular A and $\mathbf{b}$ given in (1) and the choice $\alpha = 1/\gamma$ and $\beta = (1 - \sqrt{1/\gamma})^2$. Compare your result with the rate in (2) when $\gamma \gg 1$. [*Hint: To simplify the algebra you may find it helpful to write α in terms of β.*]

Paper 4, Section II

40C Numerical Analysis

(a) Define the *Rayleigh quotient* of a matrix $A \in \mathbb{R}^{n\times n}$ at a vector $\mathbf{x} \in \mathbb{R}^n$. Describe the method of *Rayleigh quotient iteration* to compute an eigenvalue of a matrix.

In the remainder of the question $A \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{R}$ is a simple eigenvalue of A. $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, with $\|\mathbf{u}\|_2 = \|\mathbf{v}\|_2 = 1$, are respectively the left and right eigenvectors of A associated with the eigenvalue λ. We define $s(\lambda) = 1/|\mathbf{u}^T\mathbf{v}|$ to be the *sensitivity* of the eigenvalue λ.

When A is to be regarded as depending on a parameter t the notation $A(t)$ will be used, with corresponding use of $\lambda(t)$, $\mathbf{u}(t)$ and $\mathbf{v}(t)$.

(b) Let $E \in \mathbb{R}^{n\times n}$ be a perturbation matrix and let $\lambda(t)$ be an eigenvalue of $A(t) = A(0) + tE$ with $t \in \mathbb{R}$. Assuming $\lambda(t)$ is differentiable at $t = 0$, show that

$$|\lambda'(0)| \leqslant \frac{\|E\|_2}{|\mathbf{u}(0)^T\mathbf{v}(0)|}, \tag{1}$$

where $\|E\|_2$ is the operator norm of E.

[*Hint: consider* $\mathbf{u}(0)^T A(t)\mathbf{v}(t)$.]

(c) What can you say about the sensitivity $s(\lambda)$ if A is a symmetric matrix? More generally, what can you say if A is a normal matrix?

(d) Let

$$A = \begin{pmatrix} \lambda_1 & 1 & & \\ & \lambda_2 & 1 & \\ & & \ddots & 1 \\ & & & \lambda_n \end{pmatrix},$$

where $\lambda_1 = 1$, and $\lambda_i = 1 - 1/i$ for $i \geqslant 2$. Show that for the eigenvalue $\lambda = \lambda_1 = 1$, the sensitivity $s(\lambda)$ is at least $n!$.

(e) Consider applying Rayleigh quotient iterations to compute the eigenvalue λ of a matrix A. Upon termination of the algorithm, we obtain $\tilde{\mathbf{v}} \in \mathbb{R}^n$, $\|\tilde{\mathbf{v}}\|_2 = 1$ and $\tilde{\lambda} \in \mathbb{R}$ such that

$$\|A\tilde{\mathbf{v}} - \tilde{\lambda}\tilde{\mathbf{v}}\|_2 = \epsilon$$

where ϵ is the machine precision. Show that $|\tilde{\lambda} - \lambda| \lesssim \epsilon s(\lambda)$.

[*Hint: construct a perturbation matrix* E *such that* $(A + E)\tilde{\mathbf{v}} = \tilde{\lambda}\tilde{\mathbf{v}}$ *and use the approximation* $|\lambda(1) - \lambda(0)| \approx |\lambda'(0)|$.]

Paper 1, Section II

34B Principles of Quantum Mechanics

(a) Write down the Hamiltonian for a quantum harmonic oscillator of frequency ω in terms of the creation and annihilation operators $A^\dagger$ and A. You may work in units where $\hbar = 1$. Define the *number operator* N and state all commutation relations amongst A, $A^\dagger$ and N. Show that the eigenvalues of N are: (i) real, (ii) non-negative and (iii) integers.

(b) Consider a system of two independent harmonic oscillators of frequency $\omega = 1$ and $\omega = 2$. The $\omega = 1$ oscillator has creation and annihilation operators $A^\dagger$ and A, while the $\omega = 2$ oscillator has creation and annihilation operators $B^\dagger$ and B.

(i) Find the five lowest eigenvalues of the Hamiltonian H_0 of the combined system and determine the degeneracy of each of them.

(ii) The system is perturbed so that it is now described by the new Hamiltonian $H = H_0 + \lambda H'$, where $\lambda \in \mathbb{R}$ and $H' = A^\dagger A^\dagger B + AAB^\dagger$. Using degenerate perturbation theory, calculate to order λ the energies of the eigenstates associated with the level $E_0 = \frac{9}{2}$. Write down the perturbed eigenstates, to order λ, associated with these perturbed energies. By explicit evaluation show that they are in fact exact eigenstates of H with these energies as eigenvalues.

[In part (b) you may use without proof that for the harmonic oscillator

$$A\,|n\rangle = \sqrt{n}\,|n-1\rangle \qquad \text{and} \qquad A^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,.]$$

Paper 2, Section II

35B Principles of Quantum Mechanics

A two-state quantum system has Hamiltonian H_0 with eigenvectors $|-\rangle$ and $|+\rangle$, and corresponding eigenvalues E_- and $E_+ > E_-$. The system is perturbed by the Hermitian operator ΔH with matrix elements

$$\langle +|\,\Delta H\,|-\rangle = i\lambda\,, \qquad\qquad \langle +|\,\Delta H\,|+\rangle = \langle -|\,\Delta H\,|-\rangle = 0,$$

where λ is a real constant.

(i) Starting from the Schrödinger equation for $H_0 + \Delta H$ and explicitly deriving any necessary results, determine the corrections to the energy eigenstates and eigenvalues in perturbation theory up to linear order in λ.

(ii) Find the exact eigenstates and eigenvalues and show that they agree with the results of perturbation theory up to linear order in λ.

(iii) Determine the radius of convergence of perturbation theory in λ. [*Hint: the square root function has a branch point when its argument vanishes.*]

Paper 3, Section II

33B Principles of Quantum Mechanics

(a) Consider a composite system of several distinguishable particles. Describe how the multiparticle state is constructed from single-particle states. For the case of two identical particles, describe how considering the interchange symmetry leads to the definition of bosons and fermions.

(b) Consider two non-interacting, identical particles, each with spin 1. The single particle, spin-independent Hamiltonian $H(\mathbf{X}_i, \mathbf{P}_i)$ has non-degenerate eigenvalues E_n and wavefunctions $\psi_n(\mathbf{x}_i)$ where $i = 1, 2$ labels the particle and $n = 0, 1, 2, 3, \ldots$. In terms of these single-particle wavefunctions and single-particle spin states $|1\rangle$, $|0\rangle$ and $|-1\rangle$, write down all of the two-particle states and energies for (i) the ground states and (ii) the first excited states.

(c) For the system in part (b), assume now that E_n is a linear function of n. Find the degeneracy of the N^{th} energy level of the two-particle system for: (i) N even and (ii) N odd.

Paper 4, Section II

33B Principles of Quantum Mechanics

(a) A composite system is made of two sub-systems with total angular momenta j_1 and j_2, respectively. Let $\mathbf{J} = \{J_x, J_y, J_z\}$ be the angular momentum operator of the composite system and $|j, m\rangle$ a basis of eigenstates of $\mathbf{J}^2$ and J_z.

(i) Write $\mathbf{J}$ and the associated ladder operators $J_\pm$ in terms of the angular momentum operators $\mathbf{J}_{1,2}$ of each sub-system.

(ii) State the possible values of j in terms of j_1 and j_2 and specify under what conditions it is possible to have $j = 0$.

(iii) Write down all the states of definite j and m that have $m \geqslant j_1 + j_2 - 1$, in terms of the states of the sub-systems $|j_1, m_1\rangle$ and $|j_2, m_2\rangle$.

(iv) Given a pure state, define what it means for the state to be a *product state* and what it means for the state to be an *entangled state*. Specify whether each of the states in (iii) is a product state or an entangled state.

(b) Let $j = j_1 + j_2$. For each of the two states of the system $|j, j\rangle$ and $|j, j-1\rangle$ compute the reduced density matrix of subsystem 1 and the associated entanglement entropy. Comment on the value of the entanglement entropy when $j_1 = j_2$.

(c) Explain why, if it exists, the state with $j = 0$ must be of the form

$$|0, 0\rangle = \sum_{m=-j_1}^{j_1} \alpha_m \, |j_1, m\rangle_1 \, |j_1, -m\rangle_2 \; .$$

By considering $J_+ \, |0, 0\rangle$, determine a relation between α_{m+1} and α_m, and hence find α_m.

[*Units in which $\hbar = 1$ have been used throughout. The states $|j, m\rangle$ obey*

$$J_\pm \, |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)} \, |j, m \pm 1\rangle \; .]$$

Paper 1, Section II

29K Principles of Statistics

Consider a parametric model $\{f(\cdot,\theta) : \theta \in \Theta\}$ satisfying the usual regularity conditions.

(a) Let $\widehat{\theta}_n$ denote a maximum likelihood estimator based on i.i.d. samples $X_1,\ldots,X_n$ from the distribution P_{θ_0}. What is the limiting distribution of $\sqrt{n}(\widehat{\theta}_n - \theta_0)$?

For the remainder of this question, let $\Theta = \mathbb{R}^p$.

(b) Define the *Wald statistic* $W_n(\theta)$ based on a parameter $\theta \in \Theta$. Suppose $\theta_0 \in \Theta$. What is the limiting distribution of $W_n(\theta_0)$ based on i.i.d. samples from P_{θ_0}? State an asymptotically valid confidence region for θ_0 with coverage probability $1-\alpha$.

(c) Suppose $k \leqslant p$. For a fixed $\theta^* \in \Theta$, suppose we wish to test the null hypothesis

$$H_0 : \theta_i = \theta_i^*, \quad \forall 1 \leqslant i \leqslant k,$$

i.e., the first k coordinates of θ are equal to the first k coordinates of θ^*. Define a test statistic (a generalization of the Wald statistic) whose limiting distribution under H_0 is a chi-squared distribution with k degrees of freedom, and rigorously prove the correctness of the limiting distribution. Deduce an asymptotically valid level-α hypothesis test based on the statistic.

[*You may quote any result from the lectures that you need, without proof.*]

Paper 2, Section II

29K Principles of Statistics

(a) Define the correlation $\rho_{X,Y}$ between two random variables (X, Y).

Now suppose $\begin{pmatrix} X \\ Y \end{pmatrix}$ follows a bivariate normal distribution with mean $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and nonsingular covariance matrix $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$.

(b) Derive a formula for an MLE $\widehat{\Sigma} = \begin{pmatrix} \widehat{\Sigma}_{11} & \widehat{\Sigma}_{12} \\ \widehat{\Sigma}_{12} & \widehat{\Sigma}_{22} \end{pmatrix}$ for Σ by solving for a root of the score function. [You may use, without proof, the facts that $\frac{\partial}{\partial A} \log |A| = A^{-1}$ and $\frac{\partial}{\partial A} v^T A v = v v^T$, for a symmetric matrix A and vector v. You do not need to prove that a root of the score function actually corresponds to an MLE.]

(c) Derive an expression for the limiting distribution of $\sqrt{n} \left(\begin{pmatrix} \widehat{\Sigma}_{11} \\ \widehat{\Sigma}_{12} \\ \widehat{\Sigma}_{22} \end{pmatrix} - \begin{pmatrix} \Sigma_{11} \\ \Sigma_{12} \\ \Sigma_{22} \end{pmatrix} \right)$.

[Your answer should be in terms of the entries of Σ. You may use, without proof, the following relations for the bivariate normal:

$$\mathbb{E}[X^4] = 3\Sigma_{11}^2, \quad \mathbb{E}[X^3 Y] = 3\Sigma_{11}\Sigma_{12}, \quad \mathbb{E}[X^2 Y^2] = \Sigma_{11}\Sigma_{22} + 2\Sigma_{12}^2,$$

$$\mathbb{E}[X Y^3] = 3\Sigma_{22}\Sigma_{12}, \quad \mathbb{E}[Y^4] = 3\Sigma_{22}^2.]$$

(d) Now consider the plug-in MLE $\widehat{\rho} := \frac{\widehat{\Sigma}_{12}}{\sqrt{\widehat{\Sigma}_{11}\widehat{\Sigma}_{22}}}$. Derive an expression for the limiting distribution of $\sqrt{n}(\widehat{\rho} - \rho_{X,Y})$.

[*You may quote any result from the lectures that you need, without proof.*]

Paper 3, Section II

28K Principles of Statistics

Consider a classification problem where data are drawn from two different distributions $N(\mu_0, \Sigma_0)$ or $N(\mu_1, \Sigma_1)$, where $\mu_0, \mu_1 \in \mathbb{R}^p$ and $\Sigma_0, \Sigma_1 \in \mathbb{R}^{p\times p}$ are positive definite matrices.

Let $\pi_0 \in (0, 1)$ and $\pi_1 = 1 - \pi_0$.

(a) Define the *Bayes classifier* δ_{π_0}, and show that the decision boundary is linear when $\Sigma_0 = \Sigma_1$, and otherwise quadratic.

(b) Show that for any (Σ_0, Σ_1), the classifier described in (a) is the unique Bayes rule for the prior (π_0, π_1).

(c) Show that there exists some $\pi^* \in (0, 1)$ such that the Bayes classifier corresponding to the prior $(\pi^*, 1 - \pi^*)$ is minimax. Is the prior least favorable?

[*You may quote any result from the lectures that you need, without proof.*]

Paper 4, Section II

28K Principles of Statistics

Suppose $X \sim f$ takes values in $\mathcal{X}$, and h is a reference density on $\mathcal{X}$ from which it is possible to generate i.i.d. samples.

(a) State the steps of the *importance sampling algorithm* and explain why it can be used to approximate $\mathbb{E}[g(X)]$, where g is a function defined on $\mathcal{X}$.

Now consider the following algorithm:

1. Generate $Y_1, \ldots, Y_m$ i.i.d. from h. Let
$$q_i = \frac{f(Y_i)/h(Y_i)}{\sum_{j=1}^m f(Y_j)/h(Y_j)}, \qquad \forall 1 \leqslant i \leqslant m.$$

2. Let X^* be a random variable generated from the discrete distribution on $\{Y_1, \ldots, Y_m\}$, such that $\mathbb{P}(X^* = Y_k) = q_k$ for all $1 \leqslant k \leqslant m$.

(b) Show that X^* converges in distribution to f, i.e., for all $t \in \mathbb{R}$, we have
$$\mathbb{P}(X^* \leqslant t) \overset{a.s.}{\to} F(t),$$
as $m \to \infty$, where F is the cumulative distribution function corresponding to f.

(c) Suppose F is continuous. Prove that the convergence in part (b) is uniform:
$$\sup_{t \in \mathbb{R}} |\mathbb{P}(X^* \leqslant t) - F(t)| \overset{a.s.}{\to} 0,$$
as $m \to \infty$.

[*You may quote any result from the lectures that you need, without proof.*]

Paper 1, Section II
27K Probability and Measure

(a) State and prove Dynkin's lemma.

(b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Show that if $\mathcal{A}_1, \mathcal{A}_2$ are π-systems contained in $\mathcal{F}$ such that

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2) \quad \text{for all } A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2,$$

then the generated σ-algebras $\sigma(\mathcal{A}_1)$ and $\sigma(\mathcal{A}_2)$ are independent.

Paper 2, Section II
27K Probability and Measure

(a) Denote by $L^1(\mathbb{R}^d)$ the space of Lebesgue integrable functions on $\mathbb{R}^d$. For $f \in L^1(\mathbb{R}^d)$ with Fourier transform $\hat{f} \in L^1(\mathbb{R}^d)$, state (without proof) the Fourier inversion theorem and deduce Plancherel's identity for such f from it. Argue that if f is continuous, then the inversion formula holds everywhere.

(b) Show that the integral

$$g(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iux}\frac{4\sin^2(u/2)}{u^2}du, \;\; x \in \mathbb{R},$$

exists, and vanishes whenever $|x| > 1$. What is $\|g\|_2^2$? Justify your answers.

Paper 3, Section II
26K Probability and Measure

(a) State (without proof) Birkhoff's ergodic theorem. Show that convergence in that theorem holds in $L^1(\mu)$, whenever μ is a probability measure. [You may use convergence results for integrals without proof, provided they are clearly stated.]

(b) Now consider $(0, 1]$ equipped with its Borel σ-algebra $\mathcal{B}$ and Lebesgue measure μ. For $A \in \mathcal{B}$, $a \in (0, 1] \setminus \mathbb{Q}$, and

$$\theta(x) = x + a \bmod 1, \quad x \in (0, 1],$$

determine the μ-almost everywhere limit of $S_n(1_A)/n$ as $n \to \infty$, where

$$S_n(1_A) = 1_A + 1_A \circ \theta + \dots 1_A \circ \theta^{n-1}.$$

[You may use without proof that θ is ergodic.]

(c) If $A = (a, b]$ for $0 < a < b < 1$, show that convergence in the last limit in fact occurs *everywhere* on $(0, 1]$. [*Hint: Use your result from (b) with* $A_k = (a + k^{-1}, b - k^{-1}]$ *for all* k *large enough.*]

Paper 4, Section II

26K Probability and Measure

(a) Let $(Y_n : n \in \mathbb{N})$ be an infinite sequence of i.i.d. random variables such that $\mathbb{E}|Y_1| = \infty$. Show that $\limsup_{n\to\infty} |Y_1 + \cdots + Y_n|/n = \infty$ almost surely.

(b) Show that one can find $(Y_n : n \in \mathbb{N})$ as in part (a) but such that $(Y_1 + \cdots + Y_n)/n$ converges weakly to some random variable Z.

[*You may use theorems from lectures provided you state them clearly.*]

Paper 1, Section I
10D Quantum Information and Computation

(a) Assume that you are given a device that is able to clone arbitrary quantum states. Consider two states $|\phi\rangle, |\psi\rangle$ with $|\phi\rangle \neq |\psi\rangle$. Show how the given device can be used to distinguish between these states with arbitrarily high success probability. [You may use without proof any results from the course provided these are clearly stated.]

(b) Assume you are given a device that is able to distinguish the states $|\phi\rangle$ and $|\psi\rangle$ perfectly. Show how this can be used to clone these states. [You can assume that you are able to prepare any computational basis state and implement any unitary operator U.]

(c) Let $\{|\phi_0\rangle, |\phi_1\rangle\}$ and $\{|\psi_0\rangle, |\psi_1\rangle\}$ be two sets of states. Show that there exists a unitary operator U and states $|e_0\rangle$ and $|e_1\rangle$ such that

$$U|\phi_0\rangle|0\rangle = |\psi_0\rangle|e_0\rangle$$
$$U|\phi_1\rangle|0\rangle = |\psi_1\rangle|e_1\rangle$$

if and only if $|\langle\phi_0|\phi_1\rangle| \leqslant |\langle\psi_0|\psi_1\rangle|$.

[*Hint: You can use the fact that for sets of states* $\{|\xi_0\rangle, |\xi_1\rangle\}$ *and* $\{|\eta_0\rangle, |\eta_1\rangle\}$ *with* $\langle\xi_0|\xi_1\rangle = \langle\eta_0|\eta_1\rangle$ *there exists a unitary operator* U *such that* $U|\xi_0\rangle = |\eta_0\rangle$ *and* $U|\xi_1\rangle = |\eta_1\rangle$.]

Paper 2, Section I

10D Quantum Information and Computation

(a) Consider the Bell states

$$|\Phi^+_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad \text{and} \quad |\Phi^-_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)\,. \tag{1}$$

Show that $\langle\Phi^+_{AB}|\, Q \otimes I\, |\Phi^+_{AB}\rangle = \langle\Phi^-_{AB}|\, Q \otimes I\, |\Phi^-_{AB}\rangle$ for any positive semidefinite linear operator Q acting on qubit A.

(b) Suppose you are now given a quantum state which can either be $|\Phi^+_{AB}\rangle$ or $|\Phi^-_{AB}\rangle$ with equal probability.

(i) If you have access to both qubits A and B, can you determine which of the two states you have by doing a measurement on both qubits?

(ii) If you can only access qubit A, can you determine which of the two states you have by doing a measurement on it alone?

(iii) Suppose instead that qubit A is with Alice and qubit B is with Bob. Alice and Bob are at distant locations. They are allowed to do local measurements on the qubits in their possession and can communicate classically with each other. Can they determine the joint state of the two qubits?

(c) Suppose Alice uses the *quantum dense coding protocol* and a third party, Charlie, intercepts the qubit that Alice sends to Bob. Can Charlie infer which of the four bit strings 00, 01, 10 and 11 Alice is trying to send? Justify your answer.

Paper 3, Section I

10D Quantum Information and Computation

(a) Given two positive integers N and a which are coprime to each other (with $1 < a < N$), define the *order* of $a \bmod N$.

(b) For such a pair of integers (a, N), the *modular exponential function* $f : \mathbb{Z} \to \mathbb{Z}_N$, is defined as $f : k \mapsto a^k \bmod N$, where $\mathbb{Z}_N := \{0, 1, \ldots, N-1\}$. Prove that f is a periodic function and determine its period (clearly stating any theorem that you use).

(c) Suppose that we would like to factorise $N = 33$ and we pick $a = 10$. Following the argument presented in the lecture for Shor's algorithm, show how the order of $a \bmod N$ can be used to factorise N. Find the order of $a \bmod N$ by hand and hence factorise N.

(d) Recall that Shor's algorithm for factoring an integer N involves an application of the quantum Fourier transform on m qubits and a subsequent measurement of these m qubits which yields an integer c, where $0 \leqslant c < 2^m$. Suppose we want to factor the number $N = 21$; we pick $a = 8$, $m = 9$ and get the measurement result $c = 256$. Show how you can find the order of $a \bmod N$ from this measurement result. [You should clearly state any results that you use from the lectures.]

Paper 4, Section I

10D Quantum Information and Computation

[In this question you do not need to draw any circuits and you can assume that Alice can perform a measurement on two qubits in the Bell basis.]

(a) Suppose that Alice and Bob share the quantum state

$$|\psi^+_{AB}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$

and can communicate classically. Alice wants to send an arbitrary qubit state to Bob. State the steps that Alice and Bob need to execute to achieve this goal.

(b) Suppose Alice, Bob and Charlie share the following state of three qubits:

$$|\Psi_{ABC}\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right),$$

where the qubits A, B and C are with Alice, Bob and Charlie, respectively. Moreover, Alice has the qubit state $|\alpha\rangle = a\,|0\rangle + b\,|1\rangle$, with $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$. She now performs the Bell measurement on the two qubits in her possession. Depending on the measurement outcome, she asks Bob and Charlie to perform the necessary correction operations on their individual qubits, as is done in the standard teleportation protocol. Show that the final *joint* state of Bob and Charlie at the end of this protocol is either the state $|\varphi_1\rangle := a\,|00\rangle + b\,|11\rangle$ or the state $|\varphi_2\rangle := a\,|00\rangle - b\,|11\rangle$. Show that these states are entangled if and only if $a \neq 0$ and $b \neq 0$.

Paper 2, Section II

15D Quantum Information and Computation

(a) Let $\mathbb{Z}_N = \{0, 1, 2, \ldots, N-1\}$ and let QFT_N denote the quantum Fourier transform mod N. What is the action of QFT_N on $|x\rangle$, where $x \in \mathbb{Z}_N$?

(b) Show that $\mathrm{QFT}_N^2 |x\rangle = |-x\rangle$. Hence show that $\mathrm{QFT}_N^4 = \mathbb{I}$. What can you conclude about the eigenvalues of QFT_N?

(c) Let $f : \mathbb{Z}_{16} \to \mathbb{Z}_4$ be a periodic function such that $f(0) = 2$, $f(1) = 1$, $f(2) = 3$, $f(3) = 0$ and $f(x) = f(x-4)$ for all $x \in \mathbb{Z}_{16}$ (so that $f(4) = 2$ etc.).

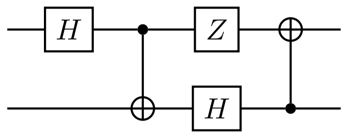
We want to determine the periodicity of the function f using the quantum Fourier transform. The periodicity determination algorithm acts on two registers and involves two measurements – one being a measurement of the second register and one being a measurement of the first register. Work through all the steps of the periodicity determination algorithm, assuming that the outcome of the first measurement is 1 and the outcome of the second measurement is 12. Does the algorithm succeed?

(d) Now consider the same setup as in part (c) but assume that the outcome of the second measurement is 8. Does the algorithm succeed?

Paper 3, Section II

15D Quantum Information and Computation

Consider the following quantum circuit C:

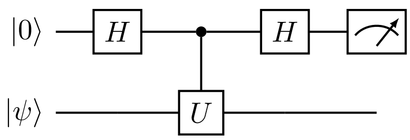


(a) Suppose the state $|0\rangle\,|0\rangle$ is sent through the circuit. What is the state at the output? Suppose each of the two qubits are measured in the computational basis. What is the distribution of measurement outcomes?

(b) Let V denote the unitary operator corresponding to the circuit C. Draw the quantum circuit corresponding to the inverse operator V^{-1}.

(c) The SWAP gate for two qubits is defined as $\mathrm{SWAP}\,|x\rangle\,|y\rangle = |y\rangle\,|x\rangle$, where $x, y \in \{0, 1\}$. Show that the SWAP gate can be implemented as a combination of CNOT gates and draw the corresponding quantum circuit.

(d) Let U be a unitary operator with eigenstate $|\psi\rangle$ such that $U\,|\psi\rangle = e^{i\theta}\,|\psi\rangle$. Consider the following quantum circuit:



Write down the final state at the end of the algorithm. What is the probability that the outcome 1 is observed when the first register is measured in the computational basis? Suppose we are promised that either $U\,|\psi\rangle = |\psi\rangle$ or $U\,|\psi\rangle = -\,|\psi\rangle$, but we have no other information about U and $|\psi\rangle$. Show that the above circuit can be used to determine which of these is the case with certainty.

Paper 1, Section II

19H Representation Theory

What is a *representation* of a finite group G? What does it mean to say that a representation is *irreducible*? What is the *degree* of a representation? What does it mean to say that two representations are *isomorphic*?

Consider the dihedral group D_{2n} of order $2n$ for $n \geqslant 3$. Show directly that every irreducible complex representation of D_{2n} has degree at most 2.

For odd $n \geqslant 3$, explicitly construct $\frac{n+3}{2}$ pairwise non-isomorphic irreducible complex representations of D_{2n}. Justify your answer.

For even $n \geqslant 4$, explicitly construct $\frac{n+6}{2}$ pairwise non-isomorphic irreducible complex representations of D_{2n}. Justify your answer.

Paper 2, Section II

19H Representation Theory

Consider the subset H of $\mathrm{GL}_2(\mathbb{F}_{11})$ consisting of matrices of the form

$$\begin{pmatrix} a^2 & b \\ 0 & 1 \end{pmatrix} \text{ with } a, b \in \mathbb{F}_{11} \text{ and } a \neq 0.$$

Show that H is a non-abelian group of order 55 with 7 conjugacy classes and construct its character table. [*You may assume standard results from the course and that* 2 *is a generator of the cyclic group* $\mathbb{F}_{11}^{\times}$.]

Paper 3, Section II

19H Representation Theory

(a) State and prove Burnside's lemma. Deduce that if a finite group G acts 2-transitively on a set X then the corresponding permutation representation $\mathbb{C}X$ decomposes as a direct sum of two non-isomorphic irreducible representations.

(b) Let $G = S_n$ act naturally on the set $X = \{1, \ldots, n\}$. For each non-negative integer r, let X_r be the set of all r-element subsets of X equipped with the natural action of G, and π_r be the character of the corresponding permutation represention. If $0 \leqslant l \leqslant k \leqslant n/2$, show that

$$\langle \pi_k, \pi_l \rangle_G = l + 1.$$

Deduce that $\pi_r - \pi_{r-1}$ is a character of an irreducible representation for each $1 \leqslant r \leqslant n/2$.

What happens for $r > n/2$?

Paper 4, Section II

19H Representation Theory

State Schur's lemma.

What is a *complex representation* of a topological group G? What does it mean to say a complex representation (ρ, V) of G is *unitary*?

Explain why every complex representation of S^1 is unitary. Deduce that every complex representation of S^1 is a direct sum of 1-dimensional representations.

Let G be the group of 3×3 upper unitriangular real matrices

$$G := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}$$

under matrix multiplication. Let Z be the centre of G and Z_0 the cyclic subgroup of Z given by

$$Z_0 = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| z \in \mathbb{Z} \right\} \leqslant Z = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| z \in \mathbb{R} \right\}.$$

By considering elements of the form $g^{-1}h^{-1}gh$ with $g, h \in G$, show that every 1-dimensional representation of G has kernel containing Z.

Show that any complex representation (ρ, V) of G/Z_0 decomposes as a direct sum of subrepresentations $(\rho_i, V_i)_{i=1,\ldots,d}$ with the property that

$$\mathrm{Res}^{G/Z_0}_{Z/Z_0} \rho_i = \theta_i \, \mathrm{id}_{V_i}$$

for some distinct 1-dimensional representations $\theta_1, \ldots, \theta_d$ of Z/Z_0. By considering $\det \rho_i$, or otherwise, deduce that $d = 1$ and that θ_1 is the trivial representation. Hence show that G/Z_0 does not have a faithful representation.

Paper 1, Section II
24F Riemann Surfaces

Let D be a domain in $\mathbb{C}$. What is a *germ* on D? Define the *space of germs* $\mathcal{G}$ over D. Briefly describe the topology, the forgetful map $\pi : \mathcal{G} \to D$ and the complex structure on $\mathcal{G}$, all without proof. Define the evaluation map $\mathcal{E} : \mathcal{G} \to \mathbb{C}$, and prove that $\mathcal{E}$ is analytic.

Let D be the result of removing the eighth roots of unity from $\mathbb{C}$, and consider the function element $w = \sqrt{z^8 - 1}$ defined over D. Give an explicit gluing construction of the component R of the space of germs corresponding to w. You should construct the evaluation map and the forgetful map on R, and exhibit an analytic embedding $\Phi : R \hookrightarrow \mathcal{G}$. [You do not need to prove that the image of Φ is a component of $\mathcal{G}$.]

Assume that R can be embedded into a compact Riemann surface $\overline{R}$ by adding finitely many points. Assume, furthermore, that the forgetful map π extends to a meromorphic function $\bar{\pi} : \overline{R} \to \mathbb{C}_\infty$. How many points are in $\overline{R} \setminus R$? What is the genus of $\overline{R}$? [You may use standard theorems from the course, as long as you state them carefully.]

Paper 2, Section II
24F Riemann Surfaces

State the valency theorem, and define the *degree* $\deg f$ of an analytic map f of compact Riemann surfaces.

Consider a rational function f with derivative f'. Define the *degree* of f, and prove that $\deg f - 1 \leqslant \deg f' \leqslant 2 \deg f$. Give examples to show that these bounds can be achieved, for every possible value of $\deg f \geqslant 1$.

Consider a non-constant elliptic function g with derivative g'. Define the *degree* of g, and prove that $\deg g + 1 \leqslant \deg g' \leqslant 2 \deg g$. Give examples to show that these bounds can be achieved, for every odd value of $\deg g \geqslant 3$. [You may use properties of standard examples of elliptic functions without proof.]

Paper 3, Section II
23F Riemann Surfaces

State the uniformisation theorem.

Write down a list of all Riemann surfaces uniformised by $\mathbb{C}$ and $\mathbb{C}_\infty$, and prove that your list is complete. [You may assume that, if a Riemann surface R is uniformised by a Riemann surface X, then R is conformally equivalent to the quotient of X by a group of conformal equivalences of X acting freely and properly discontinuously. You may also assume standard facts about the groups of conformal equivalences of $\mathbb{C}$ and $\mathbb{C}_\infty$.]

Prove that any domain $D \subseteq \mathbb{C}$ with a complement containing more than one point is uniformised by the open unit disc $\mathbb{D}$.

Suppose there is a holomorphic embedding $\mathbb{C}_* \to R$, where R is a compact Riemann surface. Prove that R is conformally equivalent to the Riemann sphere.

Paper 1, Section I

5J Statistical Modelling

Consider a possibly biased coin. Suppose the probability of flipping a head is $0 < p < 1$ and p is unknown. Let $r > 0$ be given. In a sequence of flips, let X be the total number of tails when r heads are reached. Show that

$$\mathbb{P}(X = x) = \binom{x+r-1}{x}(1-p)^x p^r, \ x = 0, 1, \ldots .$$

Show that this is a one-parameter exponential family. Find its natural parameter, sufficient statistic, and cumulant function, and compute the mean and variance of X in terms of p.

Paper 2, Section I

5J Statistical Modelling

Explain the following R commands in words, then write down the model that is being fitted.

```
> n <- 100
> p <- 2
> X <- matrix(rnorm(n * p), nrow = n, ncol = p)
> Y <- rbinom(n, size = 1, prob = 0.5)
> sum(Y)
[1] 48
> fit1 <- glm(Y ~ X, family = binomial)
> sum(predict(fit1, type = "response"))
[1] 48
```

Explain why the output of the last command should be exactly the same as the output of `sum(Y)` by writing down the likelihood function of the model.

Do you expect the following command to output exactly 48, too? If not, do you expect it to be very different from 48? Justify your answers.

```
> fit2 <- glm(Y ~ X, family = binomial(probit))
> sum(predict(fit2, type = "response"))
```

Paper 3, Section I

5J Statistical Modelling

Write down the density function of a one-parameter exponential family with natural parameter θ and sufficient statistic Y. Define the *deviance* $D(\theta_1, \theta_2)$ from θ_1 to θ_2, and show that it is equal to

$$D(\theta_1, \theta_2) = 2\{(\theta_1 - \theta_2)\mu_1 - K(\theta_1) + K(\theta_2)\},$$

where μ_1 is the mean parameter corresponding to θ_1 and $K(\cdot)$ is the cumulant function of the exponential family.

Derive the deviance from the Poisson distribution with mean μ_1 to the Poisson distribution with mean μ_2, and find the second order Taylor approximation of the deviance as $\mu_2 \to \mu_1$. [*Hint: Recall that if Y follows a Poisson distribution with mean μ, then* $\mathbb{P}(Y = k) = \mu^k e^{-\mu}/k!$, $k = 0, 1, \ldots$.]

Paper 4, Section I

5J Statistical Modelling

Below is a simplified 1993 dataset of US cars. The columns list make, model, price (in \$1000), miles per gallon, number of passengers, length and width in inches, and weight (in pounds). The data are displayed in R as follows (abbreviated):

```
> cars
         make   model price mpg psngr length width weight
1       Acura Integra  15.9  31     5    177    68   2705
2       Acura  Legend  33.9  25     5    195    71   3560
3        Audi      90  29.1  26     5    180    67   3375
          ...             ...                         ...
91 Volkswagen Corrado  23.3  25     4    159    66   2810
92      Volvo     240  22.7  28     5    190    67   2985
93      Volvo     850  26.7  28     5    184    69   3245
```

It is reasonable to assume that prices for different makes are independent. How would you instruct R to model the logarithm of the price as a linear combination of an error term and

(i) an intercept;

(ii) an intercept and all other quantitative properties of the cars;

(iii) an intercept, all other quantitative properties of the cars, and the make of the cars?

Suppose the fitted models are assigned to objects `fit1`, `fit2`, and `fit3`, respectively. Suppose R provides the following analysis of variance table for these models:

```
> anova(fit1, fit2, fit3)

[...]

  Res.Df    RSS Df Sum of Sq       F    Pr(>F)
1     92 8584.0
2     87 3349.1  5    5234.9 69.7334 < 2.2e-16 ***
3     56  840.8 31    2508.3  5.3891 2.541e-08 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

What are your conclusions about the statistical models in `fit1`, `fit2` and `fit3` based on this table? Explain how you can determine the number of unique car manufacturers in this dataset from this table.

Paper 1, Section II

13J Statistical Modelling

Let X be a fixed $n \times p$ design matrix with full column rank. Let H be the projection matrix onto the column space of X. Suppose the n-vector of response Y satisfies $Y \sim \mathrm{N}(\mu, \sigma^2 I_n)$ where the n-vector μ is fixed but unknown. Let $Y^* \sim \mathrm{N}(\mu, \sigma^2 I_n)$ be another random vector that has the same distribution as Y but is independent of Y.

(i) Show that

$$\mathbb{E}(\|HY - Y^*\|^2) = \|(I-H)\mu\|^2 + (n+p)\sigma^2.$$

Explain why the above identity is an example of the bias-variance tradeoff. [You may use without proof the fact that H is a projection matrix with rank p.]

(ii) Suppose σ^2 is known. Show that Mallows' C_p, given by

$$C_p = \|Y - HY\|^2 + 2p\sigma^2,$$

is an unbiased estimator of $\mathbb{E}(\|HY - Y^*\|^2)$.

For the rest of this question, suppose $\mu = X\beta$ for some unknown p-vector β and σ^2 is unknown.

(iii) Write down the $(1-\alpha)$-level confidence ellipsoid for β.

(iv) Recall *Cook's distance* for the observation (X_i, Y_i) (where X_i^T is the ith row of X) is a measure of the influence of (X_i, Y_i) on the fitted values. Give the precise definition of Cook's distance and give its interpretation in terms of the confidence ellipsoid for β.

(v) In the model above with $n = 100$ and $p = 4$, you notice that one observation has Cook's distance 3.1. Would you be concerned about the influence of this observation? Justify your answer.

[*Hint: You may find some of the following facts useful:*

1. *If* $Z \sim \chi^2_4$, *then* $\mathbb{P}(Z \leqslant 1.06) = 0.1$, $\mathbb{P}(Z \leqslant 7.78) = 0.9$.
2. *If* $Z \sim F_{4,96}$, *then* $\mathbb{P}(Z \leqslant 0.26) = 0.1$, $\mathbb{P}(Z \leqslant 2.00) = 0.9$.
3. *If* $Z \sim F_{96,4}$, *then* $\mathbb{P}(Z \leqslant 0.50) = 0.1$, $\mathbb{P}(Z \leqslant 3.78) = 0.9$.]

Paper 4, Section II
13J Statistical Modelling

The data frame `worldcup22` contains information about the matches played in a sports competition, including for each team in the match the starting formations (indicated by letters A-L), the expected goals (xg) and the actual goals. In the questions below we will assume that the match results are independent.

```
> worldcup22
     team1   team2 team1_xg team2_xg team1_form team2_form team1_goal team2_goal
1    Qatar Ecuador      0.3      1.2          I          H          0          2
2  England IR Iran      2.1      1.4          E          J          6          2
       ...                       ...                                         ...
63 Croatia Morocco      0.7      1.2          E          F          2          1
64   Japan  France      3.3      2.2          F          E          3          3
> fit1 <- glm(team1_goal ~ team1_form + team2_form, worldcup22,
              family = poisson)
```

(i) Let Y denote the response vector and X denote the design matrix for `fit1`. Write down the likelihood function that is maximized by the command above. [Recall that if Y follows a Poisson distribution with mean μ, then $\mathbb{P}(Y = k) = \mu^k e^{-\mu}/k!$, $k = 0, 1, \ldots$.]

(ii) Comment on the following abbreviated summary of `fit1`. Is there enough information to conclude that the formation of `team1` does not affect its goals? If not, what is the name of the statistical procedure you can use to test this hypothesis?

```
> summary(fit1)
              Estimate Std. Error z value Pr(>|z|)
(Intercept)      1.890      0.581     3.3    0.001 **
team1_formB     -0.672      0.595    -1.1    0.259
team1_formC    -17.865   2446.075     0.0    0.994
team1_formD      0.595      1.293     0.5    0.645
team1_formE     -0.361      0.441    -0.8    0.413
team1_formF     -0.098      0.414    -0.2    0.812
team1_formG     -1.120      1.089    -1.0    0.304
team1_formH     -0.332      0.490    -0.7    0.498
team1_formI     -1.855      1.104    -1.7    0.093 .
team1_formJ      0.285      0.830     0.3    0.731
team2_formK    -18.831   3467.859     0.0    0.996
team2_formB     -1.199      0.565    -2.1    0.034 *
team2_formC     -1.792      1.080    -1.7    0.097 .
team2_formL     -0.905      0.558    -1.6    0.105
team2_formE     -1.482      0.478    -3.1    0.002 **
team2_formF     -1.464      0.504    -2.9    0.004 **
team2_formH     -0.728      0.494    -1.5    0.140
team2_formI     -0.980      0.588    -1.7    0.095 .
team2_formJ     -0.143      0.612    -0.2    0.816
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```


[QUESTION CONTINUES ON THE NEXT PAGE]

(iii) Expected goals (xg) is a new metric in sports analytics that computes the number of goals a team should have scored based on the quality of the chances created. State the following two hypotheses mathematically: (a) H_1: `team1_goal` has mean `team1_xg`; (b) H_2: `team1_goal` follows a Poisson distribution with mean `team1_xg`. Then name the result in probability theory that suggests `team1_goal` should approximately follow a Poisson distribution.

(iv) An analyst fitted the following model to test H_1. Does the model fit suggest evidence against H_1? Give one reason why we should be skeptical about the standard errors in the table.

```
> fit2 <- lm(team1_goal ~ team1_xg - 1, worldcup22)
> summary(fit2)
Coefficients:
        Estimate Std. Error t value Pr(>|t|)
team1_xg  1.15790    0.08643    13.4   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

(v) The analyst then fitted the following model and computed the 95% confidence interval for the coefficients. Explain why the observation that the confidence interval for `log(team1_xg)` contains 1 does not directly imply that H_2 cannot be rejected at the 5% significance level.

```
> fit3 <- glm(team1_goal ~ log(team1_xg), worldcup22, family = poisson)
> confint(fit3)
                  2.5 %    97.5 %
(Intercept)  -0.1387542 0.3836497
log(team1_xg)  0.6691166 1.3395731
```

Paper 1, Section II

36A Statistical Physics

(a) What is meant by the *microcanonical*, *canonical* and *grand canonical ensembles*? Under what conditions is the choice of ensemble irrelevant?

(b) Consider a classical particle of mass m moving non-relativistically in two-dimensional space enclosed inside a circle of radius R and attached by a spring to the centre. The particle therefore moves in a potential

$$V(r) = \begin{cases} \frac{1}{2}\kappa r^2 & \text{for } r < R\,, \\ \infty & \text{for } r \geqslant R\,, \end{cases}$$

where κ is the spring constant and $r^2 = x^2 + y^2$. The particle is coupled to a heat reservoir at temperature T.

(i) Calculate the partition function for the particle.

(ii) Calculate the average energy $\langle E \rangle$ and the average potential energy $\langle V \rangle$ of the particle.

(iii) Compute $\langle E \rangle$ in the two limits $\frac{1}{2}\kappa R^2 \gg k_B T$ and $\frac{1}{2}\kappa R^2 \ll k_B T$. How do these two results compare with what is expected from equipartition of energy?

(iv) Compute the partition function for a collection of N identical non-interacting such particles.

Paper 2, Section II

37A Statistical Physics

A simple one-dimensional model of a rubber molecule consists of a chain of n links, where n is fixed. Each link has a fixed length a and can be oriented in either the positive or negative direction. A unique state i of the molecule is specified by giving the orientation of each link and the molecule's length in this state is l_i. If n_+ links are oriented in the positive direction and n_- in the negative direction, then $n = n_+ + n_-$ and the length of the molecule is $l = (n_+ - n_-)a$. All configurations have the same energy.

(a) What is the range of possible values of l? What is the number of states of the molecule for fixed n_+ and n_-?

(b) Now consider an ensemble with $A \gg 1$ copies of the molecule in which a_i members are in state i. Write down an expression for the mean length L. By introducing Lagrange multipliers τ and α show that the most probable configuration for the $\{a_i\}$ with given L is found by maximising

$$\ln\left(\frac{A!}{\prod_i a_i!}\right) + \tau \sum_i a_i\, l_i - \alpha \sum_i a_i\,.$$

Hence show that the most probable configuration has

$$p_i = e^{\tau l_i}/Z\,,$$

where p_i is the probability for finding an ensemble member in state i and Z is the partition function which should be defined.

(c) Show that Z can be expressed as

$$Z = \sum_l g(l)\, e^{\tau l}$$

where the meaning of $g(l)$ should be explained. Hence show that

$$Z = \sum_{n_+=0}^{n} \frac{n!}{n_+!n_-!} \left(e^{\tau a}\right)^{n_+} \left(e^{-\tau a}\right)^{n_-}\;, \qquad n_+ + n_- = n\,.$$

(d) Show that the free energy $G = -k_B T \ln Z$ for the system is

$$G = -nk_B T \ln\left(2\cosh \tau a\right)\,,$$

where k_B is the Boltzmann constant and T is the temperature. Hence show that

$$L = -\frac{1}{k_B T}\left(\frac{\partial G}{\partial \tau}\right) \quad \text{and} \quad \tanh \tau a = \frac{L}{na}\,.$$

(e) Why is the tension f in the rubber molecule equal to $k_B T \tau$? [Here f and L are analogous to, respectively, pressure p and volume V in three-dimensional systems, and G is the Gibbs free energy because the setup corresponds to a system with fixed tension rather than with a fixed length.]

(f) Now assume that $na \gg L$. Show that the chain satisfies Hooke's law $f \propto L$. What happens if f is held constant and T is increased?

Paper 3, Section II

35A Statistical Physics

(a) State the formula for the Bose-Einstein distribution for the mean occupation numbers n_r of discrete single-particle states r with energies $E_r \geqslant 0$ in a gas of identical ideal Bosons in terms of $\beta = 1/k_B T$ and the chemical potential μ. Write down expressions for the total particle number N and the total energy E when the single-particle states can be treated as continuous with energies $E \geqslant 0$ and density of states $g(E)$.

(b) Consider the bosonic vibrational modes (phonons) in a two-dimensional crystal with dispersion relation $\omega = C|\mathbf{k}|^\alpha$, where ω is the frequency, $\mathbf{k}$ is the wavevector, and $C > 0$ and $0 < \alpha < 2$ are constants. The crystal is square with area A.

(i) Show that the density of states is

$$g(\omega) = B\omega^b \,,$$

where B and b are constants that you should determine. [You may assume that the phonons have two polarizations.]

(ii) Calculate the Debye frequency ω_D by identifying the number of single-phonon states with the total number of degrees of freedom $2n$, where n is the number of atoms in the crystal. Find the Debye temperature T_D.

(iii) Derive an expression for the total energy, leaving your answer in integral form with the integral over $x = \beta\hbar\omega$.

(iv) Now consider the case $\alpha = 1/2$. Calculate the heat capacity at constant volume C_V in the limit $T \gg T_D$. Show that $C_V \sim T^d$ in the limit $T \ll T_D$, where d is a real number that you should determine. Comment on these two results.

Paper 4, Section II

35A Statistical Physics

(a) Give *Clausius' statement* of the second law of thermodynamics and *Kelvin's statement* of the second law of thermodynamics. Show that these two statements are equivalent.

Throughout the rest of this question you should consider a classical ideal gas and assume that the number of particles is fixed.

(b) Write down the equation of state for an ideal gas. Write down an expression for its internal energy in terms of the heat capacity at constant volume C_V.

(c) Describe the meaning of an *adiabatic process*. Using the first law of thermodynamics, derive the relationship between p and V for an adiabatic process occurring in an ideal gas.

(d) Consider a cycle involving an ideal gas and consisting of the following four reversible steps:

$A \to B$: Adiabatic compression;
$B \to C$: Expansion at constant pressure with heat in Q_1;
$C \to D$: Adiabatic expansion;
$D \to A$: Cooling at constant volume with heat out Q_2.

(i) Sketch this cycle in the (p, V)-plane and in the (T, S)-plane. Derive equations for the curves DA and BC in the (T, S)-plane.

(ii) Derive an expression for the efficiency, $\eta = W/Q_1$, where W is the work out, in terms of the temperatures T_A, T_B, T_C, T_D at points A, B, C, D, respectively.

Paper 1, Section II

30K Stochastic Financial Models

Let U be a smooth, increasing and concave function on $\mathbb{R}$.

(a) Given a vector space $\mathcal{X}$ of random variables, define a function F by

$$F(y) = \sup_{X \in \mathcal{X}} \mathbb{E}[U(X + y)],$$

and suppose that for all constants $y \in \mathbb{R}$ the supremum is achieved by some $X_y \in \mathcal{X}$. Show that F is increasing and concave.

(b) Given a constant m and a random variable Z such that $\mathbb{E}(Z) = 0$, define a function G by

$$G(s) = \mathbb{E}[U(m + sZ)].$$

Show that G is concave. Show that G is decreasing on $[0, \infty)$.

Now consider a market with interest rate $r \geqslant 0$ and d risky assets, such that the vector of time-n prices S_n evolves as

$$S_n = (1 + r)S_{n-1} + \xi_n.$$

Assume that the sequence $(\xi_n)_{n \geqslant 1}$ of $\mathbb{R}^d$-valued random vectors is IID and generates the filtration. Given initial wealth X_0, an investor's time-n wealth X_n evolves as

$$X_n = (1 + r)X_{n-1} + \theta_n^\top \xi_n$$

for $n \geqslant 1$, where the trading strategy $(\theta_n)_{n \geqslant 1}$ is previsible. Given a time horizon $N \geqslant 1$, the investor seeks a trading strategy to maximise $\mathbb{E}[U(X_N)]$.

(c) Write down the Bellman equation for the investor's value function V. Assuming that an optimal portfolio exists at each time-step, show that $V(n, \cdot)$ is increasing and concave for all $0 \leqslant n \leqslant N$.

(d) Now assume that for all $n \geqslant 1$, ξ_n has the multi-variate Gaussian $N(b, \Sigma)$ distribution, where $b \neq 0$ and Σ is positive definite. Assuming the existence of a unique optimal time-n portfolio θ_n^* for each $1 \leqslant n \leqslant N$, show that there exist non-negative random variables λ_n such that $\theta_n^* = \lambda_n \Sigma^{-1} b$.

[You may use the mutual fund theorem or the Gaussian integration-by-parts formula without proof.]

Paper 2, Section II

30K Stochastic Financial Models

Let $(M_n)_{n\geqslant 0}$ be a martingale with respect to a filtration $(\mathcal{F}_n)_{n\geqslant 0}$.

(a) Given an $\mathcal{F}_0$-measurable X_0 and a previsible process $(A_n)_{n\geqslant 1}$, let

$$X_n = X_0 + \sum_{k=1}^{n} A_k(M_k - M_{k-1})$$

for $n \geqslant 1$. Assuming that X_n is integrable for each $n \geqslant 0$, show that $(X_n)_{n\geqslant 0}$ is a martingale.

(b) Let T be a stopping time and let

$$Y_n = M_{\min\{n,T\}}$$

for $n \geqslant 0$. Show that $(Y_n)_{n\geqslant 0}$ is a martingale.

(c) Let $(X_n)_{n\geqslant 0}$ be as in part (a) and suppose that both X_n and M_n are square-integrable for each $n \geqslant 0$. Suppose that $\mathrm{Var}(M_n|\mathcal{F}_{n-1}) > 0$ almost surely for each $n \geqslant 1$. Show that

$$A_n = \frac{\mathrm{Cov}(X_n, M_n|\mathcal{F}_{n-1})}{\mathrm{Var}(M_n|\mathcal{F}_{n-1})}$$

for all $n \geqslant 1$.

Let $(\xi_n)_{n\geqslant 1}$ be a sequence of independent random variables such that for all $n \geqslant 1$, $\mathbb{P}(\xi_n = +1) = \frac{1}{2} = \mathbb{P}(\xi_n = -1)$. Suppose that the filtration $(\mathcal{F}_n)_{n\geqslant 0}$ is generated by $(\xi_n)_{n\geqslant 1}$.

(d) Show that there exists a previsible process $(B_n)_{n\geqslant 1}$ such that

$$M_n = M_0 + \sum_{k=1}^{n} B_k \xi_k$$

for all $n \geqslant 0$.

(e) Now show for any bounded stopping time T that

$$\mathbb{E}[M_T^2] = M_0^2 + \mathbb{E}\left[\sum_{k=1}^{T} B_k^2\right].$$

Paper 3, Section II

29K Stochastic Financial Models

Let $(W_t)_{t \geqslant 0}$ be a Brownian motion, and let a and b be positive constants.

(a) Let $B_0 = 0$ and $B_t = tW_{1/t}$ for $t > 0$. Show that $(B_t)_{t \geqslant 0}$ is a Brownian motion. [You may use without proof a characterisation of Brownian motion as a Gaussian process. You may also use without proof the fact that $W_t/t \to 0$ almost surely as $t \to \infty$.]

(b) Prove that $\mathbb{P}(\sup_{0 \leqslant s \leqslant t}(W_s - as) \leqslant b) = \mathbb{P}(\sup_{u \geqslant 1/t}(W_u - bu) \leqslant a)$.

(c) Use the reflection principle and the Cameron–Martin theorem to show that

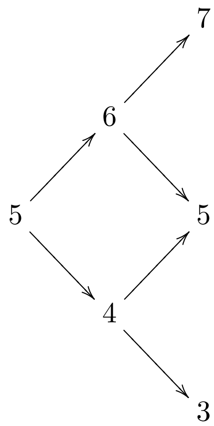
$$\mathbb{P}(\sup_{0 \leqslant s \leqslant t}(W_s - as) \leqslant b) = \mathbb{P}(W_t - at \leqslant b) - e^{-2ab}\mathbb{P}(W_t - at \leqslant -b).$$

(d) Let $T = \sup\{t \geqslant 0 : W_t - at > b\}$ with the convention that $\sup \emptyset = 0$. Find $\mathbb{P}(T \leqslant t)$ in terms of the standard normal distribution function Φ.

Paper 4, Section II

29K Stochastic Financial Models

Consider the following two-period market model. There is a single risky stock with prices $(S_n)_{n \in \{0,1,2\}}$ given by



where in each period the price is equally likely to go up as to go down.

(a) Suppose that the interest rate $r = 1/6$. Find an arbitrage $(\varphi_n)_{n \in \{1,2\}}$.

For the rest of the problem, suppose $r = 1/8$.

(b) Find the time-0 no-arbitrage price of a European put option maturing at $T = 2$ with strike $K = 5$. How many shares of the stock should be held in the first period to replicate the payout of the put?

(c) Find the time-0 no-arbitrage price of a European call option maturing at $T = 2$ with strike $K = 5$.

(d) Now find the time-0 no-arbitrage price of an American put option maturing at $T = 2$ with strike $K = 5$. What is an optimal exercise policy?

Paper 1, Section I

2F Topics In Analysis

(a) State and prove the theorem of Liouville on approximation of algebraic numbers.

(b) If u, v are coprime positive integers and p, q are coprime positive integers with $q > v$, show that

$$\left|\frac{p}{q} - \frac{u}{v}\right| > \frac{1}{q^2}.$$

(c) Show that, if $a_j \in \mathbb{Q}$, $a_j > 0$ and $\sum_{j=1}^{\infty} a_j$ converges, then we can find a strictly increasing sequence of positive integers $n(j)$ such that $\sum_{j=1}^{\infty} a_{n(j)}$ is transcendental.

Paper 2, Section I

2F Topics In Analysis

In this question we consider Γ, the collection of closed paths γ not passing through 0, that is to say, continuous functions $\gamma : [0,1] \to \mathbb{C} \setminus 0$ with $\gamma(0) = \gamma(1)$.

Define the *winding number* $w(\gamma, 0)$ of $\gamma \in \Gamma$. If $\gamma \in \Gamma$, $\phi : [0,1] \to \mathbb{C}$ is continuous with $\phi(0) = \phi(1)$ and $|\gamma(t)| > |\phi(t)|$ for all $t \in [0,1]$, what can we say about $w(\gamma + \phi, 0)$?

Explain what it means to say that γ_0, γ_1 are *homotopic by paths in* Γ.

State a theorem on the winding number of homotopic paths and use it to prove the fundamental theorem of algebra and the non-existence of retractions for discs.

Paper 3, Section I

2F Topics In Analysis

State Runge's theorem on polynomial approximation.

Which of the following statements are true and which false? Give reasons.

(i) Let $E = \{x + iy : x, y \geqslant 0\}$ and Ω be an open set containing E. Then, if $f : \Omega \to \mathbb{C}$ is analytic, we can find a sequence of polynomials converging uniformly on E to f.

(ii) Let $E = \{x + iy : x, y \geqslant 0\}$ and Ω be an open set containing E. Then, if $f : \Omega \to \mathbb{C}$ is analytic, we can find a sequence of polynomials converging pointwise on E to f.

(iii) Suppose Ω is open, K_1, K_2 are compact subsets of Ω, $f : \Omega \to \mathbb{C}$ is analytic and there exist polynomials $P_{j,n}$ with $P_{j,n} \to f$ uniformly on K_j. Then there exist polynomials P_n with $P_n \to f$ uniformly on $K_1 \cup K_2$.

(iv) Let $I = \{x + iy : 1 \geqslant x \geqslant 0, y = 0\}$. If $f : I \to \mathbb{C}$ is continuous, then we can find polynomials P_n such that $P_n \to f$ uniformly on I.

Paper 4, Section I

2F Topics In Analysis

We say that a function $f : X \to X$ has a fixed point if there exists an $x \in X$ with $f(x) = x$.

(i) Use the intermediate value theorem to show that, if $f : [0,1] \to [0,1]$ is continuous, it has a fixed point. Show also that, if $0, 1 \in f([0,1])$, then f is surjective.

(ii) Suppose that A and B are homeomorphic subsets of $\mathbb{R}^2$. Show that, if every continuous function $g : A \to A$ has a fixed point, then so does every continuous function $f : B \to B$.

(iii) State Brouwer's fixed point theorem for the closed unit disc $\bar{D}$.

(iv) Show that the closed unit disc is not homeomorphic to the annulus

$$A = \{(x, y) \in \mathbb{R}^2 : 1 \leqslant x^2 + y^2 \leqslant 2\}.$$

(v) Suppose that B is a subset of $\mathbb{R}^2$ containing at least two points. If every continuous function $g : B \to B$ has a fixed point, does it follow that B is homeomorphic to the closed unit disc? Give reasons.

Paper 2, Section II

11F Topics In Analysis

(a) State and prove the Baire Category Theorem.

(b) Consider the set $C^\infty([0,1])$ of infinitely differentiable functions on $[0,1]$. Show that

$$d(f, g) = \sum_{r=0}^{\infty} 2^{-r} \min\{1, \|f^{(r)} - g^{(r)}\|_\infty\}$$

is a well defined metric on $C^\infty([0,1])$ and that it is complete.

(c) Show that, if we use this metric, then there is a set E of first category for which the following is true. If $f \notin E$, $q \in (0,1)$ is rational and M is a positive integer, then there exists an $m \geqslant M$ such that

$$|f^{(m)}(q)| > m! \times m^m.$$

(d) If $f \notin E$, show that the Taylor series for f has radius of convergence 0 at every rational point $q \in (0,1)$. Explain briefly why this means that, for any point $x \in [0,1]$, there is no Taylor series which converges to f in a neighbourhood of that point.

Paper 4, Section II

12F Topics In Analysis

Let $C([0,1])$ denote the space of continuous real functions on $[0,1]$ equipped with the uniform norm $\|\cdot\|_\infty$.

(a) Consider $\mathbb{R}^{n+1}$ with the standard Euclidean norm $\|\cdot\|$, and let T be the map $T:\mathbb{R}^{n+1}\to C([0,1])$ given by $T(\mathbf{a})=\sum_{r=0}^{n} a_r t^r$. Let S be the map $S:\mathbb{R}^{n+1}\to\mathbb{R}$ given by $S(\mathbf{a})=\|T\mathbf{a}\|_\infty$. Show that there exists a $\delta>0$ such that

$$|S(\mathbf{a})|\geqslant\delta \text{ whenever } \|\mathbf{a}\|=1.$$

Conclude that $\|T(\mathbf{a})\|_\infty\to\infty$ as $\|\mathbf{a}\|\to\infty$.

(b) If $f\in C([0,1])$ and $n\geqslant 0$, show that there exists a (not necessarily unique) 'best fit' polynomial P of degree at most n such that

$$\|P-f\|_\infty\leqslant\|Q-f\|_\infty \text{ whenever } Q \text{ is a polynomial of degree at most } n.$$

(c) State Chebychev's equiripple criterion and show that it is a sufficient condition for a polynomial to be best fit.

(d) Let $g\in C([0,1])$, $M=\|g\|_\infty$ and suppose that

$$0=u_0<v_0<u_1<v_1<\ldots<v_{m-1}<u_m<v_m=1$$

are such that

$$\begin{array}{lll} M\geqslant g(t)>-M & \text{for } t\in[u_{2j},v_{2j}], & (2j\leqslant m)\\ -M\leqslant g(t)<M & \text{for } t\in[u_{2j+1},v_{2j+1}], & (2j+1\leqslant m)\\ -M<g(t)<M & \text{for } t\in[v_{j-1},u_j], & (j\leqslant m). \end{array}$$

Let $w_j=(v_{j-1}+u_j)/2$ and set $Q(t)=(-1)^{m-1}\prod_{j=1}^{m-1}(t-w_j)$. Show that, if $\eta>0$ is sufficiently small, we have

$$\|\eta Q-g\|_\infty<M.$$

Deduce that Chebychev's criterion is also a necessary condition for a polynomial to be best fit.

Paper 1, Section II

40C Waves

(a) Starting from the equations governing sound waves linearized about a state with density ρ_0 and sound speed c_0, derive the acoustic energy equation, giving expressions for the kinetic energy density K, the potential energy density due to compression W and the wave-energy flux $\mathbf{I}$.

(b) The radius $R(t)$ of a sphere oscillates according to

$$R(t) = a + \mathrm{Re}(\epsilon e^{i\omega t}),$$

where ϵ and ω are real, with $0 < \epsilon \ll a$.

(i) Find an expression for the velocity potential $\phi(r,t)$ in the region outside the sphere.

(ii) Show that for an appropriate time-average, which you should define carefully, the time-averaged rate of working by the surface of the sphere is

$$2\pi a^2 \rho_0 \omega^2 \epsilon^2 c_0 \frac{\omega^2 a^2}{c_0^2 + \omega^2 a^2}.$$

(iii) Calculate the value at $r = a$ of the dimensionless ratio $c_0 \langle K + W \rangle / |\langle \mathbf{I} \rangle|$, where angle brackets denote the time average used above.

(iv) Comment briefly on the limits $c_0 \ll \omega a$ and $c_0 \gg \omega a$, explaining their physical meaning and considering the relative magnitudes of the time-averaged kinetic energy, potential energy and acoustic energy flux.

Paper 2, Section II

40C Waves

(a) A uniform elastic solid with wave speeds c_P and c_S (using the usual notation) occupies the region $z < 0$. An SV-wave with unit amplitude displacement

$$\mathbf{u}_I = \mathrm{Re}\{(\cos\theta, 0, -\sin\theta) e^{ik_I(x\sin\theta + z\cos\theta) - i\omega t}\}$$

is incident from $z < 0$ on a rigid boundary at $z = 0$. Find the form and amplitudes of the reflected waves.

(b) Derive a condition on the incident angle θ for the reflected P-wave to be evanescent. Show by explicit calculation that if the P-wave is evanescent:

(i) the reflected SV-wave also has unit amplitude and

(ii) the P-wave has zero acoustic energy flux in the z-direction if time-averaged in an appropriate way, which you should specify carefully.

Paper 3, Section II

39C Waves

(a) The function $\phi(x,t)$ satisfies the equation

$$\frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x} + \frac{1}{9}\frac{\partial^9 \phi}{\partial x^9} = 0,$$

where $U > 0$ is a constant.

(i) Find the dispersion relation for waves of frequency ω and wavenumber k.

(ii) Sketch both the phase velocity c_p and the group velocity c_g as functions of k.

(iii) Do wave crests move faster or slower than a wave packet?

(b) Suppose that $\phi(x,0)$ is real and given by a Fourier transform as

$$\phi(x,0) = \int_{-\infty}^{\infty} A(k)e^{ikx}dk.$$

(i) Use the method of stationary phase to obtain an approximation for $\phi(Vt,t)$ for fixed $V > U$ and large t.

(ii) If the initial condition is now restricted further to be even, so that $\phi(x,0) = \phi(-x,0)$, deduce an approximation for the sequence of times at which $\phi(Vt,t) = 0$.

(iii) What can be said about $\phi(Vt,t)$ if $V < U$? [Detailed calculation is not required in this case.]

[*Hint: You may assume that* $\int_{-\infty}^{\infty} e^{-au^2}\,du = \sqrt{\frac{\pi}{a}}$ *for* $\mathrm{Re}(a) \geqslant 0, a \neq 0$.]

Paper 4, Section II

39C Waves

For adiabatic motion of an ideal gas, the pressure p is given in terms of the density ρ by a relation of the form

$$p(\rho) = p_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \qquad (\dagger)$$

where p_0, ρ_0 and γ are positive constants, with $\gamma > 1$. For such a gas, you are given that the compressive internal energy per unit volume W can be expressed as

$$W(\rho) = \frac{p(\rho)}{\gamma - 1}.$$

(a) For one-dimensional motion with speed u, write down expressions for the mass flux and the momentum flux. Using the expressions for the energy flux $u(p + W + \frac{1}{2}\rho u^2)$ and the mass flux, deduce that if the motion is steady then

$$\frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{1}{2}u^2 = C, \qquad (\star)$$

for some constant C.

(b) A one-dimensional shock wave propagates at constant speed along a tube containing the gas. Upstream of the shock the gas is at rest with pressure p_0 and density ρ_0. Downstream of the shock the pressure is maintained at the constant value $p_1 = (1 + \beta)p_0$ with $\beta > 0$. Show that

$$\frac{\rho_1}{\rho_0} = \frac{2\gamma + (\gamma + 1)\beta}{2\gamma + (\gamma - 1)\beta}, \qquad (\ddagger)$$

assuming that $(\star)$ holds throughout the flow.

(c) For small β, show that the density ratio $(\ddagger)$ from part (b) satisfies approximately the adiabatic relation $(\dagger)$, correct to $\mathcal{O}(\beta^2)$.

END OF PAPER