UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB 2022

List of Courses

Paper 2, Section I
2G Analysis and Topology

Show that a topological space X is connected if and only if every continuous integer-valued function on X is constant.

Let $\mathcal{A}$ be a family of connected subsets of a topological space X such that $\bigcup_{A\in\mathcal{A}} A = X$. Assume that $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{A}$. Prove that X is connected.

Deduce, or otherwise show, that if X and Y are connected topological spaces, then $X \times Y$ is also connected in the product topology.

Paper 4, Section I
2G Analysis and Topology

Let (f_n) be a sequence of continuous real-valued functions on a topological space X. Assume that there is a function $f\colon X \to \mathbb{R}$ such that every $x \in X$ has a neighbourhood U on which (f_n) converges to f uniformly. Show that f is continuous at every $x \in X$. Further show that (f_n) converges to f uniformly on every compact subset of X.

Paper 1, Section II
10G Analysis and Topology

Define the terms *Cauchy sequence* and *complete metric space*. Prove that every Cauchy sequence in a metric space is bounded.

Show that a metric space (M, d) is complete if and only if given any sequence (F_n) of non-empty, closed subsets of M satisfying

- $F_n \supset F_{n+1}$ for all $n \in \mathbb{N}$ and
- $\mathrm{diam} F_n = \sup\{d(x, y) : x, y \in F_n\} \to 0$ as $n \to \infty$,

the intersection $\bigcap_{n\in\mathbb{N}} F_n$ is non-empty.

State the contraction mapping theorem.

Let (Λ, ρ) and (M, d) be non-empty metric spaces, and assume that (M, d) is complete. Let $T\colon \Lambda \times M \to M$ be a function with the following properties:

- there exists $0 \leqslant k < 1$ such that $d(T(\lambda, x), T(\lambda, y)) \leqslant kd(x, y)$ for all $\lambda \in \Lambda$ and all $x, y \in M$;
- for each $x \in M$, the function $\Lambda \to M$, given by $\lambda \mapsto T(\lambda, x)$, is continuous.

Show that there is a unique function $x^*\colon \Lambda \to M$ such that $T(\lambda, x^*(\lambda)) = x^*(\lambda)$ for all $\lambda \in \Lambda$. Show further that the function x^* is continuous.

Paper 2, Section II

10G Analysis and Topology

Define the notion of *uniform convergence* for a sequence (f_n) of real-valued functions on an arbitrary set S and the notion of *uniform continuity* for a function $h\colon M \to N$ between metric spaces.

Let $C_0(\mathbb{R}^d)$ denote the set of all continuous functions $f\colon \mathbb{R}^d \to \mathbb{R}$ satisfying $f(x) \to 0$ as $\|x\| \to \infty$, i.e. for all $\varepsilon > 0$ there exists $K > 0$ such that $|f(x)| < \varepsilon$ whenever $\|x\| > K$ (where $\|x\|$ denotes the usual Euclidean length of x). Briefly explain why every function in $C_0(\mathbb{R}^d)$ is bounded. Prove that $C_0(\mathbb{R}^d)$ is a complete metric space in the uniform metric. Is it true that every member of $C_0(\mathbb{R}^d)$ is uniformly continuous? Give a proof or counterexample.

Let $\varepsilon\colon \mathbb{R} \to [0,\infty)$ be a continuous function with $\varepsilon(0) = 0$. For $n \in \mathbb{N}$ define $f_n\colon \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \sqrt{x^2 + \varepsilon(x/n)}$. Must (f_n) converge pointwise? Must (f_n) converge uniformly? Do your answers change if we further assume that for some $M \geqslant 0$ and for all $t \in \mathbb{R}$ we have $\varepsilon(t) \leqslant M|t|$? Justify your answers.

Paper 3, Section II

11G Analysis and Topology

Let $f\colon U \to \mathbb{R}^n$ be a function where U is an open subset of $\mathbb{R}^m$, and let $a \in U$. Define what it means that *f is differentiable at a* and define *the derivative of f at a*. Define what it means that *f is continuously differentiable at a*. Show that a linear map $\mathbb{R}^m \to \mathbb{R}^n$ is continuously differentiable at every point of $\mathbb{R}^m$.

State and prove the mean value inequality. Let U be an open, connected subset of $\mathbb{R}^m$. Let $f\colon U \to \mathbb{R}^n$ be a differentiable function such that $Df|_a$ is the zero map for all $a \in U$. Show that f is a constant function.

State the inverse function theorem. Consider the curve C in $\mathbb{R}^2$ defined by the equation

$$x^2 + y + \cos(xy) = 1.$$

Show that there exist an open neighbourhood U of $(0,0)$ in $\mathbb{R}^2$, an open interval I in $\mathbb{R}$ containing 0 and a continuous function $g\colon I \to \mathbb{R}$ such that $U \cap C$ is the graph of g, *i.e.*,

$$\{(x,y) \in \mathbb{R}^2 : x \in I,\ y = g(x)\} = U \cap C.$$

Paper 4, Section II

10G Analysis and Topology

Define the notions of *compact space, Hausdorff space* and *homeomorphism.*

Let X be a topological space and R be an equivalence relation on X. Define the quotient space X/R and show that the quotient map $q\colon X \to X/R$ is continuous. Let Y be another topological space and $f\colon X \to Y$ be a continuous function such that $f(x) = f(y)$ whenever xRy in X. Show that the unique function $F\colon X/R \to Y$ with $F \circ q = f$ is continuous.

Show that the quotient of a compact space is compact. Give an example to show that the quotient of a Hausdorff space need not be Hausdorff.

Let $f\colon X \to Y$ be a continuous bijection from the compact space X to the Hausdorff space Y. Carefully quoting any necessary results, show that f is a homeomorphism.

Let $X = [0,1]^2$ be the closed unit square in $\mathbb{R}^2$. Define an equivalence relation R on X by $(x_1, y_1)R(x_2, y_2)$ if and only if one of the following holds:

(i) $x_1 = x_2$ and $y_1 = y_2$, or

(ii) $\{x_1, x_2\} = \{0, 1\}$ and $y_1 = y_2$, or

(iii) $y_1 = y_2 \in \{0, 1\}$.

Show that the quotient space X/R is homeomorphic to the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.

Paper 4, Section I

3G Complex Analysis

Define what it means for two domains in $\mathbb{C}$ to be *conformally equivalent.*

For each of the following pairs of domains, determine whether they are conformally equivalent. Justify your answers.

(i) $\mathbb{C} \setminus \{0\}$ and $\{z \in \mathbb{C} : 0 < |z| < 1\}$;

(ii) $\mathbb{C}$ and $\{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$;

(iii) $\{z \in \mathbb{C} : \operatorname{Im}(z) > 0,\ |z| < 1\}$ and $\{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$.

Paper 3, Section II

13G Complex Analysis

State Rouché's theorem. State the open mapping theorem and prove it using Rouché's theorem. Show that if f is a non-constant holomorphic function on a domain Ω, then $|f|$ has no local maximum on Ω.

Let Ω be a bounded domain in $\mathbb{C}$, and let $\overline{\Omega}$ denote the closure of Ω. Let $f : \overline{\Omega} \to \mathbb{C}$ be a continuous function that is holomorphic on Ω. Show that if $|f(z)| \leqslant M$ for all $z \in \partial\Omega$, then $|f(z)| \leqslant M$ for all $z \in \Omega$, where $\partial\Omega = \overline{\Omega} \setminus \Omega$ is the boundary of Ω.

Consider the unbounded domain $\Omega = \{z \in \mathbb{C} : \operatorname{Re} z > 1\}$. Let $f : \overline{\Omega} \to \mathbb{C}$ be a continuous function that is holomorphic on Ω. Assume that f is bounded both on Ω and on its boundary $\partial\Omega$. Show that if $|f(z)| \leqslant M$ for all $z \in \partial\Omega$, then $|f(z)| \leqslant M$ for all $z \in \Omega$. [*Hint: Consider for large $n \in \mathbb{N}$ and for a large disc $D(0,R)$ the function $z \mapsto \big(f(z)\big)^n/z$ on $D(0,R) \cap \Omega$.*] Is the boundedness assumption of f on Ω necessary? Justify your answer.

Paper 1, Section I

3B Complex Analysis OR Complex Methods

(a) What is the Laurent series of $e^{1/z}$ about $z_0 = 0$?

(b) Let $\rho > 0$. Show that for all large enough $n \in \mathbb{N}$, all zeros of the function

$$f_n(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \ldots + \frac{1}{n!z^n}$$

lie in the open disc $\{z : |z| < \rho\}$.

Paper 1, Section II

12G Complex Analysis OR Complex Methods

(a) Let $f(z) = -\sum_{n=1}^{\infty} \frac{(1-z)^n}{n}$ for $|z-1| < 1$. By differentiating $z \exp(-f(z))$, show that f is an analytic branch of logarithm on the disc $D(1,1)$ with $f(1) = 0$. Use scaling and the function f to show that for every point a in the domain $D = \mathbb{C} \setminus \{x \in \mathbb{R} : x \geqslant 0\}$, there is an analytic branch of logarithm on a small neighbourhood of a whose imaginary part lies in $(0, 2\pi)$.

(b) For $z \in D$, let $\theta(z)$ be the unique value of the argument of z in the interval $(0, 2\pi)$. Define the function $L \colon D \to \mathbb{C}$ by $L(z) = \log|z| + i\theta(z)$. Briefly explain using part (a) why L is an analytic branch of logarithm on D. For $\alpha \in (-1, 1)$ write down an analytic branch of z^α on D.

(c) State the residue theorem. Evaluate the integral

$$I = \int_0^\infty \frac{x^\alpha}{(x+1)^2}\, dx$$

where $\alpha \in (-1, 1)$.

Paper 2, Section II

12B Complex Analysis OR Complex Methods

(a) Suppose that $f : \mathbb{C} \to \mathbb{C}$ is analytic, and is bounded in the half-plane $\{z : \mathrm{Re}(z) > 0\}$. Prove that, for any real number $c > 0$, there is a positive real constant M such that

$$|f(z_1) - f(z_2)| \leqslant M|z_1 - z_2|$$

whenever $z_1, z_2 \in \mathbb{C}$ satisfy $\mathrm{Re}(z_1) > c$, $\mathrm{Re}(z_2) > c$, and $|z_1 - z_2| < c$.

(b) Let the functions $g, h : \mathbb{C} \to \mathbb{C}$ both be analytic.

(i) State Liouville's Theorem.

(ii) Show that if g is not constant, then $g(\mathbb{C})$ is dense in $\mathbb{C}$.

(iii) Show that if $|h(z)| \leqslant |\mathrm{Re}(z)|^{-1/2}$ for all $z \in \mathbb{C}$, then h is constant.

Paper 3, Section I

3B Complex Methods

Let $f = u + iv$ be an analytic function in a connected open set $D \subset \mathbb{C}$, where $u(x, y)$ and $v(x, y)$ are real-valued functions on D, with $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$, for $z \in D$.

(a) Show that $f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$, and state the Cauchy–Riemann equations.

(b) Suppose there are real constants a, b and c such that $a^2 + b^2 \neq 0$ and

$$au(x, y) + bv(x, y) = c, \qquad z \in D.$$

Show that f is constant on D.

Paper 4, Section II

12B Complex Methods

Let $B : [0, \infty) \to \mathbb{R}^{n\times p}$ be a $n \times p$ matrix-valued function. The Laplace transform $\mathcal{L}\{B\}$ of B is defined componentwise on the matrix element functions of B.

(a) Show that if A is a constant $n \times n$ matrix and $B : [0, \infty) \to \mathbb{R}^{n\times p}$ is an $n \times p$ matrix-valued function, then $\mathcal{L}\{AB\} = A\mathcal{L}\{B\}$.

(b) Consider the ODE given by

$$y'(t) = Ay(t) + g(t), \quad y(0) = y_0 \in \mathbb{R}^n, \quad t \geqslant 0, \tag{$*$}$$

where A is a constant $n \times n$ matrix, and $g : [0, \infty) \to \mathbb{R}^n$ is a vector-valued function whose Laplace transform $G(s) = \mathcal{L}\{g\}(s)$ exists for all but one $s \in \mathbb{C}$. Show that

$$Y(s) = (sI - A)^{-1}(y_0 + G(s)),$$

and that

$$\mathcal{L}\{e^{tA}\}(s) = (sI - A)^{-1},$$

for all s that are not eigenvalues of A, where $Y = \mathcal{L}\{y\}$ is the Laplace transform of the solution y of ($*$). You may assume that y exists and is the unique solution to the ODE for all $t \geqslant 0$ with solution $y(t) = e^{tA}y_0$ when $g = 0$.

(c) Consider the ODE

$$y'(t) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} y(t) + \begin{bmatrix} e^{2t} \\ -2t \end{bmatrix}, \quad y(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad t \geqslant 0.$$

Determine the integer values $n \in \mathbb{N}$ such that $\lim_{t\to\infty} e^{-nt}y(t)$ exists and is a finite and nonzero vector in $\mathbb{R}^2$.

Paper 2, Section I

4D Electromagnetism

Define what is meant by a *capacitor* and by *capacitance*.

Consider a cylindrical capacitor consisting of two concentric cylinders of length L, linear charge density λ and radii a and $b > a$, respectively. Assuming that $L \gg b$ and that end effects may be neglected, compute the electric field E between the cylinders, the potential difference V, the capacitance C and the energy U stored in this system. Verify that $U = \frac{1}{2}QV$ where Q is the total charge.

Paper 4, Section I

5D Electromagnetism

Consider a system of electric charges distributed in such a way that there is a charge $-Q$ at the point $(x, y, z) = (0, 0, d)$, a charge $+NQ$, with N a positive integer, located at the origin of coordinates and a charge $-MQ$ for a positive integer M at the point $(0, 0, -d)$.

(a) Compute the electric potential at a distance $\mathbf{r}$ and expand in powers of $1/r$. Identify the monopole, dipole and quadrupole terms in the expansion.

(b) For which values of N and M do monopole and/or dipole terms cancel? If the monopole term cancels, what can be said about the limits for which $d \to 0$ but either Qd or Qd^2 are constants?

(c) For the case where the monopole and dipole terms cancel, compute the force on a particle of charge $-Q$ located at $\mathbf{r} = (x, 0, 0)$. Is the force attractive or repulsive?

Paper 1, Section II

15D Electromagnetism

Write down Maxwell's equations in free space for the electric field $\mathbf{E}(\mathbf{x},t)$ and magnetic field $\mathbf{B}(\mathbf{x},t)$ in the presence of an electric charge density $\rho(\mathbf{x},t)$ and current density $\mathbf{J}(\mathbf{x},t)$.

(a) Use Maxwell's equations to prove the continuity equation $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$ and then derive the conservation of electric charge $Q = \int_V \rho \, d^3\mathbf{x}$. Which assumption do you need to make in order to establish this result?

(b) In empty space, with $\rho = |\mathbf{J}| = 0$, show that each component of $\mathbf{E}$ and $\mathbf{B}$ satisfies the wave equation. Compute the speed of the waves in terms of the permittivity $\epsilon_0 \simeq 8.85 \times 10^{-12}$ m^{-3} kg^{-1} s^4 A^2 and permeability $\mu_0 \simeq 1.25 \times 10^{-6}$ NA^{-2} of free space. Explain the importance of this result.

(c) Using Maxwell's equations and the expression for the energy stored in electric and magnetic fields inside a volume V:

$$U = \frac{1}{2} \int_V \left(\epsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0} \mathbf{B}^2 \right) d^3\mathbf{x},$$

write down an equation for the variation of the energy in terms of the Poynting vector, which you should define, and provide an interpretation. [*The identity* $\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B})$ *may be useful.*]

(d) For a linearly polarised monochromatic electromagnetic wave of frequency ω and wave vector $\mathbf{k}$ the electric field can be written as $\mathbf{E} = \mathbf{E}_0 \sin(\mathbf{k} \cdot \mathbf{x} - \omega t)$. Show that the Poynting vector is parallel to the wave vector $\mathbf{k}$ and compute its magnitude. Consider the time average of the Poynting vector and relate it to the average energy stored in the electric and magnetic fields.

(e) If a mobile phone transmits electromagnetic waves with a power of 1 watt, compute the average amplitude of the Poynting vector and the amplitude of the electric field at 10 cm from the handset. You may assume that the radiation is isotropic.

Paper 2, Section II

16D Electromagnetism

Consider a relativistic particle of mass m and charge q in the presence of a constant electric field $\mathbf{E}$ and constant magnetic field $\mathbf{B}$.

(a) Write down the covariant relativistic generalisation of the Lorentz force law, explaining each of the terms. Decompose the equation in terms of the temporal and spatial components. Compare with the non-relativistic version of the law.

(b) Find the time variation of the energy $\mathcal{E}(t)$ in terms of the electric field and the particle's velocity.

(c) For $\mathbf{B} = \mathbf{0}$ and $\mathbf{E} = (0, 0, E)$ find the particle's energy $\mathcal{E}(t)$ and position $z(t)$ as functions of time t assuming that the particle was initially at the origin with momentum $\mathbf{p_0} = (p_0, 0, 0)$ (and energy $\mathcal{E}_0 = \sqrt{m^2c^4 + c^2p_0^2}$) where c is the speed of light. [*Hint: Recall* $\mathcal{E}^2 - c^2p^2 = m^2c^4$.]

(d) Determine the trajectory of the particle in the x-z plane, $z(x)$, expressing the result in terms of the constants $q, m, E, p_0, \mathcal{E}_0$. [*Hint: Recall* $dz/dx = p_z/p_x$.]

(e) Determine the limiting behaviour of $z(t)$ for both large and small t and compare the latter with the well-known non-relativistic result of a particle with constant acceleration $a = qE/m$. What are $z(x)$ and $x(t)$ in this case?

Paper 3, Section II

15D Electromagnetism

Consider a steady electric current density $\mathbf{J}(\mathbf{r})$ and the corresponding magnetic vector potential $\mathbf{A}(\mathbf{r})$.

(a) Show that each component of $\mathbf{A}(\mathbf{r})$ in Cartesian coordinates satisfies a Poisson equation $\nabla^2\mathbf{A} = -\mu_0\mathbf{J}$ and write down the general integral expression for $\mathbf{A}(\mathbf{r})$ in terms of $\mathbf{J}(\mathbf{r})$. Explain why you can assume $\nabla \cdot \mathbf{A} = 0$.

(b) Use the expression for the vector potential to derive the Biot–Savart law:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3\mathbf{r}'.$$

(c) Consider a circular loop of wire of radius R in the x-y plane with a circulating current I. Using the Biot–Savart law, determine the direction and magnitude of the corresponding magnetic field $\mathbf{B}(\mathbf{r})$ at a point on the z-axis. What is the magnetic field at the centre of the loop?

(d) If there is a second parallel loop of radius $2R$ with centre in the z-axis at a distance D from the first loop and current $2I$ circulating in the opposite direction, find the point between the wires at which the magnetic field vanishes.

Paper 2, Section I

5C Fluid Dynamics

A three-dimensional flow has a velocity field $\mathbf{u}(\mathbf{x}) = \mathbf{\Gamma}\cdot\mathbf{x}+\mathbf{U}_0$, where $\mathbf{\Gamma}$ is a constant second-rank tensor and $\mathbf{U}_0$ is a constant vector, with components

$$\mathbf{\Gamma} = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}, \quad \mathbf{U}_0 = \begin{pmatrix} P \\ Q \\ R \end{pmatrix}.$$

(a) What are the conditions on the components of $\mathbf{\Gamma}$ and $\mathbf{U}_0$ for the flow to be:

(i) incompressible?

(ii) irrotational?

(b) In the case where

$$\mathbf{\Gamma} = \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{U}_0 = \begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix}, \quad (\alpha \neq 0),$$

compute the streamline passing through the point $(1,0,0)$.

Paper 3, Section I

7C Fluid Dynamics

A two-dimensional cylinder of radius a is stationary in a uniform flow of velocity $U\mathbf{e}_x$. The flow is assumed to be steady, inviscid, two-dimensional and irrotational. There is no circulation around the cylinder.

Using a velocity potential, solve for the flow $\mathbf{u}(r,\theta)$ around the cylinder. Use Bernoulli's equation to compute the pressure on its surface as a function of the polar angle θ.

Paper 1, Section II

16C Fluid Dynamics

An incompressible viscous fluid of constant uniform viscosity μ and density ρ undergoes unidirectional flow of the form $\mathbf{u} = u(y,t)\mathbf{e}_x$ in two dimensions. Gravity is negligible.

(a) Use a small control fluid volume of size $\delta x \times \delta y$,

(i) to show that this flow satisfies mass conservation;

(ii) to derive the momentum conservation equation satisfied by $u(y,t)$ and the pressure $p(x)$.

(b) The flow is steady, is subject to a uniform pressure gradient $G = \mathrm{d}p/\mathrm{d}x$ and occurs between two rigid surfaces at $y = 0$ and $y = h$. The surface at $y = 0$ is stationary while the surface at $y = h$ translates with velocity $U\mathbf{e}_x$, where U is a constant parameter.

(i) Solve for the flow $u(y)$ in terms of G and U.

(ii) Compute the value G_0 of the applied pressure gradient G for which the shear stress at $y = 0$ is zero.

(iii) For $G = G_0$, deduce the volume flux in the x direction.

(iv) For $G = G_0$, use $u(y)$ to compute the shear stress exerted by the flow on the top plate. Show that it can also be obtained by using a force balance on a small control fluid volume of size $\delta x \times h$.

Paper 3, Section II

16C Fluid Dynamics

(a) Starting from the Euler equation for an inviscid fluid with no body force, derive the unsteady Bernoulli equation relating the pressure and the velocity potential in a time-dependent irrotational, incompressible flow.

(b) A liquid occupies the two-dimensional annular region $a(t) < r < b(t)$ between a gas bubble occupying $0 \leqslant r < a(t)$ and an infinite gas in $r > b(t)$. The flow is incompressible, irrotational and radially symmetric.

(i) If the radius of the gas bubble is prescribed (i.e. the function $a(t)$ is known), solve for the potential flow in the liquid. Deduce the time-variation of $b(t)$ and interpret your result physically.

(ii) The pressure in the gas in $r > b$ is a constant p_∞. Compute the time-varying pressure $p(r,t)$ in the liquid at $r = a(t)$.

(iii) Assuming small perturbations for the bubble radius $a(t) = a_0[1 + \epsilon(t)]$ with $|\epsilon| \ll 1$, deduce the linearised variation of the radius $b(t)$. Find the linearised variation of the pressure $p(a,t)$.

(iv) The pressure $p_0(t)$ in the bubble is uniform in space and satisfies $p_0 V = \textit{const}$, where $V(t)$ is the volume of the bubble. Deduce the relationship between ϵ and $p(a,t) - p_\infty$.

(v) Show that the bubble undergoes oscillations and compute its frequency ω.

Paper 4, Section II
16C Fluid Dynamics

(a) A body of fluid has a free surface given by $z = \eta(x, y, t)$ in Cartesian coordinates and the fluid velocity is denoted by $\mathbf{u} = (u, v, w)$. Applying the kinematic boundary condition at the free surface, derive the relationship between the value of w at the free surface and $\mathrm{D}\eta/\mathrm{D}t$.

(b) An inviscid fluid is confined in a box with sides at $x = 0, L$ and $y = 0, L$. The fluid is semi-infinite in the $-z$ direction and is bounded above by a free surface at $z = \eta(x, y, t)$. The fluid is forced to oscillate by applying a prescribed variation in the air pressure just above the free surface,

$$p(x, y, t) = p_0 \cos(\pi x/L) \cos(2\pi y/L) \cos(\omega t),$$

with ω a prescribed constant frequency.

(i) Assuming irrotational flow and small-amplitude motion of the interface, state the equation satisfied by the velocity potential ϕ in the fluid and state all the boundary conditions.

(ii) Show that a separable solution for ϕ of the form

$$\phi = Z(z) \cos(\pi x/L) \cos(2\pi y/L)\, F(t)$$

is consistent with the dynamic boundary condition and that it satisfies the boundary conditions at $x = 0, L$ and $y = 0, L$.

(iii) Solve for the function $Z(z)$.

(iv) Using the kinematic boundary condition, show that the shape of the interface is of the form

$$\eta(x, y, t) = \cos(\pi x/L) \cos(2\pi y/L)\, H(t),$$

and derive the relationship between $H(t)$ and $F(t)$.

(v) Use the dynamic boundary condition to solve for $H(t)$ and $F(t)$.

(vi) Deduce that the amplitudes H and F do not remain bounded for a specific value of the frequency ω which you should determine, and briefly interpret this phenomenon physically.

Paper 1, Section I

2F Geometry

What is a *topological surface*?

Consider

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \,:\, x^2 + y^2 + z^2 = 1\},$$

which you may assume is a topological surface. For the equivalence relation $\sim$ on S^2 generated by $(x, y, z) \sim (-x, -y, -z)$, show that $S^2/\sim$ is a topological surface. For the equivalence relation $\approx$ on S^2 generated by $(x, y, z) \approx (-x, -y, z)$, show that $S^2/\approx$ is homeomorphic to S^2.

Paper 3, Section I

2E Geometry

Let $\mathbb{H}$ be the hyperbolic upper half plane. Explain how the Riemannian metric $\frac{dx^2+dy^2}{y^2}$ on $\mathbb{H}$ can be used to compute lengths, angles and areas.

Consider the triangle in $\mathbb{H}$ with vertices at $e^{i\alpha}$, $e^{i\beta}$ and ∞, where $0 < \alpha < \beta < \pi$. Compute its area, and deduce the Gauss–Bonnet theorem for a hyperbolic polygon.

Paper 1, Section II

11F Geometry

Define in terms of allowable parametrisations what it means to say that a subset $S \subset \mathbb{R}^3$ is a *smooth surface.*

Let $\phi : \mathbb{R} \to (0, \infty)$ be a smooth function. Show that

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 \,:\, x^2 + y^2 = \phi(z)^2\}$$

is a smooth surface in $\mathbb{R}^3$.

Suppose $a < b$ and $r > 0$ are such that for all $a \leqslant a' < b' \leqslant b$ we have

$$\mathrm{Area}(\{(x, y, z) \in \Sigma \,:\, a' \leqslant z \leqslant b'\}) = 2\pi r \cdot (b' - a').$$

Show that ϕ must satisfy $r^2 = \phi(t)^2 + \phi(t)^2\phi'(t)^2$ for $a \leqslant t \leqslant b$. Assuming that $\phi(t) < r$ for $a \leqslant t \leqslant b$, show that the graph of the function $\phi|_{[a,b]}$ lies on a circle of radius r.

Paper 2, Section II

11F Geometry

Let $U \subset \mathbb{R}^2$ and $f : U \to \mathbb{R}$ be a smooth function. Derive a formula for the first and second fundamental forms of the surface in $\mathbb{R}^3$ parametrised by

$$\begin{aligned} \sigma : U &\longrightarrow \mathbb{R}^3 \\ (u,v) &\longmapsto (u,v,f(u,v)) \end{aligned}$$

in terms of f. State a formula for the Gaussian curvature in terms of the first and second fundamental forms, and hence give a formula for the Gaussian curvature of this surface.

Let $\Sigma \subset \mathbb{R}^3$ be a smooth surface and $P \subset \mathbb{R}^3$ be a plane. Supposing that Σ is tangent to P along a smooth curve $\gamma \subset \mathbb{R}^3$ and otherwise lies on one side of P, show that the Gaussian curvature of Σ is zero at all points on γ.

Paper 3, Section II

12E Geometry

Let $\sigma : V \to \Sigma$ be a smooth parametrisation of an embedded surface $\Sigma \subset \mathbb{R}^3$, and let $\gamma : (a,b) \to \Sigma;\ t \mapsto \sigma(u(t), v(t))$ be a smooth curve. Show by differentiating $\sigma_u \cdot \gamma'$ and $\sigma_v \cdot \gamma'$ that γ satisfies the geodesic equations if and only if $\gamma''(t)$ is normal to the surface. Deduce that geodesics are parametrised at constant speed.

Now assume in addition that Σ is a surface of revolution. Let $\rho(t)$ be the distance from $\gamma(t)$ to the axis of revolution, and let $\theta(t)$ be the angle between γ and the parallel at $\gamma(t)$. Prove that if γ is a geodesic then it satisfies the Clairaut relation

$$\rho(t)\cos\theta(t) = \text{constant}.$$

On the hyperboloid $\Sigma = \{x^2 + y^2 = z^2 + 1\}$ give examples of

(i) a curve parametrised at constant speed, which satisfies the Clairaut relation, but is *not* a geodesic,

(ii) a plane that meets Σ in a pair of disjoint geodesics,

(iii) a plane that meets Σ in a pair of geodesics that intersect at right angles.

Are there any geodesics entirely contained in the region $z > 0$? Are there any geodesics $\gamma \subset \Sigma$ with $\phi(\gamma) = \gamma$ for every isometry $\phi : \Sigma \to \Sigma$? Justify your answers.

Paper 4, Section II

11E Geometry

(a) Show that the Möbius maps commuting with $z \mapsto 1/\overline{z}$ are of the form

$$z \mapsto \frac{az+b}{\overline{b}z+\overline{a}}$$

where $a, b \in \mathbb{C}$ with $|a|^2 - |b|^2 \neq 0$. Which of these maps preserve the unit disc?

(b) Write down the Riemannian metric on the disc model $\mathbb{D}$ of the hyperbolic plane. Describe the geodesics passing through O and prove that they are length minimising curves. Deduce that every geodesic is part of a circle or line preserved by the transformation $z \mapsto 1/\overline{z}$. [You may assume that the maps in part (a) that preserve the unit disc are isometries.]

(c) Let $P \in \mathbb{D}$ be a point at a hyperbolic distance $\rho > 0$ from O. Let ℓ be the hyperbolic line passing through P at right angles to OP. Show that ℓ has Euclidean radius $1/\sinh\rho$ and centre at a distance $1/\tanh\rho$ from O.

(d) Consider a hyperbolic quadrilateral with three right angles, and angle θ at the remaining vertex v. Show that

$$\cos\theta = \tanh a \tanh b$$

where a and b are the hyperbolic lengths of the sides incident with v.

Paper 2, Section I

1E Groups, Rings and Modules

Let R be a commutative ring. Show that the following statements are equivalent.

(i) There exists $e \in R$ with $e^2 = e$ and $e \neq 0, 1$.

(ii) $R \cong R_1 \times R_2$ for some non-trivial rings R_1 and R_2.

Let $R = \{(a,b) \in \mathbb{Z}^2 \mid a \equiv b \pmod 2\}$. Show that R is a ring under componentwise operations. Is R an integral domain? Is R isomorphic to a product of non-trivial rings?

Paper 3, Section I

1E Groups, Rings and Modules

Let F be a finite field of order q. Let $G = \mathrm{GL}_2(F)/Z$ where $Z \leqslant \mathrm{GL}_2(F)$ is the subgroup of scalar matrices. Define an action of $\mathrm{GL}_2(F)$ on $F \cup \{\infty\}$ and use this to show that there is an injective group homomorphism

$$\phi : G \to S_{q+1}.$$

Now let $F = \mathbb{F}_2[\omega]/(\omega^2 + \omega + 1) = \{0, 1, \omega, \omega + 1\}$ be the field with $q = 4$ elements (where $\mathbb{F}_2 = \{0, 1\}$ is the field with 2 elements). Compute the order of G, find a Sylow 2-subgroup P of G, and show that $\phi(P) \leqslant A_5$.

Paper 1, Section II

9E Groups, Rings and Modules

Let R be a Noetherian integral domain with field of fractions F. Prove that the following statements are equivalent.

(i) R is a principal ideal domain.

(ii) Every pair of elements $a, b \in R$ has a greatest common divisor which can be written in the form $ra + sb$ for some $r, s \in R$.

(iii) Every finitely generated R-submodule of F is cyclic.

(iv) Every R-submodule of R^n can be generated by n elements.

Show that any integral domain that is isomorphic to $\mathbb{Z}^n$ as a group under addition is Noetherian as a ring. Find an example of such a ring that does *not* satisfy conditions (i)-(iv). Justify your answer.

Paper 2, Section II
9E Groups, Rings and Modules

(a) Let P be a Sylow p-subgroup of a group G, and let Q be any p-subgroup of G. Prove that $Q \leqslant gPg^{-1}$ for some $g \in G$. State the remaining Sylow theorems.

(b) Let G be a group acting faithfully and transitively on a set X of size 7. Suppose that

(i) for every $x \in X$ we have $\mathrm{Stab}_G(x) \cong S_4$,

(ii) for every $x, y \in X$ distinct we have $\mathrm{Stab}_G(x) \cap \mathrm{Stab}_G(y) \cong C_2 \times C_2$.

Determine the order of G and its number of Sylow p-subgroups for each prime p. [*Hint: For one of the primes p it may help to use the following fact, which you may assume. If H is a subgroup of S_p of order p then the normaliser of H in S_p has order $p(p-1)$.*]

Deduce that no proper normal subgroup of G has order divisible by 3 or order divisible by 7. Hence or otherwise prove that G is simple.

Paper 3, Section II
10E Groups, Rings and Modules

(a) Let R be a unique factorisation domain (UFD) with field of fractions F. What does it mean to say that a polynomial $f \in R[X]$ is *primitive*? Assuming that the product of two primitive polynomials is primitive, prove that for $f \in R[X]$ primitive the following implications hold.

(i) f irreducible in $R[X] \implies f$ irreducible in $F[X]$.

(ii) f prime in $F[X] \implies f$ prime in $R[X]$.

Deduce that $R[X]$ is a UFD. [You may use any standard characterisation of a UFD, provided you state it clearly.]

(b) A rational function $f \in \mathbb{C}(X,Y)$ is *symmetric* if $f(X,Y) = f(Y,X)$. Show that if $f \in \mathbb{C}(X,Y)$ is symmetric then it can be written as $f = g/h$ where $g, h \in \mathbb{C}[X,Y]$ are coprime and symmetric.

Paper 4, Section II
9E Groups, Rings and Modules

State and prove Eisenstein's criterion. Show that if p is a prime number then $f(X) = X^{p-1} + X^{p-2} + \ldots + X^2 + X + 1$ is irreducible in $\mathbb{Z}[X]$. Let $\zeta \in \mathbb{C}$ be a root of f. Prove that $\mathbb{Z}[\zeta] \cong \mathbb{Z}[X]/(f)$. [Any form of Gauss' lemma may be quoted without proof.]

Now let $p = 3$. Show that $\mathbb{Z}[\zeta]$ is a Euclidean domain. Prove that if n is even then there is exactly one conjugacy class of matrices $A \in \mathrm{GL}_n(\mathbb{Z})$ such that $A^2 + A + I = 0$. What happens if n is odd? You should carefully state any theorems that you use.

Paper 1, Section I

1F Linear Algebra

Let V and W be finite-dimensional real vector spaces, and $\mathcal{L}(V, W)$ denote the vector space of linear maps from V to W. Prove that the dimensions of these vector spaces satisfy

$$\dim(\mathcal{L}(V, W)) = \dim(V) \cdot \dim(W).$$

If $A \leqslant V$ and $B \leqslant W$ are vector subspaces, let

$$X = \{\phi \in \mathcal{L}(V, W) \,:\, \phi(A) \leqslant B\},$$

which you may assume is a vector subspace of $\mathcal{L}(V, W)$. Prove a formula for the dimension of X in terms of the dimensions of V, W, A and B.

If S and T are vector subspaces of V such that $V = S + T$, let

$$Y = \{\phi \in \mathcal{L}(V, V) \,:\, \phi(S) \leqslant S \text{ and } \phi(T) \leqslant T\},$$

which you may assume is a vector subspace of $\mathcal{L}(V, V)$. Prove a formula for the dimension of Y in terms of the dimensions of V, S, and T.

Paper 4, Section I

1F Linear Algebra

Let V be a finite-dimensional real vector space. What is a *non-degenerate bilinear form* on V?

If $B_1(-,-)$ is a non-degenerate bilinear form on V and $B_2(-,-)$ is a bilinear form on V, which may be degenerate, show that there is a linear map $\alpha : V \to V$ such that

$$B_2(v, w) = B_1(v, \alpha(w)) \text{ for all } v, w \in V.$$

Show that

$$\{w \in V \,:\, B_2(v, w) = 0 \text{ for all } v \in V\} = \mathrm{Ker}(\alpha).$$

[*You may use any results on dual vector spaces provided they are clearly stated.*]

Paper 1, Section II

8F Linear Algebra

For each of the following statements give a proof or counterexample.

(a) If A and B are 3×3 complex matrices with the same characteristic polynomial and the same minimal polynomial, then they are conjugate.

(b) There are three mutually non-conjugate complex matrices with characteristic polynomial $(2-t)^2(1-t)^5$ and minimal polynomial $(2-t)^2(1-t)^2$.

(c) If $\alpha : V \to V$ is a linear isomorphism from a finite-dimensional complex vector space to itself such that some iterate α^N with $N > 0$ is diagonalisable, then α is diagonalisable.

(d) A real matrix which is diagonalisable when considered as a complex matrix is also diagonalisable as a real matrix.

(e) Two real matrices which are conjugate when considered as complex matrices are also conjugate as real matrices.

Paper 2, Section II

8F Linear Algebra

What is the *characteristic polynomial* of a square matrix A?

State and prove the Cayley–Hamilton theorem for square complex matrices.

For square matrices X and Y let us write $[X, Y] = XY - YX$. Given another square matrix Z, show that $[X, YZ] = [X, Y]Z + Y[X, Z]$.

Suppose now that A and B are square complex matrices such that $[B, A]$ commutes with A, i.e. $[[B, A], A] = 0$. Show that for any polynomial $\varphi(t)$ we have

$$[B, \varphi(A)] = \varphi'(A)[B, A],$$

where $\varphi'(t)$ denotes the derivative of φ. For a polynomial $f(t)$, whose kth derivative is denoted by $f^{(k)}(t)$, satisfying $f(A) = 0$, show by induction that $f^{(k)}(A)[B, A]^{2^k - 1} = 0$. Deduce that some power of the matrix $[B, A]$ is zero.

Paper 3, Section II

9F Linear Algebra

Let V be a finite-dimensional real inner product space, and $\alpha : V \to V$ be a linear map. What does it mean to say that α is *self-adjoint*?

If $\alpha : V \to V$ is self-adjoint, prove that there is an orthonormal basis for V consisting of eigenvectors of α.

Let P_n denote the vector space of real polynomials of degree at most n. Show that

$$\langle f, g \rangle = \int_0^\infty f(x)g(x)e^{-x}\,dx$$

defines an inner product on this vector space, and that the linear map $\alpha : P_n \to P_n$ given by

$$\alpha(f) = xf'' + (1-x)f'$$

is self-adjoint with respect to this inner product.

Show that α has eigenvalues $0, -1, -2, -3, \ldots, -n$. When $n = 2$ determine corresponding eigenvectors.

[*Hint: You may use the identity* $\int_0^\infty x^n e^{-x}\,dx = n!$.]

Paper 4, Section II

8F Linear Algebra

If V and W are finite-dimensional vector spaces and $\gamma : V \to W$ is a linear map, what is the *matrix representation* of γ with respect to bases $\mathcal{B}$ of V and $\mathcal{C}$ of W?

If $\alpha, \beta : V \to V$ are linear maps, what does it mean to say that they are *conjugate*? How is this interpreted in terms of matrices representing α and β with respect to a basis $\mathcal{B}$ of V?

Let V be a vector space and $\beta : V \to V$ be a linear isomorphism. Write $\mathcal{L}(V,V)$ for the vector space of linear maps from V to V, and define a function by

$$\begin{aligned} \phi_\beta : \mathcal{L}(V,V) &\longrightarrow \mathcal{L}(V,V) \\ \alpha &\longmapsto \beta^{-1}\alpha\beta. \end{aligned}$$

Show that ϕ_β is a linear isomorphism, and that if β is conjugate to β' then ϕ_β is conjugate to $\phi_{\beta'}$.

Assuming that V is a 2-dimensional complex vector space, determine the Jordan Normal Form of ϕ_β in terms of that of β.

Paper 3, Section I

8H Markov Chains

A gang of thieves decides to commit a robbery every week. The gang only robs one of three possible targets: Art museums, Banks, or Casinos, which they conveniently denote by $\{A, B, C\}$. The places they rob follows a Markov chain with the following transition probability matrix:

$$P = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 3/4 & 0 & 1/4 \\ 3/8 & 1/8 & 1/2 \end{pmatrix}.$$

(a) Find the stationary distribution of this Markov chain.

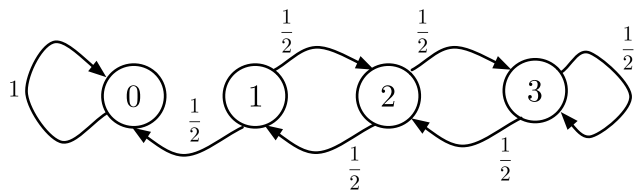
(b) Is the Markov chain reversible?

(c) Since this spate of robberies had been going on for a long time (i.e., the Markov chain is in stationarity), the police approach Detective Holmes for assistance. Detective Holmes arrives at the crime scene, which happens to be a bank. Detective Holmes asks the police, "What is the probability that these thieves robbed a bank two weeks ago, as well?" The police, not having taken Part IB Markov Chains, are stumped. Please help the police by finding this probability.

Paper 4, Section I

7H Markov Chains

Consider the Markov chain in the figure below.



(a) Let $g(i) = \mathbb{E}_i[T_0]$ be the expected time to get absorbed in state 0 starting from state i. Find $g(1), g(2)$ and $g(3)$.

(b) Suppose the Markov chain is initialised in state 1. What is the probability it will visit 3 before getting absorbed in 0?

(c) Suppose the Markov chain is initialised in state 1. What is the expected number of visits to state 3 before the chain gets absorbed in 0?

Paper 1, Section II

19H Markov Chains

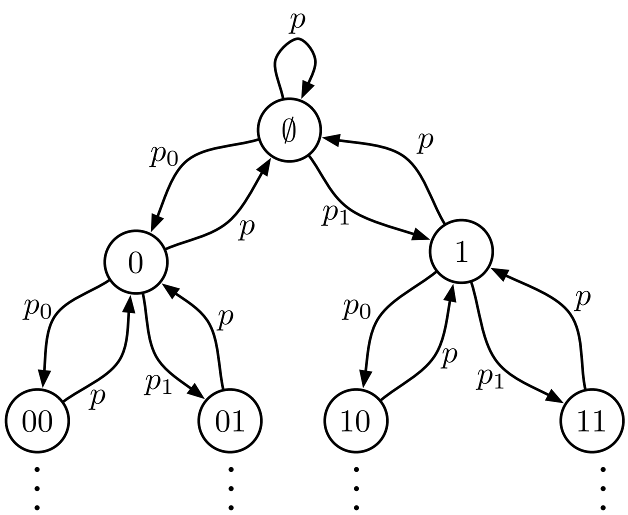
Label the vertices of a binary tree by all binary vectors, with the exception of the "root" node, which is labeled $\emptyset$. Let $p_0, p_1 > 0$ such that $p_0 + p_1 < 1$, and let $p = 1 - p_0 - p_1$. Consider a Markov chain X_n on the binary tree with transition probabilities as follows:

$$\mathbb{P}(X_{n+1} = (b_1, b_2, \ldots, b_k, i) | X_n = (b_1, b_2, \ldots, b_k)) = p_i \quad \text{for } i = 0, 1,$$
$$\mathbb{P}(X_{n+1} = (b_1, b_2, \ldots, b_{k-1}) | X_n = (b_1, b_2, \ldots, b_k)) = p$$

for any non-root vertex $(b_1, b_2, \ldots, b_k) \in \{0, 1\}^k$, and

$$\mathbb{P}(X_{n+1} = i | X_n = \emptyset) = p_i \quad \text{for } i = 0, 1,$$
$$\mathbb{P}(X_{n+1} = \emptyset | X_n = \emptyset) = p$$

for the root vertex. The figure below shows the states and the transition probabilities for the first two levels of the tree.



(a) Prove that the Markov chain is irreducible and find its period. Justify your answers.

(b) What are the conditions on p_0, p_1 so that the chain is transient/null recurrent/positive recurrent? Justify your answer.

(c) Assume that the p_0, p_1 are chosen such that the chain is positive recurrent. Let $\ell(X_n)$ denote the length of the string representing state X_n. For example, $\ell(\emptyset) = 0$ and $\ell(0010) = 4$. Prove that the following limit exists

$$\lim_{n \to \infty} \mathbb{P}(\ell(X_n) = k | X_0 = \emptyset),$$

and determine its value.

Paper 2, Section II

18H Markov Chains

Let $(X_n)_{n\geqslant 0}$ be a Markov chain with finite state space S. Let $(Y_n)_{n\geqslant 0}$ denote another Markov chain on the same state space S. Let P denote the transition probability matrix of $(X_n)_{n\geqslant 0}$ and Q denote the transition probability matrix of $(Y_n)_{n\geqslant 0}$. You are given that for each $i, j \in S$,

$$P_{ij} > 0 \implies Q_{ij} > 0.$$

For each of the following statements provide a proof or counterexample:

(a) If $(X_n)_{n\geqslant 0}$ is irreducible, then $(Y_n)_{n\geqslant 0}$ is also irreducible.

(b) If every state in $(X_n)_{n\geqslant 0}$ is aperiodic, then every state in $(Y_n)_{n\geqslant 0}$ is also aperiodic.

(c) If $(X_n)_{n\geqslant 0}$ has no transient states, then $(Y_n)_{n\geqslant 0}$ also has no transient states.

(d) For $i \in S$, let μ_i denote the mean of the first return time to i starting from i, in the Markov chain $(X_n)_{n\geqslant 0}$, and let η_i denote the mean of the first return time to i starting from i, in the Markov chain $(Y_n)_{n\geqslant 0}$. Then $\eta_i \leqslant \mu_i$.

Paper 2, Section I

3A Methods

Expand $f(x) = x^3 - \pi^2 x$ as a Fourier series on $-\pi < x < \pi$.

Use the series and Parseval's theorem for Fourier series (which you may quote without proof) to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

Paper 3, Section I

5A Methods

Calculate the Green's function $G(x;\xi)$ given by the solution to

$$\frac{d^2G}{dx^2} - G = \delta(x-\xi);\ G(0;\xi) = 0 \text{ and } G(x;\xi) \to 0 \text{ as } x \to \infty,$$

where $\xi \in (0,\infty)$, $x \in (0,\infty)$ and $\delta(x)$ is the Dirac δ-function.

Use this Green's function to calculate an explicit solution $y(x)$ to the boundary value problem

$$\frac{d^2y}{dx^2} - y = e^{-2x},$$

where $x \in (0,\infty)$, $y(0) = 0$ and $y(x) \to 0$ as $x \to \infty$.

Paper 1, Section II

13A Methods

(a) Let $y_0(x)$ be a non-trivial solution of the Sturm–Liouville problem

$$\mathcal{L}(y_0;\lambda_0)=0;\;\; y_0(0)=y_0(1)=0,$$

where

$$\mathcal{L}(y;\lambda)=\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+\left[q(x)+\lambda w(x)\right]y.$$

Show that, if $y(x)$ and $f(x)$ are related by

$$\mathcal{L}(y;\lambda_0)=f,$$

with $y(x)$ satisfying the same boundary conditions as $y_0(x)$, then

$$\int_0^1 y_0 f\,dx=0. \qquad (\star)$$

(b) Now assume that y_0 is normalised so that

$$\int_0^1 w y_0^2\,dx=1,$$

and consider the problem

$$\mathcal{L}(y;\lambda)=y^{m+1};\;\; y(0)=y(1)=0,$$

where m is a positive integer. By choosing f appropriately in $(\star)$ deduce that, if

$$\lambda-\lambda_0=\epsilon^m\mu \text{ and } y(x)=\epsilon y_0(x)+\epsilon^2 y_1(x),$$

where $0<\epsilon\ll 1$ and $\mu=O(1)$, then

$$\mu=\int_0^1 y_0^{m+2}\,dx+O(\epsilon).$$

Paper 2, Section II

14A Methods

(a) Laplace's equation in plane polar coordinates has the form

$$\nabla^2\phi = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\phi(r,\theta) = 0.$$

Using separation of variables, show that the general solution is:

$$\phi(r,\theta) = a_0 + c_0 \ln r + \sum_{n=1}^{\infty}\left(a_n r^n + c_n r^{-n}\right)\cos n\theta + \sum_{n=1}^{\infty}\left(b_n r^n + d_n r^{-n}\right)\sin n\theta,$$

for arbitrary real constants a_i, b_i, c_i and d_i.

Which (if any) constants must be zero for the solution to be regular in:

(i) the interior of a disc centred at the origin?

(ii) the exterior of a disc centred at the origin?

(iii) an annular region centred at the origin?

(b) Consider 2π-periodic functions $f(\theta)$ such that

$$f(\theta) = \sum_{n=1}^{\infty} A_n \cos n\theta,$$

for some coefficients A_n.

(i) Solve Laplace's equation $\nabla^2\phi = 0$ in the annulus $1 < r < e^2$ with boundary conditions:

$$\phi(r,\theta) = \begin{cases} f(\theta) - 1, & r = 1 \\ f(\theta) + 1, & r = e^2, \end{cases}$$

for general $f(\theta)$.

(ii) Calculate the explicit solution for the specific choice:

$$f(\theta) = \begin{cases} \frac{\pi}{2} - \theta, & 0 \leqslant \theta < \pi \\ -\frac{3\pi}{2} + \theta, & \pi \leqslant \theta < 2\pi. \end{cases}$$

Paper 3, Section II

14A Methods

(a) You are given that $f(x)$, $g(x)$ and $h(x)$ are all absolutely integrable functions with absolutely integrable Fourier transforms $\tilde{f}(k)$, $\tilde{g}(k)$ and $\tilde{h}(k)$ such that

$$\tilde{h}(k) = [\tilde{f}(k)][\tilde{g}(k)],$$

i.e. that $\tilde{h}(k)$ is the product of $\tilde{f}(k)$ and $\tilde{g}(k)$. Express $h(x)$ in terms of an integral expression involving $f(x)$ and $g(x)$.

(b) If $p'(x) = g(x)$, express $\tilde{p}(k)$ in terms of $\tilde{g}(k)$. [You may assume that the transforms are well-defined.]

(c) Express the inverse transforms of $\cos ka$ and $\sin ka$ in terms of the δ-function, where a is a positive constant.

(d) Consider the following wave problem for $u(x,t)$:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}; \; u(x,0) = f(x), \; \frac{\partial}{\partial t}u(x,0) = g(x).$$

Use parts (a)-(c) to construct d'Alembert's solution:

$$u(x,t) = \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(\xi)\, d\xi. \qquad (\star)$$

[No credit will be given for using any other approach to derive ($\star$). You may assume the expression derived in part (a) applies.]

(e) Consider the specific case

$$f(x) = 0; \; g(x) = \begin{cases} x & \text{for } |x| \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $t > 1$, identify a region of the x-t plane including the line $x = 0$ where $u(x,t) = 0$. Briefly interpret this result physically. [*Hint: You may find it useful to consider the lines* $x = 1 - t$ *and* $x = -1 + t$.]

[*The following convention is used in this question:*

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\, dx \;\; \textit{and} \;\; f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}\, dk.]$$

Paper 4, Section II

14A Methods

(a) Using Fourier transforms with respect to x, express in integral form the general solution $\theta(x,t)$ to the (unforced) heat equation with initial data $\Theta(x)$ and diffusivity $D>0$:

$$\frac{\partial\theta}{\partial t} = D\frac{\partial^2\theta}{\partial x^2};\ \theta(x,0) = \Theta(x).$$

[You may quote the convolution theorem for Fourier transforms without proof.]

(b) By constructing an appropriate Green's function, express in integral form the general solution $\theta_f(x,t)$ to the forced heat equation with homogeneous initial data:

$$\frac{\partial\theta_f}{\partial t} - D\frac{\partial^2\theta_f}{\partial x^2} = f(x,t);\ \theta_f(x,0) = 0,$$

for some function $f(x,t)$.

(c) Now consider the combined problem:

$$\frac{\partial\theta_c}{\partial t} - D\frac{\partial^2\theta_c}{\partial x^2} = -A\delta\left(x+2\sqrt{D}\right)\delta(t-1);\ \theta_c(x,0) = \delta\left(x-2\sqrt{D}\right),$$

where A is a positive real constant. Determine $\theta_c(x,t)$, and hence deduce that $\theta_c(0,2)=0$ if

$$A = \sqrt{\frac{e}{2}}.$$

[*The following convention is used in this question:*

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx \ \ and\ \ f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(k)e^{ikx}\,dk.$$

You may also quote the transform pair

$$g(x,t) = \frac{1}{\sqrt{4\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right);\ \tilde{g}(k,t) = e^{-Dk^2t},$$

as well as any relevant properties of the δ-function without proof.]

Paper 1, Section I

5B Numerical Analysis

Given a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $\boldsymbol{y} \in \mathbb{R}^m$ where $m \geqslant n$, consider the problem of finding $\boldsymbol{c}^* \in \mathbb{R}^n$ that minimises $\|A\boldsymbol{c} - \boldsymbol{y}\|_2$ for $\boldsymbol{c} \in \mathbb{R}^n$, where $\| \cdot \|_2$ is the standard Euclidean norm.

(a) Prove that $\boldsymbol{c}^*$ is a solution to the above minimisation problem if and only if $A^T A\boldsymbol{c}^* = A^T\boldsymbol{y}$.

(b) Show that if A is of full rank, then $\boldsymbol{c}^*$ is unique.

Paper 4, Section I

6B Numerical Analysis

Consider the inner product

$$\langle g, h \rangle = \int_a^b g(x)h(x)w(x)\,dx \qquad (*)$$

on $C[a,b]$, where $w(x) > 0$ for $x \in (a,b)$. Define $\|g\|^2 = \langle g, g \rangle$. Let $Q_0, Q_1, Q_2, \ldots$ be orthogonal polynomials with respect to the inner product $(*)$, and let $f \in C[a,b]$.

(a) Prove that the polynomial $p_n^* \in \mathcal{P}_n$ that minimises the squared distance $\|f - p\|^2$ among all $p \in \mathcal{P}_n$ is given by

$$p_n^*(x) = \sum_{k=0}^{n} \frac{\langle f, Q_k \rangle}{\langle Q_k, Q_k \rangle} Q_k(x).$$

(b) Hence, show that

$$\|f\|^2 = \|f - p_n^*\|^2 + \|p_n^*\|^2.$$

Paper 1, Section II

17B Numerical Analysis

Consider the ODE

$$y' = f(y), \quad y(0) = y_0 > 0, \tag{$*$}$$

where $f(y) = -\mathrm{sign}(y)$, $y(t) \in \mathbb{R}$ and $t \in [0,T]$, with $T > y_0$. The sign function is defined as

$$\mathrm{sign}(y) = \begin{cases} 1 & \text{for } y > 0 \\ 0 & \text{for } y = 0 \\ -1 & \text{for } y < 0. \end{cases}$$

(a) Does the function f satisfy a Lipschitz condition for $y \in \mathbb{R}$? Justify your answer.

(b) Show that there is a unique continuous function $y : [0,T] \to \mathbb{R}$ that is differentiable for all $t \in [0,T]$ except for some $\tilde{t} \in (0,T]$ and satisfies the ODE $(*)$ for all $t \in [0,T] \setminus \tilde{t}$.

(c) The Euler method for $(*)$ produces a sequence $\{y_n\}_{n \leqslant N}$, where $N = \lfloor \frac{T}{h} \rfloor$ and $h > 0$ is the step-size. Is

$$|y_n - y(nh)| \leqslant \mathcal{O}(h), \qquad \text{for } 0 \leqslant n \leqslant N,$$

where $y(t)$ is the solution described in part (b)? Justify your answer.

Paper 2, Section II

17B Numerical Analysis

Consider an ODE of the form

$$y' = f(y), \quad y(0) = y_0 \in \mathbb{R}, \tag{$*$}$$

where $y(t)$ exists and is unique for $t \in [0,T]$ and $T > 0$.

(a) For a numerical method approximating the solution of $(*)$, define the linear stability domain. What does it mean for such a numerical method to be A-stable?

(b) Let $a \in \mathbb{R}$ and consider the Runge–Kutta method—producing a sequence $\{y_n\}_{n \leqslant N}$, where $N = \lfloor \frac{T}{h} \rfloor$ and $h > 0$ is the step-size—defined by

$$\begin{aligned} k_1 &= f\left(y_n + \frac{1}{4}hk_1 + \left(\frac{1}{4} - a\right)hk_2\right), \\ k_2 &= f\left(y_n + \left(\frac{1}{4} + a\right)hk_1 + \frac{1}{4}hk_2\right), \\ y_{n+1} &= y_n + \frac{1}{2}h(k_1 + k_2), \quad n = 0, 1, \ldots, N-1. \end{aligned}$$

Determine the values of the parameter $a \in \mathbb{R}$ for which the Runge–Kutta method is A-stable.

Paper 3, Section II

17B Numerical Analysis

Consider $C[a,b]$ equipped with the inner product $\langle f,g\rangle = \int_a^b f(x)g(x)w(x)\mathrm{d}x$, where $w(x) > 0$ for $x \in (a,b)$. Let $\mathcal{P}_n$ denote the set of polynomials of degree less than or equal to n. For $f \in C[a,b]$ consider the quadrature formulas

$$I(f) = \int_a^b f(x)w(x)\mathrm{d}x \approx \sum_{i=0}^{n} a_i^{(n)} f(x_i^{(n)}) = I_n(f), \quad n = 0,1,2,\ldots \qquad (*)$$

with weights $a_i^{(n)} \in \mathbb{R}$ and nodes $x_i^{(n)} \in [a,b]$, which are exact on all polynomials $q \in \mathcal{P}_n$.

(a) Prove that the quadrature formula $(*)$ is exact for all $q \in \mathcal{P}_{n+1+k}$ if and only if the polynomial $Q_{n+1}(x) = \prod_{i=0}^{n}(x - x_i^{(n)})$ is orthogonal (with respect to $\langle \cdot,\cdot\rangle$) to all polynomials of degree k.

(b) Prove that no quadrature formula $(*)$ could be exact on polynomials of degree $2n+2$.

(c) Prove that if $(*)$ is exact on $\mathcal{P}_{2n}$, then $a_i^{(n)} > 0$.

(d) Show that if $a_i^{(n)} > 0$ for all i and n, then

$$I_n(f) \to I(f), \qquad n \to \infty.$$

[*Hint: Use the Weierstrass theorem: for any* $\epsilon > 0$ *there exists* $n \in \mathbb{N}$ *and a polynomial* $p_n \in \mathcal{P}_n$ *such that* $|f(x) - p_n(x)| < \epsilon$, *for* $x \in [a,b]$.]

Paper 1, Section I
7H Optimisation

What is the *minimum-cost flow problem* on a graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set E? Your answer should be in terms of

- a cost matrix $C \in \mathbb{R}^{n \times n}$,
- a vector $b \in \mathbb{R}^n$ whose i-th entry is the amount of flow that enters vertex i,
- a lower bound on the flow given by a matrix $\underline{M} \in \mathbb{R}^{n \times n}$, and
- an upper bound on the flow given by a matrix $\overline{M} \in \mathbb{R}^{n \times n}$.

Show that we can always assume $\underline{M} = 0$ by constructing an equivalent problem to the general problem above. Explain why the problems are equivalent.

Paper 2, Section I
7H Optimisation

Solve the following optimisation problem using the Lagrange sufficiency theorem:

$$\begin{aligned} \text{minimise} \quad & x^2 + y^4 + z^6 \\ \text{subject to} \quad & x + 2y + 3z = 6. \end{aligned}$$

Does strong duality hold for this problem?

Let ϕ be the value function $\phi(b) = \inf\{x^2 + y^4 + z^6 : x + 2y + 3z = b\}$. Evaluate the derivative $\phi'(6)$.

Paper 3, Section II

19H Optimisation

Let $S \subset \mathbb{R}^3$ be the set of all $(x_1, x_2, x_3) \in \mathbb{R}^3$ satisfying the following linear inequalities:

$$\begin{gathered} 0 \leqslant x_1, x_2, x_3 \leqslant 1, \\ x_1 + x_2 + x_3 \leqslant 2.5. \end{gathered}$$

(a) Show that S is a non-empty convex set.

(b) What is meant by an *extreme point* of a convex set? Find all extreme points of S.

(c) Suppose we want to solve the following linear program:

$$\begin{aligned} \text{maximise} \quad & x_1 + 2x_2 + 4x_3 \\ \text{subject to} \quad & (x_1, x_2, x_3) \in S. \end{aligned}$$

What is the solution to this problem and where is it attained?

(d) Suppose the simplex method is initialised at $(0, 0, 0)$ to solve the above linear program. Recall that depending on the choices of pivot elements made at each step, many different outcomes are possible. Here, an outcome denotes the path the simplex method takes over the basic feasible solutions of the problem.

What is the smallest number of steps in which the simplex method can find the solution? What is the largest number of steps in which the simplex method can find the solution? Calculate the total number of distinct outcomes possible when the simplex method is initialised at $(0, 0, 0)$.

It may be helpful to draw a picture.

Paper 4, Section II

18H Optimisation

Let A be the $m \times n$ payoff matrix of a two-person, zero-sum game. What is Player I's optimization problem?

Write down a sufficient condition that a vector $p \in \mathbb{R}^m$ is an optimal mixed strategy for Player I in terms of the optimal mixed strategy for Player II and the value of the game.

If $m = n$ and A is an invertible, symmetric matrix such that $A^{-1}e \geqslant 0$, where $e = (1, 1, \ldots, 1)^\top \in \mathbb{R}^m$, show that the value of the game is $(e^\top A^{-1} e)^{-1}$.

Consider the following game: Players I and II each have three cards labelled 1, 2, and 3. Each player chooses one of their cards, independently of the other player, and places it in the same envelope. If the sum of the numbers in the envelope is smaller than or equal to 4, then Player II pays Player I the sum (in £), and otherwise Player I pays Player II the sum. (For instance, if Player I chooses card 3 and Player II chooses card 2, then Player I pays Player II £5.) What is the optimal strategy for each player?

Paper 3, Section I

6D Quantum Mechanics

Consider the one-dimensional, time-independent Schrödinger equation:

$$\frac{d^2\chi(x)}{dx^2} + \frac{2m}{\hbar^2}[E - U(x)]\chi(x) = 0, \qquad x \in \mathbb{R}.$$

(a) Explain the meaning of the functions $\chi(x), U(x)$ and parameters $E, m, \hbar$.

(b) Solutions of this equation describing bound states correspond to $\chi(x) \to 0$ for $x \to \pm\infty$. Are there bound states for a potential that asymptotes to a constant U_0 (that is $U(x) \to U_0$ as $x \to \pm\infty$) for the cases $E > U_0 > 0$ and $0 < E < U_0$?

(c) Show, by contradiction or otherwise, that the energy spectrum of bound states is non-degenerate.

Paper 4, Section I

4D Quantum Mechanics

(a) Prove Ehrenfest's theorem in one-dimensional quantum mechanics:

$$\frac{d}{dt}\langle \hat{O} \rangle_\psi = \frac{i}{\hbar}\langle [\hat{H}, \hat{O}] \rangle_\psi + \langle \frac{\partial \hat{O}}{\partial t} \rangle_\psi,$$

where $\hat{O}$ is a Hermitian operator, $\hat{H}$ is the Hamiltonian and

$$\langle \hat{O} \rangle_\psi = \int \psi^*(x,t)\hat{O}\psi(x,t)dx$$

is the expectation value of the operator $\hat{O}$ in a state determined by the wave function $\psi(x,t)$.

(b) Using Ehrenfest's theorem prove that

$$m\frac{d}{dt}\langle \hat{x} \rangle_\psi = \langle \hat{p} \rangle_\psi, \qquad \frac{d}{dt}\langle \hat{p} \rangle_\psi = -\langle \frac{dU}{dx} \rangle_\psi, \qquad \frac{d}{dt}\langle \hat{H} \rangle_\psi = 0,$$

where $U(x)$ is the scalar potential. Compare with similar expressions in classical mechanics.

Paper 1, Section II

14D Quantum Mechanics

Consider a physical observable O represented by a Hermitian operator $\hat{O}$ acting on a Hilbert space $\mathcal{H}$. We define the uncertainty $\Delta_\psi O$ in a measurement of O on a state ψ as $(\Delta_\psi O)^2 = \langle \hat{O}^2 \rangle_\psi - \langle \hat{O} \rangle_\psi^2$ with the expectation value defined as $\langle \hat{O} \rangle_\psi = (\psi, \hat{O}\psi)$.

(a) Using the Schwartz inequality $|(\phi, \psi)|^2 \leqslant (\phi, \phi)(\psi, \psi)$ for two states ϕ, ψ, prove the generalised uncertainty relation for the observables A, B:

$$(\Delta_\psi A)(\Delta_\psi B) \geqslant \frac{1}{2}\left| \Big(\psi, [\hat{A}, \hat{B}]\psi\Big) \right|, \tag{$\dagger$}$$

where $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ is the commutator of $\hat{A}$ and $\hat{B}$.

(b) Given the two Hermitian operators $\hat{X}$ and $\hat{Y}$ and a real parameter λ, we define

$$f(\lambda) = \Big\langle \big(\hat{X} - i\lambda\hat{Y}\big)\big(\hat{X} + i\lambda\hat{Y}\big) \Big\rangle_\psi .$$

Minimising $f(\lambda)$ and using the fact that $f(\lambda) \geqslant 0$, provide an alternative derivation of the uncertainty relation ($\dagger$).

(c) For the position and momentum operators, $\hat{x}$ and $\hat{p} = -i\hbar\frac{\partial}{\partial x}$, respectively, find their commutator $[\hat{x}, \hat{p}]$ and derive the Heisenberg uncertainty relation $\Delta_\psi x \Delta_\psi p \geqslant \frac{1}{2}\hbar$.

(d) Show that a Gaussian wave function $\psi(x) = Ce^{-\alpha x^2}$ solves the one-dimensional Schrödinger's equation for a quadratic potential $U(x) = kx^2$ with $k > 0$. Determine the value of the constants α, C and the energy E in terms of k and the particle's mass m. Show that this wave function saturates the Heisenberg uncertainty relation ($\Delta_\psi x \Delta_\psi p = \frac{1}{2}\hbar$). Furthermore, show that in order to saturate this Heisenberg relation, the wave function has to be Gaussian. [*Hint: You may use* $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$ *and* $\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \sqrt{\frac{\pi}{4a^3}}$.]

Paper 2, Section II
15D Quantum Mechanics

(a) Consider the Schrödinger equation for the wave function $\psi(\mathbf{r},t)$ corresponding to a particle subject to a real potential energy $U(\mathbf{r},t)$. Defining the probability density $\rho(\mathbf{r},t) = |\psi(\mathbf{r},t)|^2$ and probability current density

$$\mathbf{J}(\mathbf{r},t) = -\frac{i\hbar}{2m}\left[\psi^* \nabla\psi - (\nabla\psi)^*\,\psi\right],$$

derive and interpret the continuity equation $\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0$.

(b) Consider the one-dimensional Schrödinger equation with a step potential

$$U(x) = \begin{cases} 0 & x < -a \\ U_0 & -a < x < a \\ 0 & x > a, \end{cases}$$

where $a > 0, U_0 > 0$.

(i) Using matching conditions at $x = \pm a$, find the transmitted wave function $\psi(x,t)$ and probability density $\rho(x,t)$ in the region $x > a$, for an incident wave corresponding to a particle of mass m and energy $E = U_0/2$ moving towards the potential barrier from $x < -a$. Express the results in terms of the quantity $k = \sqrt{2mE}/\hbar$.

(ii) Compute the ratio between the transmitted and the incident current densities and interpret the result in terms of the continuity equation.

Paper 4, Section II

15D Quantum Mechanics

(a) Using the canonical commutation relations $[\hat{x}_i, \hat{p}_j] = \mathrm{i}\hbar\delta_{ij}$ with $i, j = 1, 2, 3$, show that the angular momentum operators $\hat{L}_i = \epsilon_{ijk}\hat{x}_j\hat{p}_k$ satisfy the commutation relations:

$$[\hat{L}_i, \hat{L}_j] = \mathrm{i}\hbar\epsilon_{ijk}\hat{L}_k, \qquad [\hat{L}_i, \hat{x}_j] = \mathrm{i}\hbar\epsilon_{ijk}\hat{x}_k, \qquad [\hat{L}_i, \hat{p}_j] = \mathrm{i}\hbar\epsilon_{ijk}\hat{p}_k.$$

Using these relations show that $[\hat{L}^2, \hat{L}_i] = 0$ where $\hat{L}^2 = \hat{L}_i\hat{L}_i$. Show further that for a spherically symmetric system $[\hat{L}^2, \hat{H}] = 0$, where the Hamiltonian $\hat{H}$ takes the form $\hat{H} = \frac{\hat{p}^2}{2m} + U(\hat{r})$. Can the operators $\hat{H}, \hat{L}^2, \hat{L}_3$ be simultaneously diagonalised? Justify your answer.

(b) Consider the Schrödinger equation for the Hydrogen atom in which the potential energy is $U(r) = -\frac{q^2}{r}$. Concentrating on the wave function with zero eigenvalues for both $\hat{L}_3$ and $\hat{L}^2$, the equation for the radial component of the wave function, $R(r)$, reduces to:

$$R'' + \frac{2}{r}R' + \left(\frac{\beta}{r} - \gamma^2\right) R = 0,$$

where $\beta = \frac{2mq^2}{\hbar^2}$ and $\gamma^2 = -\frac{2mE}{\hbar^2}$, with E denoting the energy.

(i) Considering the $r \to \infty$ limit, explain why $R \sim e^{-\gamma r}$.

(ii) Consider then the series solution

$$R(r) = f(r)e^{-\gamma r}, \qquad f(r) = \sum_n a_n r^n.$$

Derive the recurrence relation

$$a_n = \frac{2\gamma n - \beta}{n(n+1)} a_{n-1},$$

then argue why the energy is quantised and determine the ground state energy.

(iii) Using the ground state wave function $R(r) = Ce^{-\gamma r}$, determine the normalisation factor C and estimate the expectation value of the radius $\langle r \rangle_R$. Compare with the Bohr radius.

Paper 1, Section I

6H Statistics

(a) Define the *generalized likelihood ratio test statistic* and state Wilks' Theorem.

(b) The following experiment was conducted in the late 1800s to determine whether the use of carbolic acid was helpful in amputations. Out of 75 amputations, with and without carbolic acid, the following data were collected:

	Carbolic acid used	Carbolic acid not used
Patient lived	34	19
Patient died	6	16

Describe a hypothesis test to determine whether the use of carbolic acid affects the rate of patient mortality: What is the null hypothesis, and what is the alternative? What is an appropriate statistic and what is the critical region for a test of size α? You need not calculate the value of the statistic.

Paper 2, Section I

6H Statistics

Let $\sigma > 0$ be a fixed, known constant, and let $X_1, \ldots, X_n$ be i.i.d. $\mathcal{N}(\theta, \sigma^2)$ random variables.

(a) Compute a maximum likelihood estimator $\hat{\theta}_{MLE}$ for θ.

(b) What is a 95% confidence interval for θ based on $\hat{\theta}_{MLE}$?

Now suppose θ has a prior distribution $\theta \sim \mathcal{N}(\mu, \nu^2)$, where $\mu \in \mathbb{R}$ and $\nu > 0$ are both known. The posterior distribution of θ, given the observations $\{X_1, \ldots, X_n\}$ is known to be

$$\mathcal{N}\left(\frac{\frac{n\bar{X}}{\sigma^2} + \frac{\mu}{\nu^2}}{\frac{n}{\sigma^2} + \frac{1}{\nu^2}}, \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\nu^2}}\right), \quad \text{where} \quad \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

(c) What is a 95% credible interval for θ based on the posterior?

(d) Compare the answers to parts (b) and (c) as $n \to \infty$.

Paper 1, Section II

18H Statistics

Suppose X_1 and X_2 are i.i.d. $\mathcal{N}(\mu, 1)$ random variables.

(a) Write down the joint probability density function of (X_1, X_2).

(b) Prove that $T = X_1 + X_2$ is a sufficient statistic for μ. Is it a minimal sufficient statistic? Justify your answer.

(c) Suppose we wish to estimate $\theta := \mu^2$. Prove that $S = X_1^2 - 1$ is an unbiased estimator of θ. Find the mean square error of S. You may use the fact that $\mathbb{E}[Z^4] = 3$ for $Z \sim \mathcal{N}(0, 1)$.

(d) What is the probability density function of X_1 conditioned on T?

(e) Use the Rao–Blackwell theorem to derive an estimator with strictly smaller mean square error than S for estimating θ. Calculate the mean square error for the new estimator you derive and compare it with the mean square error of S calculated in part (c).

Paper 3, Section II

18H Statistics

(a) Define a *uniformly most powerful (UMP) test* when $X \sim f(\cdot|\theta)$ for $\theta \in \Theta$, and the two hypotheses correspond to

$$\begin{aligned} H_0 &: \theta \in \Theta_0 \subseteq \Theta \\ H_1 &: \theta \in \Theta_1 \subseteq \Theta. \end{aligned}$$

(b) Let $f(x|\theta)$ be the logistic location probability density function

$$f(x|\theta) = \frac{e^{(x-\theta)}}{(1 + e^{(x-\theta)})^2}, \qquad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

(i) Based on one observation X, find the most powerful size-α test of $H_0 : \theta = 0$ versus $H_1 : \theta = 1$. You may use any results from the lectures without proof provided you state them clearly.

(ii) Prove that the test in part (i) is UMP of size α for testing $H_0 : \theta \leqslant 0$ versus $H_1 : \theta > 0$.

Paper 4, Section II

17H Statistics

An ecologist takes data $\{(x_i, y_i)\}_{i=1}^n$, where $x_i \geqslant 0$ is the size of an area and $y_i \in \mathbb{N}$ is the number of moss plants in the area. For fixed $\{x_i\}_{i=1}^n$, we model the data by $Y_i \sim \text{Poisson}(\theta x_i)$, where the Y_i are independent of each other.

(a) Write down a linear model relating the y_i to the x_i. Derive a formula for the least squares estimator $\hat{\theta}_{LS}$. Is the estimator biased?

(b) Compute the maximum likelihood estimator $\hat{\theta}_{MLE}$. Is the estimator biased?

(c) Compare the variances of $\hat{\theta}_{LS}$ and $\hat{\theta}_{MLE}$.

(d) Suppose we wish to test the hypotheses $H_0 : \theta = 1$ versus $H_1 : \theta = 2$. Describe a hypothesis test with test statistic $\hat{\theta}_{MLE}$, which has approximate size 0.05 when $\sum_{i=1}^n x_i$ is large. Describe a hypothesis test with test statistic $\hat{\theta}_{LS}$, which has approximate size 0.05 when each x_i is large. *[Hint: A Poisson*(λ) *distribution may be approximated by a* $\mathcal{N}(\lambda, \lambda)$ *distribution when* λ *is large.]*

Paper 1, Section I

4C Variational Principles

Briefly explain how to use a Lagrange multiplier to find the extrema of a function $f(\mathbf{x})$ subject to a constraint $g(\mathbf{x}) = 0$.

Find the maximum volume of a cuboid of side lengths $x \geqslant 0$, $y \geqslant 0$, and $z \geqslant 0$ whose space diagonal has length L.

Paper 3, Section I

4C Variational Principles

Consider a function $f\colon \mathbb{R}^n \to \mathbb{R}$, not necessarily differentiable. What does it mean for f to be *convex* in a domain D?

If f is once differentiable, state an equivalent condition involving ∇f at two points $\mathbf{x}$ and $\mathbf{y}$ in D.

If f is twice differentiable, state an equivalent condition involving the Hessian $\mathbf{H}$.

Compute the largest domain on which the function $f(x, y) = x^3 + y^3 + Axy$ is convex in $\mathbb{R}^2$ (A is a constant) and sketch it.

Paper 2, Section II
13C Variational Principles

(a) For a functional of the form

$$\mathcal{L}[y] = \int_a^b F(x, y, y', y'')\,\mathrm{d}x,$$

derive the Euler–Lagrange equation satisfied by the solution $y(x)$ leading to a stationary value of $\mathcal{L}$. Show that all boundary terms cancel if the solution is assumed to have fixed values for y and y' at the end points.

(b) A diving board of length L at a swimming pool takes the shape $y(x)$ that minimises the energy

$$\mathcal{E} = \int_0^L \left[\frac{1}{2}A(y'')^2 + \rho g y\right] \mathrm{d}x,$$

where $A > 0$ is the bending rigidity, $\rho > 0$ the mass density and $g > 0$ the acceleration due to gravity (A, ρ, g are constants).

(i) Derive the ODE satisfied by $y(x)$.

(ii) The board is clamped at the origin (i.e. $y(0) = 0$, $y'(0) = 0$) while at $x = L$, it is torque free (i.e. $y''(L) = 0$) and a vertical force of magnitude F is applied to it (i.e. $-Ay'''(L) = F$). Solve for $y(x)$ and show that it may be written as $y(x) = y_0 + y_F$, where y_0 is the solution when $F = 0$ and y_F is proportional to F.

(iii) Compute the vertical displacement at the end of the board, $\Delta = y(L)$, and show that it can be written as $\Delta = h_0 + h$, where h_0 is the displacement when $F = 0$ and h is proportional to F.

(iv) For the solution in part (ii) compute the corresponding value of the energy $\mathcal{E}$ and show that it can be written as $\mathcal{E} = E_0 + E$, with E_0 independent of F and E quadratic in F.

(v) Relate $\frac{\mathrm{d}E}{\mathrm{d}F}$ to h and interpret your result.

Paper 4, Section II

13C Variational Principles

(a) Consider a functional of the form

$$\mathcal{L}[u,v] = \iint_\Omega f(x,y,u,v,u_x,u_y,v_x,v_y)\,\mathrm{d}x\mathrm{d}y,$$

where u and v are functions of x and y [we use the notation a_b to denote the partial derivative $\partial a/\partial b$]. Assuming small variations $u \to u + \delta u$ and $v \to v + \delta v$ and using integration by parts, derive the two Euler–Lagrange equations satisfied by u and v in Ω associated with an extremum of $\mathcal{L}$ (you may ignore all contributions from boundary terms).

(b) An elastic material deforms in two dimensions with a displacement field $\mathbf{u}(\mathbf{x}) = [u(x,y), v(x,y)]$, that minimises the total elastic energy

$$\mathcal{J} = \iint_\Omega \left[\frac{1}{2}\mu(\nabla\mathbf{u} : \nabla\mathbf{u}^T) + \frac{1}{2}(\lambda+\mu)(\nabla\cdot\mathbf{u})^2\right]\mathrm{d}x\mathrm{d}y$$

where $\nabla\mathbf{u}$ is the displacement gradient tensor, defined as

$$\nabla\mathbf{u} = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix},$$

where μ and λ are two material constants and where we use the notation $\mathbf{A} : \mathbf{B}$ to refer to the trace of the matrix product $\mathbf{AB}$.

(i) Show that

$$\mathcal{J} = \iint_\Omega \left[\left(\frac{\lambda}{2}+\mu\right)(u_x^2+v_y^2) + \frac{\mu}{2}(u_y^2+v_x^2) + (\lambda+\mu)u_x v_y\right]\mathrm{d}x\mathrm{d}y.$$

(ii) Derive the two Euler–Lagrange equations satisfied by u and v and show that they can be combined into a single equation for $\mathbf{u}$.

(iii) In the one-dimensional limit where $v = 0$, $\partial u/\partial y = 0$ with boundary conditions $u(0) = 0$, $u(L) = \Delta$, show that the solution to the equation obtained in (ii) is linear in x.

END OF PAPER