UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA 2023

List of Courses

Analysis
Differential Equations
Dynamics and Relativity
Groups
Numbers and Sets
Probability
Vector Calculus
Vectors and Matrices

Paper 1, Section I

3E Analysis

Let $a \in \mathbb{R}$ and let f and g be real-valued functions defined on $\mathbb{R}$. State and prove the chain rule for $F(x) = g\big(f(x)\big)$.

Now assume that f and g are non-constant on any interval. Must the function $F(x) = g\big(f(x)\big)$ be non-differentiable at $x = a$ if

(i) f is differentiable at a and g is not differentiable at $f(a)$?

(ii) f is not differentiable at a and g is differentiable at $f(a)$?

(iii) f is not differentiable at a and g is not differentiable at $f(a)$?

Justify your answers.

Paper 1, Section I

4E Analysis

State the comparison test. Prove that if $\sum_{n=0}^{\infty} a_n z_0^n$ converges and $|z_1| < |z_0|$, then $\sum_{n=0}^{\infty} a_n z_1^n$ converges absolutely.

Define the *radius of convergence* of a complex power series. [You do not need to show that the radius of convergence is well-defined.]

If $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R_1 and $\sum_{n=0}^{\infty} b_n z^n$ has radius of convergence R_2, show that the radius of convergence R of the series $\sum_{n=0}^{\infty} a_n b_n z^n$ satisfies $R \geqslant R_1 R_2$.

Paper 1, Section II

9E Analysis

(a) Let $x_1 > 0$ and define a sequence (x_n) by

$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{1}{x_{n-1}}\right) \text{ for } n > 1.$$

Prove that $\lim_{n\to\infty} x_n = 1$.

Show that if a real sequence (x_n) satisfies

$$0 \leqslant x_{m+n} \leqslant x_m + x_n \quad \text{for all } m, n = 1, 2, \ldots,$$

then the sequence (x_n/n) is (i) bounded and (ii) convergent.

(b) Suppose that a series $\sum_{n=1}^{\infty} a_n$ of real numbers converges but not absolutely. Let

$$P_n = \sum_{i=1}^{n} \big(|a_i| + a_i\big), \qquad N_n = \sum_{i=1}^{n} \big(|a_i| - a_i\big).$$

Show that $\lim_{n\to\infty} P_n/N_n = 1$.

State the alternating series test. Let (b_n) be a sequence of positive real numbers such that

$$\lim_{n\to\infty} n\left(\frac{b_n}{b_{n+1}} - 1\right) = p,$$

where p is a positive real number. Show that the series $\sum_{n=1}^{\infty}(-1)^n b_n$ converges.

Paper 1, Section II

10E Analysis

State and prove the intermediate value theorem.

Give, with justification, an example of a function $\phi : [a,\infty) \to \mathbb{R}$ such that, for any $b > a$, ϕ takes on $[a,b]$ every value between $\phi(a)$ and $\phi(b)$ but ϕ is not continuous on $[a,b]$.

If a function $f : [a,b] \to \mathbb{R}$ is monotone on $[a,b]$ and takes every value between $f(a)$ and $f(b)$, show that f is continuous on $[a,b]$.

Let $g : (a,b) \to \mathbb{R}$ be a continuous function and suppose that there are sequences $x_n \to a$ and $y_n \to a$ as $n \to \infty$ such that $g(x_n) \to l$ and $g(y_n) \to L$ with $l < L$. Show that for each $\lambda \in [l, L]$ there is a sequence $z_n \to a$ such that $g(z_n) \to \lambda$.

Paper 1, Section II

11E Analysis

(a) State the mean value theorem. Deduce that

$$\frac{a-b}{a} < \log\frac{a}{b} < \frac{a-b}{b} \quad \text{for } 0<b<a.$$

(b) Let $f:\mathbb{R}\to\mathbb{R}$ be an n-times differentiable function, where $n>0$. Show that for each $a\in\mathbb{R}$ and $h>0$ there exists $b\in(a,a+nh)$ such that

$$\frac{1}{h^n}\Delta_h^n f(a) = f^{(n)}(b),$$

where $\Delta_h^{k+1}f(x) = \Delta_h^1\big(\Delta_h^k f(x)\big)$ and $\Delta_h^1 f(x) = f(x+h)-f(x)$.

(c) Let $I\subset\mathbb{R}$ be an open (non-empty) interval and $a\in I$. Suppose that a function $\varphi: I\to\mathbb{R}$ has a finite limit at a and $\lim_{x\to a}\varphi(x)=\varphi(a)+1$. Can φ be the derivative of some differentiable function f on I? Justify your answer.

Paper 1, Section II

12E Analysis

Define the *upper* and *lower integral* of a function on $[a,b]$ and what it means for a function to be *(Riemann) integrable* on $[a,b]$.

(a) Let $\lfloor y\rfloor = \max\{i\in\mathbb{Z} : i\leqslant y\}$. Show that the function

$$u(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \quad \text{if } x\neq 0, \qquad u(0)=0,$$

is integrable on $[0,1]$. [You may assume that every continuous function on a closed bounded interval is integrable.]

(b) Let $f:[A,B]\to\mathbb{R}$ be a continuous function and $A<a<x<B$. Prove that

$$\lim_{h\to 0}\frac{1}{h}\int_a^x \big(f(t+h)-f(t)\big)dt = f(x)-f(a).$$

[Any version of the fundamental theorem of calculus from the course can be assumed if accurately stated.]

(c) Show that if a function $g:[a,b]\to\mathbb{R}$ is integrable, then there exists a sequence of continuous functions $\varphi_n:[a,b]\to\mathbb{R}$ such that $\int_\alpha^\beta g(x)dx = \lim_{n\to\infty}\int_\alpha^\beta \varphi_n(x)dx$ for any subinterval $[\alpha,\beta]\subseteq[a,b]$.

Paper 2, Section I

1A Differential Equations

Find the general solution $y(x)$ of the differential equation

$$y''' - 4y'' + 4y' = xe^{2x}.$$

Paper 2, Section I

2A Differential Equations

(a) Find the solution $y(x)$ of

$$x^2 y' - \cos(2y) = 1$$

subject to $y \to 9\pi/4$ as $x \to \infty$. [If your answer involves inverse trigonometric functions, then you should specify their range.]

(b) Find the general solution $u(x)$ of the equation

$$xu' = x + u.$$

Paper 2, Section II

5A Differential Equations

(a) Consider the linear differential equation

$$y' + p(x)y = f(x), \qquad (*)$$

where $p(x)$ and $f(x)$ are given nonzero functions. Show how to express the general solution $y(x)$ in terms of two integrals involving $p(x)$ and $f(x)$, to be specified.

If $y_1(x)$ and $y_2(x)$ are distinct solutions of $(*)$, express the general solution of $(*)$ in terms of $y_1(x)$ and $y_2(x)$.

(b) Find the general solution $y(x)$ of the differential equation

$$xy' - (2x^2 + 1)y = x^2.$$

Show that there is only one solution of this equation with $y(x)$ bounded as $x \to \infty$, and determine its limiting value. Sketch this solution.

Paper 2, Section II

6A Differential Equations

The function $y(x,\mu)$ satisfies

$$\frac{\partial y}{\partial x} = y + \mu(x + y^2)\,, \quad y(0,\mu) = 1\,, \qquad (*)$$

and the function $u(x,\mu)$ is defined by $u = \partial y/\partial \mu$. Show that

$$\frac{\partial u}{\partial x} = u + x + y^2 + 2\mu y u\,, \quad u(0,\mu) = 0\,.$$

Determine $y(x,0)$ and then $u(x,0)$.

For small μ, the solution of $(*)$ can be approximated by a series

$$y(x,\mu) = y_0(x) + \mu y_1(x) + \mu^2 y_2(x) + \cdots\,.$$

Specify the functions $y_0(x), y_1(x)$ and $y_2(x)$.

Paper 2, Section II

7A Differential Equations

The Dirac δ-function can be defined by the properties $\delta(t) = 0$ for $t \neq 0$ and $\int_a^b f(t)\delta(t)\,dt = f(0)$ for any $a < 0 < b$ and function $f(t)$ that is continuous at $t = 0$. The function $H(t)$ is defined by

$$H(t) = \begin{cases} 1 & \text{for} \quad t \geqslant 0 \\ 0 & \text{for} \quad t < 0\,. \end{cases}$$

(a) Prove that

(i) $\delta(pt) = \delta(t)/|p|$ for any nonzero real constant p;

(ii) for any differentiable function $f(t)$

$$\int_{-\infty}^{\infty} f(t)\delta'(t)\,dt = -f'(0)\,;$$

(iii) $H'(t) = \delta(t)$.

(b) An electronic system has two time-dependent variables $x(t)$ and $y(t)$, and two inputs to which a constant unit signal is applied, each starting at a particular time. The differential equations governing the system take the form

$$\begin{aligned} \dot{x} + 2y &= H(t)\,, \\ \dot{y} - 2x &= H(t-\pi)\,. \end{aligned}$$

At $t = -\pi$, the system has $x = 1$ and $y = 0$. Find $x(t)$ for $t < 0$. Show that $x(t)$ can be written for $t > 0$ as

$$x(t) = a\sin 2t + b + q(t)\sin^2 t\,,$$

where the constants a and b and the function $q(t)$ are to be specified. Sketch $q(t)$ for $0 < t < 2\pi$.

Paper 2, Section II

8A Differential Equations

(a) Classify the equilibrium point of the system

$$\frac{dx}{dt} = 4x + 2y\,, \quad \frac{dy}{dt} = -x + y\,.$$

Sketch the phase portrait showing both the direction of any straight-line trajectories and the shapes of a representative selection of non-straight trajectories to indicate the direction of motion in each part of phase space.

(b) Consider the second-order differential equation for $x(t)$

$$\ddot{x} + 3\dot{x} - 4\log\frac{x^2+1}{2} = 0\,.$$

(i) Rewrite the equation as a system of two first-order equations for $x(t)$ and $y(t)$, where $y = \dot{x}$, and find the equilibrium points of that system.

(ii) Use linearisation to classify the equilibrium points.

(iii) On a sketch of the (x, y)-plane, show the regions where $\dot{x}$ and $\dot{y}$ are both positive, both negative, or one positive and one negative.

(iv) Using the information obtained in parts (i)–(iii), sketch the trajectories of the system, including the trajectories through $(1, 0)$.

Paper 4, Section I

3C Dynamics and Relativity

A rocket moves vertically upwards in a uniform gravitational field and emits exhaust gas downwards with time-dependent speed $U(t)$ relative to the rocket. Derive the rocket equation

$$m(t)\frac{dv}{dt} + U(t)\frac{dm}{dt} = -m(t)g\,,$$

where $m(t)$ and $v(t)$ are respectively the rocket's mass and upward speed at time t.

Suppose now that $m(t) = m_0 - \alpha t$ and $U(t) = U_0 m_0/m(t)$, where m_0, U_0 and α are constants. What is the condition for the rocket to lift off from rest at $t = 0$? Assuming that this condition is satisfied, find $v(t)$.

State the dimensions of all the quantities involved in your expression for $v(t)$, and verify that the expression is dimensionally consistent.

[*You may neglect any relativistic effects.*]

Paper 4, Section I

4C Dynamics and Relativity

In two-dimensional space-time an inertial frame S' has velocity v relative to another inertial frame S. State the Lorentz transformation that relates coordinates (x', t') in S' to coordinates (x, t) in S, assuming that the frames coincide when $t = t' = 0$.

Show that if $x_\pm = x \pm ct$ and $x'_\pm = x' \pm ct'$ then the Lorentz transformation can be expressed in the form

$$x'_+ = \lambda(v)x_+ \quad \text{and} \quad x'_- = \lambda(-v)x_-\,, \quad \text{where} \quad \lambda(v) = \left(\frac{c-v}{c+v}\right)^{1/2}. \qquad (*)$$

Deduce that $x^2 - c^2t^2 = x'^2 - c^2t'^2$.

Use $(*)$ to verify that successive Lorentz transformations with velocities v_1 and v_2 result in another Lorentz transformation with velocity v_3, to be determined in terms of v_1 and v_2.

Paper 4, Section II

9C Dynamics and Relativity

Find the moment of inertia of a uniform-density sphere with mass M and radius a with respect to an axis passing through its centre.

Such a sphere is released from rest on a plane inclined at an angle α to the horizontal. Let t_s and t_r be the times taken for the sphere to travel a distance l along the plane assuming either sliding without friction or rolling without slipping, respectively. Discuss whether energy is conserved in each of the two cases. Show that $t_s/t_r = \sqrt{5/7}$.

The uniform-density sphere is replaced by a sphere of the same mass whose density varies radially such that its moment of inertia is $\gamma M a^2$ for some constant γ. Determine the new value for t_s/t_r.

Paper 4, Section II

10C Dynamics and Relativity

(a) Write down the 4-momentum of a particle of rest mass m and 3-velocity $\mathbf{v}$, and the 4-momentum of a photon of frequency ω (having zero rest mass) moving in the direction of the unit 3-vector $\mathbf{e}$.

Show that if P_1 and P_2 are timelike future-pointing 4-vectors then $P_1 \cdot P_2 \geqslant 0$ (where the dot denotes the Lorentz-invariant scalar product). Hence or otherwise show that the law of conservation of 4-momentum forbids a photon to spontaneously decay into an electron–positron pair. [Electrons and positrons have equal and non-zero rest masses.]

(b) In the laboratory frame an electron travelling with 3-velocity $\mathbf{u}$ collides with a positron at rest. They annihilate, producing two photons of frequencies ω_1 and ω_2 that move off at angles θ_1 and θ_2 to $\mathbf{u}$, respectively. Explain why the 3-momenta of the photons and $\mathbf{u}$ lie in a plane.

By considering energy and two components of 3-momentum in the laboratory frame, or otherwise, show that

$$\frac{1 + \cos(\theta_1 + \theta_2)}{\cos\theta_1 + \cos\theta_2} = \sqrt{\frac{\gamma - 1}{\gamma + 1}}$$

where $\gamma = 1/\sqrt{1 - u^2/c^2}$.

Paper 4, Section II

11C Dynamics and Relativity

Consider a system of N particles with position vectors $\mathbf{r}_i(t)$ and masses m_i, where $i = 1, 2, \ldots, N$. Particle i experiences an external force $\mathbf{F}_i$ and an internal force $\mathbf{F}_{ij}$ from particle j, for each $j \neq i$. Stating clearly any assumptions you need, show that

$$\frac{d\mathbf{P}}{dt} = \mathbf{F} \quad \text{and} \quad \frac{d\mathbf{L}}{dt} = \mathbf{G}\,,$$

where $\mathbf{P}$ is the total momentum, $\mathbf{F}$ is the total external force, $\mathbf{L}$ is the total angular momentum about a fixed point $\mathbf{a}$, and $\mathbf{G}$ is the total external torque about $\mathbf{a}$.

Does the result $\dfrac{d\mathbf{L}}{dt} = \mathbf{G}$ still hold if the fixed point $\mathbf{a}$ is replaced by the moving centre of mass of the system? Justify your answer.

Suppose now that all the particles have the same mass m and that the external force on particle i is $-k\dfrac{d\mathbf{r}_i}{dt}$, where k is a constant. Show that

$$\mathbf{L}(t) = \mathbf{L}(0)e^{-kt/m}\,.$$

Paper 4, Section II

12C Dynamics and Relativity

A particle of mass m moves in a plane under an attractive force of magnitude $mf(r)$ towards the origin O. You may assume that the acceleration $\mathbf{a}$ in polar coordinates (r, θ) is given by

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\boldsymbol{\theta}}\,,$$

where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are the unit vectors in the directions of increasing r and θ respectively, and the dot denotes d/dt.

(a) Show that $l = r^2\dot{\theta}$ is a constant of the motion. Introducing $u = 1/r$, show that

$$\dot{r} = -l\frac{du}{d\theta}$$

and derive the geometric orbit equation

$$l^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = f\left(\frac{1}{u}\right)\,.$$

(b) Suppose now that

$$f(r) = \frac{3r + 9}{r^3}\,,$$

and that initially the particle is at distance $r_0 = 1$ from O, and moving with speed $v_0 = 4$ in the direction of decreasing r and increasing θ that makes an angle $\pi/3$ with the radial vector pointing towards O.

Show that $l = 2\sqrt{3}$ and find u as a function of θ. Hence, or otherwise, show that the particle returns to its original position after one revolution about O and then flies off to infinity.

Paper 3, Section I

1D Groups

Let G be a finite group and N a normal subgroup of G. Let C_n denote the cyclic group of order n.

Are the following statements true or false? Justify your answers.

(i) If $G/N \cong C_2$ and $N \cong C_2$ then $G \cong C_4$.

(ii) If $G/N \cong C_3$ and $N \cong C_2$ then $G \cong C_6$.

(iii) Let H be a finite group and M a normal subgroup of H. If $G/N \cong H/M$ and $N \cong M$ then $G \cong H$.

Paper 3, Section I

2D Groups

Prove that a Möbius map is determined by the image of just 3 points.

Paper 3, Section II

5D Groups

State and prove Lagrange's theorem.

Let H and K be subgroups of a finite group G. Show that $H \cap K$ is a subgroup of G. What can be said about $H \cap K$ if H and K have co-prime orders? Justify your answer.

Let G be a finite group and x an element of G. Define the *order* of x in G and denote it by $o(x)$. Let k be a positive integer. Prove that $x^k = e$ if and only if $o(x)$ divides k. (Here e denotes the identity element of G.)

Now suppose x and y are elements of G with co-prime orders. Further suppose $xy = yx$. Prove that $o(xy) = o(x)o(y)$.

Let x and y be two non-identity elements of G.

(i) If $o(x)$ and $o(y)$ are co-prime is it always true that $o(xy) = o(x)o(y)$?

(ii) If $xy = yx$ is it always true that $o(xy) = o(x)o(y)$?

State Cauchy's theorem. Hence, or otherwise, show that there are exactly two groups of order 26 up to isomorphism.

Paper 3, Section II

6D Groups

Let N be a normal subgroup of a group G and let G/N denote the set of left cosets of N in G. Explain how G/N is given a well-defined group structure.

Let $x, y \in G$. The *commutator* of x and y is defined by $[x, y] = x^{-1}y^{-1}xy$. Let $\overline{G}$ be the set of finite products of commutators of G, that is elements of $\overline{G}$ are of the form $[x_1, y_1][x_2, y_2] \ldots [x_k, y_k]$, where $x_i, y_i \in G$ for $1 \leqslant i \leqslant k$. Prove that $\overline{G}$ is a normal subgroup of G.

Show that $G/\overline{G}$ is an abelian group. Further, show that if N is a normal subgroup of G and G/N is abelian, then $\overline{G}$ is a subgroup of N.

Determine $\overline{G}$ when G is each of the following groups. Justify your answers.

(i) D_8 the dihedral group of order 8.

(ii) A_5 the alternating group of degree 5.

(iii) S_5 the symmetric group of degree 5.

Paper 3, Section II

7D Groups

Let H and K be subgroups of a finite group G. Show that

$$|HK| = \frac{|H||K|}{|H \cap K|}\,,$$

where $HK = \{hk : h \in H, k \in K\}$.

Let a and b be co-prime. If $|G : H| = a$ and $|G : K| = b$, show that $HK = G$.

Let $x \in G$. Define the *conjugacy class* of x in G and denote it by $\mathrm{Conj}_G(x)$. Define the *centraliser* of x in G and denote it by $C_G(x)$.

Let $x, y \in G$. Suppose $|\mathrm{Conj}_G(x)| = a$ and $|\mathrm{Conj}_G(y)| = b$ with a and b co-prime. Show that $C_G(x)C_G(y) = G$. Prove that

$$\mathrm{Conj}_G(xy) = \mathrm{Conj}_G(x)\mathrm{Conj}_G(y)\,,$$

where $\mathrm{Conj}_G(x)\mathrm{Conj}_G(y) = \{uv : u \in \mathrm{Conj}_G(x), v \in \mathrm{Conj}_G(y)\}$. [*Hint: Observe that* $g^{-1}xgh^{-1}yh$ *may be written as* $h^{-1}(hg^{-1}xgh^{-1}y)h$, *where* $g, h \in G$.]

Paper 3, Section II

8D Groups

State and prove the first isomorphism theorem. [You may assume that images of homomorphisms are subgroups and that kernels of homomorphisms are normal subgroups.]

Define the groups $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{R})$. Prove that $\mathrm{SL}_n(\mathbb{R})$ is a normal subgroup of $\mathrm{GL}_n(\mathbb{R})$ and identify $\mathrm{GL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{R})$.

Let $\mathrm{M}_2(\mathbb{Z})$ denote the set of 2×2 matrices with entries in $\mathbb{Z}$. Let

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}) : ad - bc \neq 0 \ \text{ and } \ \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \in \mathrm{M}_2(\mathbb{Z}) \right\}.$$

Check that G is a group and that it is infinite. Show that

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}) : ad - bc = \pm 1 \right\}.$$

Consider the following subset of G,

$$H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G : a, d \equiv 1 \bmod 2, \ \ b, c \equiv 0 \bmod 2 \right\}.$$

By considering a suitable homomorphism, or otherwise, show that H is a normal subgroup of finite index in G.

Paper 4, Section I

1F Numbers and Sets

A *permutation* of the integers $\{1, \ldots, n\}$ is a bijection from this set to itself. The permutation σ is said to be *up-down* if $\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \ldots$; it is said to be *down-up* if, instead, $\sigma(1) > \sigma(2) < \sigma(3) > \sigma(4) < \ldots$.

(a) Define a bijection between the set of up-down and the set of down-up permutations of $\{1, \ldots, n\}$.

(b) Let A_n be the number of up-down permutations of $\{1, \ldots, n\}$ for $n \geqslant 1$, and define $A_0 = 1$. Show that these numbers satisfy the equation

$$2A_{n+1} = \sum_{k=0}^{n} \binom{n}{k} A_k A_{n-k} \quad \text{for } n \geqslant 1\,.$$

[*Hint: Consider the possible up-down or down-up permutations for which a given element of* $\{1, \ldots, n+1\}$ *maps to* $n+1$.]

Paper 4, Section I

2E Numbers and Sets

State and prove the Chinese remainder theorem.

Find all solutions x of the simultaneous congruences

$$\begin{cases} x \equiv 4 \mod 6\,, \\ x \equiv 2 \mod 8\,. \end{cases}$$

Prove that for every positive integer d there exist integers a and b such that $4a^2 + 9b^2 - 1$ is divisible by d.

Paper 4, Section II

5F Numbers and Sets

The Chebyshev polynomials are defined for $x \in \mathbb{R}$ by the recurrence relation

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x) \qquad \text{for } n \geqslant 1. \end{aligned}$$

(a) Prove that $T_n\big(\cos(y)\big) = \cos(ny)$ for all integers $n \geqslant 0$.

(b) Prove that $\cos(\pi/n)$ is algebraic for all integers $n \geqslant 1$.

(c) For each integer $n \geqslant 1$, determine whether $\cos(\pi/n)$ is rational or not. [*Hint: After stating the known answers for small n, it is useful to consider the form of* $T_n(x)$ *for odd n.*]

(d) In each of the following cases, prove that the sequence (a_n) is bounded and determine whether it has a limit:

(i) $a_n = \sum_{k=1}^{n} \big(1 - \cos(\pi/k)\big)$,

(ii) $a_n = \sum_{k=0}^{n} \cos(ky)$, with $\cos(y) \neq 1$.

Paper 4, Section II

6E Numbers and Sets

If p is a prime number, prove that $(p-1)! \equiv -1 \mod p$.

If $n > 4$ is a composite number, prove that $(n-1)! \equiv 0 \mod n$.

State the Fermat–Euler theorem and deduce from it Fermat's little theorem.

If p is any prime, prove that if $a \equiv b \mod p$, then $a^{p^n} \equiv b^{p^n} \mod p^{n+1}$ for all integers $n \geqslant 1$.

Let $a > 1$ be an integer. A *pseudo-prime of base a* is a composite number $n > 1$ satisfying $a^{n-1} \equiv 1 \mod n$. By considering the numbers $\dfrac{a^{2p}-1}{a^2-1}$, where p is prime, or otherwise, prove that for each a there are infinitely many pseudo-primes of base a.

Paper 4, Section II

7D Numbers & Sets

(a) Let X be a set and let $f : X \to X$ be an injective function. Show that $f^n : X \to X$ is injective, where f^n denotes the n-fold composite of f with itself.

The image of f is given by $\{f(x) : x \in X\}$ and denoted $f(X)$. Show that

$$X \supseteq f(X) \supseteq f^2(X) \supseteq f^3(X) \supseteq \cdots$$

Suppose there exists $k \in \mathbb{N}$ such that $f^k(X) = f^{k+1}(X)$. Show that $f^k(X) = f^{k+m}(X)$ for all $m \in \mathbb{N}$. Hence, or otherwise, find a subset A of X such that $f : A \to A$ is bijective.

(b) Let $X = \{x_1, x_2, \ldots, x_n\}$ and let W_k be the set of words in elements of X of length k, that is $W_k = \{w_1 \ldots w_k : w_i \in X \text{ for } 1 \leqslant i \leqslant k\}$. Let P_n be the set of bijections $f : X \to X$. We define a relation $\sim$ on W_k as follows. Suppose $w, z \in W_k$, then $w \sim z$ if and only if there exists $f \in P_n$ such that $w_1 \ldots w_k = f(z_1) \ldots f(z_k)$, where $w = w_1 \ldots w_k$ and $z = z_1 \ldots z_k$. Show that $\sim$ defines an equivalence relation on W_k.

List the equivalence classes of W_3 for each $n \in \mathbb{N}$.

List the equivalence classes of W_4 when $n = 3$.

Let $n = 4$ and $g \in P_4$ be such that

$$g : x_1 \mapsto x_2,\ x_2 \mapsto x_3,\ x_3 \mapsto x_4 \text{ and } x_4 \mapsto x_1.$$

Let $F = \{g, g^2, g^3, g^4\}$. We define a new equivalence relation $\underset{F}{\sim}$ on W_k. Suppose $w, z \in W_k$, then $w \underset{F}{\sim} z$ if and only if there exists $f \in F$ such that $w_1 \ldots w_k = f(z_1) \ldots f(z_k)$. Are the equivalence classes of W_3 under $\underset{F}{\sim}$ the same as the equivalence classes under $\sim$? Justify your answer. [You may assume that $\underset{F}{\sim}$ is an equivalence relation.]

Paper 4, Section II

8D Numbers & Sets

Prove that a countable union of countable sets is countable.

Infinite binary sequences are sequences of the form $a_1a_2a_3\ldots$, where $a_i \in \{0,1\}$ for $i \in \mathbb{N}$. Are the sets consisting of the following countable? Justify your answers.

(i) All infinite binary sequences.

(ii) Infinite binary sequences with either a finite number of $1s$ or a finite number of $0s$.

(iii) Infinite binary sequences with infinitely many $1s$ and infinitely many $0s$.

A function $f:\mathbb{Z}\to\mathbb{N}$ is called *periodic* if there exists a positive integer k such that $f(x+k)=f(x)$ for every $x\in\mathbb{Z}$. Is the set of periodic functions $f:\mathbb{Z}\to\mathbb{N}$ countable? Justify your answer.

Is the set of bijections from $\mathbb{N}$ to $\mathbb{N}$ countable? Justify your answer.

Paper 2, Section I

3F Probability

(a) State and prove Markov's inequality.

(b) Let X be a standard normal random variable. Compute the moment generating function $M_X(t) = \mathbb{E}(e^{tX})$.

(c) Prove that, for all $v > 0$,

$$\int_v^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \leqslant e^{-v^2/2}.$$

Paper 2, Section I

4F Probability

A $2k$-*spalindrome* is a sequence of $2k$ digits that contains k distinct digits and reads the same backwards as forwards.

(a) What is the probability that a sequence of $2k$ digits, chosen independently and uniformly at random from $\{0, 1, \ldots, 9\}$, is a $2k$-spalindrome?

(b) Suppose now a sequence of $3k$ digits is chosen independently and uniformly at random from $\{0, 1, \ldots, 9\}$. What is the probability that this longer sequence contains a $2k$-spalindrome? [*Hint: Consider the event that the subsequence starting in position ℓ is a $2k$-spalindrome.*]

Paper 2, Section II

9F Probability

In a group of people, each pair are friends with probability 1/2, and friendships between different pairs of people are independent. Each person's birthday is distributed independently and uniformly among the 365 days of the year. Birthdays are independent of friendships.

The number of people in the group, N, has a Poisson distribution with mean 365.

(a) What is the expectation of the number of pairs of friends with the same birthday?

(b) Let Z_i be the number of people born on the ith day of the year. Find the joint probability mass function of $(Z_1, \ldots, Z_{365})$.

(c) What is the probability that no pair of friends have the same birthday? [You may express your answer in terms of the constant

$$C = \sum_{n=2}^{\infty} \frac{2^{-n(n-1)/2}}{n!} \approx 0.27\,. \quad]$$

Paper 2, Section II

10F Probability

Let X be a random variable with probability density function

$$f(x)=\frac{x^{n-1}e^{-x}}{(n-1)!}\quad \text{for } x\geqslant 0,$$

where n is a positive integer.

(a) Find the moment generating function $M_X(t)$ for $t<1$.

(b) Find the mean and variance of X.

(c) Prove that, for every $q\geqslant 0$,

$$\int_0^{n+q\sqrt{n}}\frac{x^{n-1}e^{-x}}{(n-1)!}\,dx\to\Phi(q)\quad \text{as } n\to\infty\,,$$

where Φ is the distribution function of a standard normal random variable. [You may cite any result from the course, provided that it is clearly stated.]

Paper 2, Section II

11F Probability

Let T_1 and T_2 be independent exponential random variables with means λ_1^{-1} and λ_2^{-1}, respectively. Let $V=\min(T_1,T_2)$ and $W=\max(T_1,T_2)$.

(a) Find the distribution of V. What is the probability that $V=T_1$?

(b) Find $\Pr(V\leqslant t\mid V>s)$ for $t>s>0$.

From now on, suppose that $\lambda_1=\lambda_2=\lambda$.

(c) Prove that V and $W-V$ are independent. What is the distribution of $W-V$?

(d) Hence, find the distribution of $2V/(W+V)$.

Paper 2, Section II

12F Probability

Let X be a random variable taking values in $\{0, 1, 2, \ldots\}$, with $\Pr(X \geqslant 2) > 0$.

(a) Define the *probability generating function* G_X of X. Show that the first and second derivatives of G_X are positive and non-decreasing on $(0, 1]$.

Now consider a branching process which starts with a population of 1. For each $n \geqslant 1$, each individual in generation n gives rise to an independent number of offspring, distributed as X, which together form generation $n + 1$.

(b) Let d be the probability that the population eventually becomes extinct. Prove that d is the smallest non-negative solution to $t = G_X(t)$.

(c) Let $\mathbb{E}(X) = \mu$. Show that if $\mu > 1$ then $d < 1$.

(d) Suppose that $\mu > 1$ and that X has variance σ^2. Show that for $t \in [0, 1]$,

$$G_X(t) \leqslant 1 - \mu(1-t) + \frac{1}{2}(\sigma^2 + \mu^2 - \mu)(1-t)^2.$$

Hence find an upper bound $d^* < 1$ for the extinction probability d, where d^* is given in terms of μ and σ^2.

Paper 3, Section I

3B Vector Calculus

What does it mean for a vector field $\mathbf{F}$ in $\mathbb{R}^3$ to be *irrotational*?

Given a field $\mathbf{F}$ that is irrotational everywhere, and given a fixed point $\mathbf{x}_0$, write down the definition of a scalar potential $V(\mathbf{x})$ that satisfies $\mathbf{F} = -\boldsymbol{\nabla} V$ and $V(\mathbf{x}_0) = 0$. Show that this potential is well-defined.

Given vector fields $\mathbf{A}_0$ and $\mathbf{B}$ with $\boldsymbol{\nabla} \times \mathbf{A}_0 = \mathbf{B}$, write down the form of the general solution $\mathbf{A}$ to $\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$. State a necessary condition on $\mathbf{B}$ for such an $\mathbf{A}_0$ to exist.

Paper 3, Section I

4B Vector Calculus

Cartesian coordinates x, y, z and cylindrical polar coordinates ρ, ϕ, z are related by

$$x = \rho\cos\phi, \quad y = \rho\sin\phi\,.$$

Find scalars h_ρ, h_ϕ and unit vectors $\mathbf{e}_\rho, \mathbf{e}_\phi$ such that $\mathrm{d}\mathbf{x} = h_\rho \mathbf{e}_\rho\,\mathrm{d}\rho + h_\phi \mathbf{e}_\phi\,\mathrm{d}\phi + \mathbf{e}_z \mathrm{d}z\,.$

A region V is defined by

$$\rho_0 \leqslant \rho \leqslant \rho_0 + \Delta\rho\,, \qquad \phi_0 \leqslant \phi \leqslant \phi_0 + \Delta\phi\,, \qquad z_0 \leqslant z \leqslant z_0 + \Delta z\,,$$

where $\rho_0, \phi_0, z_0, \Delta\rho, \Delta\phi$ and Δz are positive constants. Write down, or calculate, the scalar areas of its six faces and its volume ΔV.

For a vector field $\mathbf{F}(\mathbf{x}) = F(\rho)\mathbf{e}_\rho$, calculate the value of

$$\lim_{\Delta\rho\to 0} \frac{1}{\Delta V} \int_{\partial V} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S\,,$$

where ∂V and $\mathbf{n}$ are the surface and outward normal of the region V.

Paper 3, Section II

9B Vector Calculus

The vector fields $\mathbf{u}(\mathbf{x},t)$ and $\mathbf{w}(\mathbf{x},t)$ obey the evolution equations

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} - \boldsymbol{\nabla}P\,,$$
$$\frac{\partial \mathbf{w}}{\partial t} = (\mathbf{w}\cdot\boldsymbol{\nabla})\mathbf{u} - (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{w}\,,$$

where P is a given scalar field. Show that the scalar field $h = \mathbf{u}\cdot\mathbf{w}$ obeys an evolution equation of the form

$$\frac{\partial h}{\partial t} = (\mathbf{w}\cdot\boldsymbol{\nabla})f + (\mathbf{u}\cdot\boldsymbol{\nabla})g\,,$$

where the scalar fields f and g should be identified.

Suppose that $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$ and $\mathbf{w} = \boldsymbol{\nabla}\times\mathbf{u}$. Show that, if $\mathbf{u}\cdot\mathbf{n} = \mathbf{w}\cdot\mathbf{n} = 0$ on the surface S of a fixed volume V with outward normal $\mathbf{n}$, then

$$\frac{dH}{dt} = 0, \text{ where } H = \int_V h\,dV\,.$$

Suppose that $\mathbf{u} = (a^2-\rho^2)\rho\sin z\,\mathbf{e}_\phi + a\rho^2\sin z\,\mathbf{e}_z$ in cylindrical polar coordinates ρ,ϕ,z, where a is a constant, and that $\mathbf{w} = \boldsymbol{\nabla}\times\mathbf{u}$. Show that $h = -2a\rho^4\sin^2 z$, and calculate the value of H when V is the cylinder $0 \leqslant \rho \leqslant a$, $0 \leqslant z \leqslant \pi$.

$$\left[\textit{In cylindrical polar coordinates } \boldsymbol{\nabla}\times\mathbf{F} = \frac{1}{\rho}\begin{vmatrix} \mathbf{e}_\rho & \rho\mathbf{e}_\phi & \mathbf{e}_z \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}.\right]$$

Paper 3, Section II

10B Vector Calculus

Show that

$$\boldsymbol{\nabla} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}\, \boldsymbol{\nabla} \cdot \mathbf{b} - \mathbf{b}\, \boldsymbol{\nabla} \cdot \mathbf{a} + (\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{a} - (\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{b}\,.$$

State Stokes' theorem for a vector field in $\mathbb{R}^3$, specifiying the orientation of the integrals.

The vector fields $\mathbf{m}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ satisfy the conditions $\mathbf{m} = \mathbf{n}$ and $\mathbf{v} \cdot \mathbf{n} = 0$ on an open surface S with unit normal $\mathbf{n}(\mathbf{x})$. By applying Stokes' theorem to the vector field $\mathbf{m} \times \mathbf{v}$, show that

$$\int_S (\delta_{ij} - n_i n_j) \frac{\partial v_i}{\partial x_j} dS = \oint_C \left[\mathbf{v} \cdot (d\mathbf{x} \times \mathbf{n})\right], \qquad (*)$$

where C is the boundary of S. Describe the orientation of $d\mathbf{x} \times \mathbf{n}$ relative to S and C.

Verify $(*)$ when S is the hemisphere $r = R$, $z \geqslant 0$ and $\mathbf{v} = r \sin\theta\, \mathbf{e}_\theta$ in spherical polar coordinates r, θ, ϕ.

[*You may use the formulae* $(\mathbf{e}_r \cdot \boldsymbol{\nabla})\mathbf{e}_\theta = \mathbf{0}$ *and*

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{1}{r^2} \frac{\partial (r^2 F_r)}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial (\sin\theta\, F_\theta)}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial F_\phi}{\partial \phi}\,,$$

and you may quote formulae for dS *and* $d\mathbf{x}$ *in these coordinates without derivation.*]

Paper 3, Section II

11B Vector Calculus

(a) Verify the identity

$$\boldsymbol{\nabla} \cdot (\kappa\psi\boldsymbol{\nabla}\phi) = \psi\boldsymbol{\nabla} \cdot (\kappa\boldsymbol{\nabla}\phi) + \kappa\boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\phi\,,$$

where $\kappa(\mathbf{x})$, $\phi(\mathbf{x})$ and $\psi(\mathbf{x})$ are differentiable scalar functions.

Let V be a region in $\mathbb{R}^3$ that is bounded by a closed surface S. The function $\phi(\mathbf{x})$ satisfies

$$\boldsymbol{\nabla} \cdot (\kappa\boldsymbol{\nabla}\phi) = 0 \text{ in } V \text{ and } \phi = f(\mathbf{x}) \text{ on } S\,,$$

where κ and f are given functions and $\kappa > 0$. Show that ϕ is unique.

The function $w(\mathbf{x})$ also satisfies $w = f(\mathbf{x})$ on S. By writing $w = \phi + \psi$, show that

$$\int_V \kappa|\boldsymbol{\nabla}w|^2\,dV \geqslant \int_V \kappa|\boldsymbol{\nabla}\phi|^2\,dV\,.$$

(b) A steady temperature field $T(\mathbf{x})$ due to a distribution of heat sources $H(\mathbf{x})$ in a medium with spatially varying thermal diffusivity $\kappa(\mathbf{x})$ satisfies

$$\boldsymbol{\nabla} \cdot (\kappa\boldsymbol{\nabla}T) + H = 0\,.$$

Show that the heat flux $\int_S \mathbf{q} \cdot d\mathbf{S}$ across a closed surface S, where $\mathbf{q} = -\kappa\boldsymbol{\nabla}T$, can be expressed as an integral of the heat sources within S.

By using this version of Gauss's law, or otherwise, find the temperature field $T(r)$ for the spherically symmetric case when

$$\kappa(r) = r^\alpha, \quad -1 < \alpha < 2, \qquad\qquad H(r) = \begin{cases} H_0 & \text{if } r \leqslant 1 \\ 0 & \text{if } r > 1 \end{cases}$$

subject to the condition that $T \to 0$ as $r \to \infty$. What goes wrong if $\alpha \leqslant -1$?

Deduce that if $w(r)$ satisfies $w(1) = 1$ and $w(r) \to 0$ as $r \to \infty$ (sufficiently rapidly for the integral to converge) then

$$\int_1^\infty r^{\alpha+2}\left(\frac{dw}{dr}\right)^2 dr \geqslant \alpha + 1\,.$$

Paper 3, Section II

12B Vector Calculus

(a) State the transformation law for the components of an nth-rank tensor $T_{ij\ldots k}$ under a rotation of the basis vectors, being careful to specify how any rotation matrix relates the new basis $\{\mathbf{e}'_i\}$ to the original basis $\{\mathbf{e}_j\}$, $i, j = 1, 2, 3$.

If $\phi(\mathbf{x})$ is a scalar field, show that $\partial^2\phi/\partial x_i \partial x_j$ transforms as a second-rank tensor.

Define what it means for a tensor to be *isotropic*. Write down the most general isotropic tensors of rank k for $k = 0, 1, 2, 3$.

(b) Explain briefly why T_{ijkl}, defined by

$$T_{ijkl} = \int_{\mathbb{R}^3} x_i x_j e^{-r^2} \frac{\partial^2}{\partial x_k \partial x_l}\left(\frac{1}{r}\right) dV \,, \quad \text{where } r = |\mathbf{x}|,$$

is an isotropic fourth-rank tensor.

Assuming that

$$T_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} \,,$$

use symmetry, contractions and a scalar integral to determine the constants α, β and γ.

[*Hint:* $\nabla^2(1/r) = 0$ for $r \neq 0$.]

Paper 1, Section I
1A Vectors and Matrices

The principal value of the logarithm of a complex variable is defined to have its argument in the range $(-\pi, \pi]$.

(a) Evaluate $\log(-i)$, stating both the principal value and the other possible values.

(b) Show that i^{-2i} represents an infinite set of real numbers, which should be specified.

(c) By writing $z = \tan w$ in terms of exponentials, show that

$$\tan^{-1} z = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right).$$

Use this result to evaluate the principal value of

$$\tan^{-1}\left(\frac{2\sqrt{3}-3i}{7}\right).$$

Paper 1, Section I
2C Vectors and Matrices

For an $n \times n$ complex matrix A, define the *Hermitian conjugate* $A^{\dagger}$. State the conditions (i) for A to be unitary (ii) for A to be Hermitian.

Let A, B, C and D be $n \times n$ complex matrices and $\mathbf{x}$ a complex n-vector. A matrix N is defined to be normal if $N^{\dagger}N = NN^{\dagger}$.

(a) For A nonsingular, show that $B = A^{-1}A^{\dagger}$ is unitary if and only if A is normal.

(b) Let C be normal. Show that $|C\mathbf{x}| = 0$ if and only if $|C^{\dagger}\mathbf{x}| = 0$.

(c) Let D be normal. Deduce from part (b) that if $\mathbf{e}$ is an eigenvector of D with eigenvalue λ then $\mathbf{e}$ is also an eigenvector of $D^{\dagger}$ and find the corresponding eigenvalue.

Paper 1, Section II
5A Vectors and Matrices

(a) The position vector $\mathbf{r}$ of a general point on a surface in $\mathbb{R}^3$ is given by the equations below
(i) $(\mathbf{r}-\mathbf{a})\cdot\mathbf{n}=0$,
(ii) $\mathbf{r}=\mathbf{b}+\lambda(\mathbf{d}-\mathbf{b})+\mu(\mathbf{f}-\mathbf{b})$,
(iii) $|\mathbf{r}-\mathbf{c}|=\rho$.

Identify each surface and describe the meaning of the constant vectors $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{f}$ and $\mathbf{n}$, together with the scalars λ,μ and $\rho>0$.

(b) Find the equation for the line of intersection of the planes $2x+3y-z=3$ and $x-3y+4z=3$ in the form $\mathbf{r}\times\mathbf{m}=\mathbf{u}\times\mathbf{m}$, where $\mathbf{m}$ is a unit vector and $\mathbf{u}\cdot\mathbf{m}=0$.

Find the minimum distance from this line to the line that is inclined at equal angles to the positive x-, y-, and z- axes and passes through the origin.

(c) The intersection of the surface $\mathbf{r}\cdot\mathbf{n}=p$, where $\mathbf{n}$ is a unit vector and p is a real number, and the sphere of radius A centred on the point with position vector $\mathbf{g}$ is a circle of radius R.

Find the position vector $\mathbf{h}$ of the centre of the circle and determine R as a function of $A,p,\mathbf{g}$ and $\mathbf{n}$. Discuss geometrically the condition on R to be real.

Paper 1, Section II
6C Vectors and Matrices

(a) Consider the matrix

$$M=\begin{pmatrix}2&1&0\\0&1&-1\\0&2&4\end{pmatrix}.$$

Determine whether or not M is diagonalisable.

(b) Prove that if A and B are similar matrices then A and B have the same eigenvalues with the same corresponding algebraic multiplicities. Is the converse true? Give either a proof (if true) or a counterexample with a brief reason (if false).

(c) State the Cayley–Hamilton theorem for an $n\times n$ matrix A and prove it for the case that A is a 2×2 diagonalisable matrix.

Suppose B is an $n\times n$ matrix satisfying $B^k=0$ for some $k>n$ (where 0 denotes the zero matrix). Show that $B^n=0$.

Paper 1, Section II
7B Vectors and Matrices

(a) Let A be an $n \times n$ non-singular matrix and let $G = A^\dagger A$ and $H = AA^\dagger$, where dagger denotes the Hermitian conjugate.

By considering $|A\mathbf{x}|$ for a vector $\mathbf{x}$, or otherwise, show that the eigenvalues of G are positive real numbers.

Show that if $\mathbf{e}_i$ is an eigenvector of G with eigenvalue λ_i then $\mathbf{f}_i = A\mathbf{e}_i$ is an eigenvector of H. What is the value of $|\mathbf{f}_i|/|\mathbf{e}_i|$?

(b) Using part (a), explain how to construct unitary matrices U and V from the eigenvectors of G and H such that $V^\dagger AU = D$, where D is a diagonal matrix to be specified in terms of the eigenvalues of G. [You may assume that, for any $n \times n$ Hermitian matrix, it is possible to find n orthogonal eigenvectors.]

(c) Find U, V and D for the case

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & -2 \\ -1 & -1 \end{pmatrix} .$$

Paper 1, Section II
8B Vectors and Matrices

Define what it means for an $n \times n$ real matrix to be *orthogonal*.

Show that the eigenvalues of an orthogonal matrix have unit modulus, and show that eigenvectors with distinct eigenvalues are orthogonal.

Let Q be a 3×3 orthogonal matrix with $\det Q = -1$. Show that -1 is an eigenvalue of Q.

Let $\mathbf{n}$ be a nonzero vector satisfying $Q\mathbf{n} = -\mathbf{n}$ and consider the plane Π through the origin that is perpendicular to $\mathbf{n}$. Show that Q maps Π to itself.

Show that Q acts on Π as a rotation through some angle θ, and show that $\cos\theta = \frac{1}{2}(\operatorname{tr} Q + 1)$.

Show also that $\det(Q - I) = 4(\cos\theta - 1)$.

[*You may quote the form of relationship between two matrix representations A and A' of a linear map α with respect to different bases, but should explain results derived from it.*]

END OF PAPER