UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS   Part II

Friday, 10 June, 2022 9:00am to 12:00pm

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most six questions*** *from Section I and from any number of questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet ***for each question*** *that you have attempted, and place it at the front of your answer to that question.*

Complete a green master cover sheet listing ***all the questions*** *that you have attempted.*

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into ***a single bundle****, with the master cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.*

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1I Number Theory

Compute the continued fraction expansion of $\sqrt{29}$.

Find integers x and y satisfying $x^2 - 29y^2 = -1$.

2G Topics in Analysis

Consider the continuous map $f : [0,1] \to \mathbb{C}$ given by $f(t) = t - 1/2$. Show that there does not exist a continuous function $\phi : [0,1] \to \mathbb{R}$ with $f(t) = |f(t)| \exp(i\phi(t))$.

Show that, if $g : [0,1] \to \mathbb{C} \setminus \{0\}$ is continuous, there exists a continuous function $\theta : [0,1] \to \mathbb{R}$ with $g(t) = |g(t)| \exp(i\theta(t))$. [You may assume that this result holds in the special case when $\Re g(t) > 0$ for all $t \in [0,1]$.]

Show that $r(g) = \theta(1) - \theta(0)$ is uniquely defined.

If $u(t) = g(t^2)$ and $v(t) = g(t)^2$, find $r(u)$ and $r(v)$ in terms of $r(g)$.

Give an example with $g_1, g_2 : [0,1] \to \mathbb{C} \setminus \{0\}$ continuous such that $g_1(0) = g_2(0)$ and $g_1(1) = g_2(1)$, but $r(g_1) \neq r(g_2)$.

3K Coding and Cryptography

In this question we work over $\mathbb{F}_2$.

What is a *general feedback shift register of length* d with *initial fill* $(x_0, \ldots, x_{d-1})$? What does it mean for such a register to be *linear*?

Describe the Berlekamp–Massey method for breaking a cipher stream arising from a linear feedback shift register.

Use the Berlekamp–Massey method to find a linear recurrence with first eight terms $1, 1, 0, 0, 1, 0, 1, 1$.

4I Automata and Formal Languages

Define what it means for a context-free grammar (CFG) to be in *Chomsky normal form*.

What are an ϵ*–production* and a *unit production*?

Let G_1 be the CFG

$$\begin{aligned} S &\rightarrow \epsilon \,|\, aTa \,|\, bTa \\ T &\rightarrow Ta \,|\, Tb \,|\, c \end{aligned}$$

and let G_2 be the CFG

$$\begin{aligned} S &\rightarrow XZ \,|\, YZ \\ T &\rightarrow TX \,|\, TY \,|\, c \\ X &\rightarrow a,\ Y \rightarrow b,\ Z \rightarrow TX. \end{aligned}$$

What is the relationship between the language of G_1 and the language of G_2? Justify your answer carefully.

5J Statistical Modelling

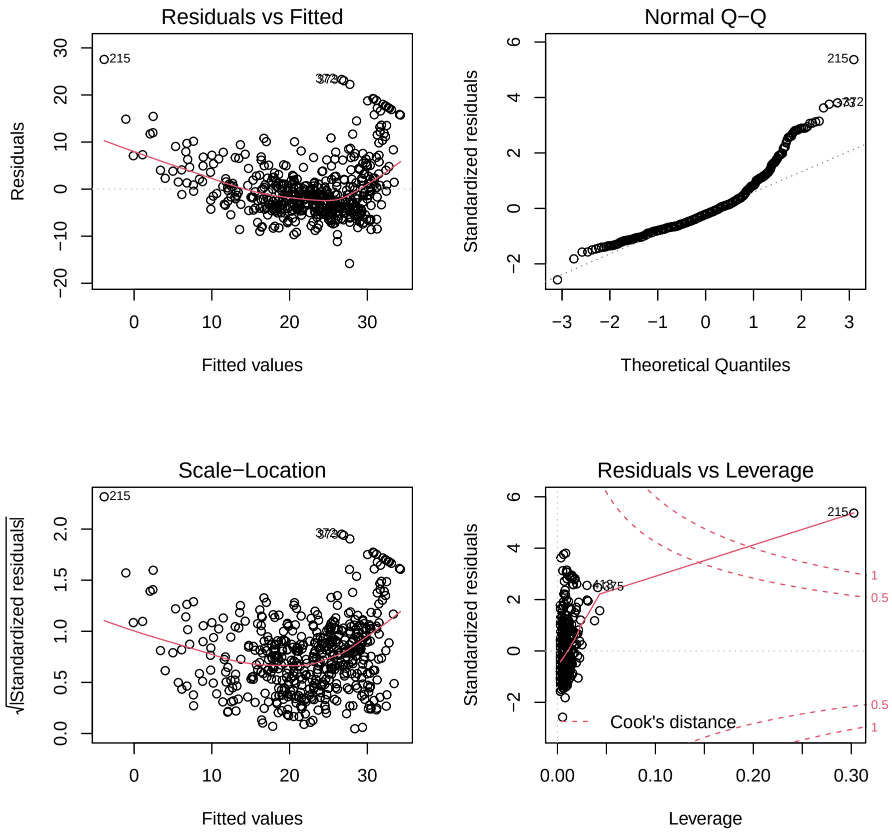
The `Boston` dataset records `medv` (median house value), `age` (average age of houses), `lstat` (percent of households with low socioeconomic status), and other covariates for 506 census tracts in Boston.

```
> head(Boston[, c("medv", "age", "lstat")])
  medv  age lstat
1 24.0 65.2  4.98
2 21.6 78.9  9.14
3 34.7 61.1  4.03
4 33.4 45.8  2.94
5 36.2 54.2  5.33
6 28.7 58.7  5.21
```

Describe the mathematical model fitted in the `R` code below and give three observations from the output of the code that you think are the most noteworthy.

```
> summary(fit <- lm(medv ~ lstat * age , data = Boston))

Residuals:
    Min      1Q  Median      3Q     Max
-15.806  -4.045  -1.333   2.085  27.552

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) 36.0885359  1.4698355  24.553  < 2e-16 ***
lstat       -1.3921168  0.1674555  -8.313 8.78e-16 ***
age         -0.0007209  0.0198792  -0.036   0.9711
lstat:age    0.0041560  0.0018518   2.244   0.0252 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 6.149 on 502 degrees of freedom
Multiple R-squared:  0.5557,    Adjusted R-squared:  0.5531
F-statistic: 209.3 on 3 and 502 DF,  p-value: < 2.2e-16

>
> par(mfrow = c(2, 2))
> plot(fit)
```

[QUESTION CONTINUES ON THE NEXT PAGE]


Residuals vs Fitted
Residuals
Fitted values
215
372
Normal Q−Q
Standardized residuals
Theoretical Quantiles
215
372
Scale−Location
√|Standardized residuals|
Fitted values
215
372
Residuals vs Leverage
Standardized residuals
Leverage
215
175
Cook's distance
1
0.5
0.5
1

6C Mathematical Biology

An allosteric enzyme E reacts with substrate S to produce a product P according to the mechanism

$$E + S \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C_1$$
$$C_1 + S \xrightarrow{k_2} C_2 \xrightarrow{k_3} C_1 + P \ ,$$

where the k_is are rate constants, and C_1 and C_2 are enzyme-substrate complexes.

(a) With lowercase letters denoting concentrations, write down the differential equation model based on the Law of Mass Action for the dynamics of e, s, c_1, c_2 and p.

(b) Show that the quantity $c_1 + c_2 + e$ is conserved and comment on its physical meaning.

(c) Using the result in (b), assuming initial conditions $s(0) = s_0$, $e(0) = e_0$, $c_1(0) = c_2(0) = p(0) = 0$, and rescaling with $\epsilon = e_0/s_0$, $\tau = k_1 e_0 t$, $u = s/s_0$, and $v_i = c_i/e_0$, show that the reaction mechanism can be reduced to

$$\frac{du}{d\tau} = f(u, v_1, v_2)\,,$$
$$\epsilon \frac{dv_1}{d\tau} = g_1(u, v_1, v_2)\,,$$
$$\epsilon \frac{dv_2}{d\tau} = g_2(u, v_1, v_2)\,.$$

Determine f, g_1 and g_2 and express them in terms of the three dimensionless quantities $\alpha = k_{-1}/k_1 s_0$, $\beta = k_2/k_1$ and $\gamma = k_3/k_1 s_0$.

(d) On time scales $\tau \gg \epsilon$, show that the rate of production of P can be expressed in terms of the rescaled substrate concentration u in the form

$$\frac{dp}{dt} = A \frac{u^2}{\alpha + u + (\beta/\gamma)u^2} \ ,$$

where A is a constant. Compare this relation to the Michaelis-Menten form by means of a sketch.

7E Further Complex Methods

What type of equation has solutions described by the following Papperitz symbol?

$$P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha_1 & \alpha_2 & \alpha_3 & z \\ \beta_1 & \beta_2 & \beta_3 & \end{Bmatrix}$$

Explain the meaning of each of the quantities appearing in the symbol.

The hypergeometric function $F(a, b, c; z)$ is defined by

$$F(a, b; c; z) = P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{Bmatrix}$$

with $F(a, b; c; z)$ analytic at $z = 0$ and satisfying $F(a, b; c; 0) = 1$.

Explain carefully why there are constants A and B such that

$$F(a, b; c; z) = Az^{-a}F(a, 1+a-c; 1+a-b; z^{-1}) + Bz^{-b}F(b, 1+b-c; 1+b-a; z^{-1}).$$

[*You may neglect complications associated with special cases such as $a = b$.*]

8B Classical Dynamics

A particle of mass $m_1 = 3m$ is connected to a fixed point by a massless spring of natural length l and spring constant k. A second particle of mass $m_2 = 2m$ is connected to the first particle by an identical spring. The masses move along a vertical line in a uniform gravitational field g, such that mass m_i is a distance $z_i(t)$ below the fixed point and $z_2 > z_1 > 0$.

[*You may assume that the potential energy of a spring of length $l+x$ is $\frac{1}{2}kx^2$, where k is the spring constant and l is the natural length.*]

Write down the Lagrangian of the system.

Determine the equilibrium values of z_i.

Let q_i be the departure of z_i from its equilibrium value. Show that the Lagrangian can be written as

$$L = \frac{1}{2}T_{ij}\dot{q}_i\dot{q}_j - \frac{1}{2}V_{ij}q_iq_j + \text{constant}\,,$$

and determine the matrices T and V.

Calculate the angular frequencies and eigenvectors of the normal modes of the system.

In what sense are the eigenvectors orthogonal?

9A Cosmology

Consider a closed Friedmann-Robertson-Walker universe filled with a fluid endowed with an energy density $\rho \geqslant 0$ and pressure $P \geqslant 0$. For such a universe the Friedmann equation reads

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\rho - \frac{c^2}{R^2\, a^2}\,,$$

where $a(t)$ is the scale factor.

What is the meaning of R? Show that a closed universe cannot expand forever.

[*Hint: Use the continuity equation to show that*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho\, a^3) \leqslant 0\,. \qquad]$$

10D Quantum Information and Computation

(a) Let B_n denote the set of all n-bit strings and write $N = 2^n$. The *Grover iteration operator* on n qubits is given by

$$Q = -H_n I_0 H_n I_{x_0}.$$

Give a definition of the constituent operators H_n, I_0 and I_{x_0} and state a geometrical interpretation of the action of Q on the space of n qubits.

(b) The quantum oracle for the identity function $\mathcal{I} : B_n \to B_n$, $\mathcal{I}(x) = x$ is the unitary operation $U_{\mathcal{I}}$ on $2n$ qubits defined by $U_{\mathcal{I}}(|\,x\rangle\,|\,y\rangle) = |\,x\rangle\,|\,y \oplus \mathcal{I}(x)\rangle$ for all $x, y \in B_n$. Here $\oplus$ denotes the sum of n-bit strings bitwise mod 2 separately at each of the n positions in the string, *i.e.* the group operation in $(\mathbb{Z}_2)^n$.

Show how the action of $U_{\mathcal{I}}$ can be represented by a circuit of CX gates.

(c) Suppose we are given a quantum oracle for $\mathcal{I}$ but it is known to be faulty on one of its inputs. Instead of the full identity function it implements instead the function $f : B_n \to B_n$ given by

$$f(x) = \begin{cases} x & \text{for all } x \neq x_0 \\ x \oplus a & \text{for } x = x_0 \end{cases}$$

where $a \in B_n$ is the n-bit string $00\ldots01$ and where $x_0 \in B_n$ is unknown, *i.e.* the given quantum oracle actually implements U_f. By providing a suitable input state for a circuit involving U_f and further gates independent of f, show how I_{x_0} on n qubits may be implemented in terms of U_f.

(d) Hence or otherwise show that for sufficiently large N, x_0 may be determined with some constant probability greater than $\frac{1}{2}$ using $O(\sqrt{N})$ queries to the oracle U_f.

SECTION II

11I Number Theory

(a) Define the *Legendre symbol* and state *Euler's criterion.* State and prove *Gauss' lemma.* Determine the primes p for which the congruence $x^2 \equiv 2 \pmod{p}$ is soluble.

(b) Let $\pi_k(x)$ be the number of primes p less than or equal to x with $p \equiv k \pmod{8}$.

(i) By considering the prime factorisation of $n^2 - 2$ for suitable n, show that $\pi_7(x) \to \infty$ as $x \to \infty$.

(ii) By considering the prime factorisation of $n^2 - 2$ for all n in a suitable range, show that for all x sufficiently large we have

$$\pi_1(x) + \pi_7(x) + 1 \geqslant \frac{\log x}{6 \log 3}\,.$$

12G Topics in Analysis

(a) State Brouwer's fixed point theorem for the closed unit disc D. For which of the following $E \subset \mathbb{R}^2$ is it the case that every continuous function $f : E \to E$ has a fixed point? Give a proof or a counterexample.

(i) E is the union of two disjoint closed discs.

(ii) $E = \{(x, 0) \,:\, 0 < x < 1\}$.

(iii) $E = \{(x, 0) \,:\, 0 \leqslant x \leqslant 1\}$.

(iv) $E = \{\mathbf{x} \in \mathbb{R}^2 \,:\, 1 \leqslant |\mathbf{x}| \leqslant 2\}$.

(b) Show that if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous function with the property that $|f(\mathbf{x})| \leqslant 1$ whenever $|\mathbf{x}| = 1$, then f has a fixed point.

[*Hint: Consider* $T \circ f$ *where for* $\mathbf{x} \in \mathbb{R}^2$, $T\mathbf{x}$ *is the element of* D *closest to* $\mathbf{x}$.]

(c) Let

$$E = \{(p_1, p_2, q_1, q_2) \,:\, 0 \leqslant p_i, q_i \leqslant 1 \text{ and } p_1 + p_2 = 1, q_1 + q_2 = 1\}$$

and suppose A, $B : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ are given by

$$A(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} p_i q_j \text{ and } B(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{2} \sum_{j=1}^{2} b_{ij} p_i q_j$$

with a_{ij} and b_{ij} constant. Let

$$u_1(\mathbf{p}, \mathbf{q}) = \max\{0, A((1,0), \mathbf{q}) - A(\mathbf{p}, \mathbf{q})\}\,, \; u_2(\mathbf{p}, \mathbf{q}) = \max\{0, A((0,1), \mathbf{q}) - A(\mathbf{p}, \mathbf{q})\}\,.$$

By considering $(\mathbf{p}', \mathbf{q}')$ with

$$\mathbf{p}' = \frac{\mathbf{p} + \mathbf{u}(\mathbf{p}, \mathbf{q})}{1 + u_1(\mathbf{p}, \mathbf{q}) + u_2(\mathbf{p}, \mathbf{q})}$$

and $\mathbf{q}'$ defined appropriately, show that we can find a $(\mathbf{p}^*, \mathbf{q}^*) \in E$ with

$$\forall (\mathbf{p}, \mathbf{q}) \in E, \quad A\big(\mathbf{p}^*, \mathbf{q}^*\big) \geqslant A\big(\mathbf{p}, \mathbf{q}^*\big) \text{ and } B\big(\mathbf{p}^*, \mathbf{q}^*\big) \geqslant B\big(\mathbf{p}^*, \mathbf{q}\big).$$

Carefully explain the result in terms of a two-person game.

13J Statistical Modelling

Consider the following R code:

```
> n <- 1000000
> sigma_z <- 1; sigma_x1 <- 0.5; sigma_x2 <- 1; sigma_y <- 2; beta <- 2
> Z <- sigma_z * rnorm(n)
> X1 <- Z + sigma_x1 * rnorm(n)
> X2 <- Z + sigma_x2 * rnorm(n)
> Y <- beta * Z + sigma_y * rnorm(n)
> lm(Y ~ Z)

Call:
lm(formula = Y ~ Z)

Coefficients:
(Intercept)            Z
  -0.003089     1.999780

> lm(Y ~ X1)

Call:
lm(formula = Y ~ X1)

Coefficients:
(Intercept)           X1
  -0.002904     1.600521

> lm(Y ~ X2)

Call:
lm(formula = Y ~ X2)

Coefficients:
(Intercept)           X2
  -0.002672     0.997499
```

Describe the phenomenon you see in the output above, then give a mathematical explanation for this phenomenon. Do you expect the slope coefficient in the second model to be generally smaller than that in the first model? Do you think modifying (for example, doubling) the value of `sigma_y` will substantially alter the slope coefficient in the second model? Justify your answer.

14C Mathematical Biology

Consider the standard system of reaction-diffusion equations

$$\begin{aligned} u_t &= D_u \nabla^2 u + f(u,v) \\ v_t &= D_v \nabla^2 v + g(u,v)\,, \end{aligned}$$

where D_u and D_v are diffusion constants and $f(u,v)$ and $g(u,v)$ are such that the system has a stable homogeneous fixed point at $(u,v) = (u_*, v_*)$.

(a) Show that the condition for a Turing instability can be expressed as

$$f_u + d g_v > 2\sqrt{dJ}\,,$$

where $d = D_u/D_v$ is the diffusivity ratio and $J = f_u g_v - f_v g_u > 0$ is the determinant of the stability matrix of the homogeneous system evaluated at (u_*, v_*).

(b) Show that this result implies that a Turing instability at equal diffusivities ($d = 1$) is not possible.

(c) Show that the result in (b) also follows directly from the structure of the reaction-diffusion equations linearised about the homogeneous fixed point in the case $D_u = D_v$.

(d) Using the example

$$\begin{pmatrix} -1 & -1 \\ 1+\delta & 1-\delta \end{pmatrix},$$

for the stability matrix of the homogeneous system, show that the diffusivity ratio at which Turing instability occurs can be made as close to unity as desired by taking δ sufficiently small.

15B Classical Dynamics

An isolated three-body system consists of particles with masses m_1, m_2 and m_3 and position vectors $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$ and $\mathbf{r}_3(t)$. The particles move under the action of their mutual gravitational attraction. Write down the Lagrangian L of the system.

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be defined by

$$\mathbf{a} = \mathbf{r}_1 - \mathbf{r}_2\,, \qquad \mathbf{b} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} - \mathbf{r}_3\,, \qquad \mathbf{c} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3}{m_1 + m_2 + m_3}\,.$$

By expressing $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, or otherwise, show that the total kinetic energy can be written as

$$\frac{1}{2}\alpha|\dot{\mathbf{a}}|^2 + \frac{1}{2}\beta|\dot{\mathbf{b}}|^2 + \frac{1}{2}\gamma|\dot{\mathbf{c}}|^2\,,$$

and obtain expressions for α, β and γ.

Show that the total potential energy can be expressed as a function of $\mathbf{a}$ and $\mathbf{b}$ only. What does this imply for the evolution of $\mathbf{c}$? Give a physical interpretation of this result.

Show also that the total angular momentum of the system about the origin is

$$\alpha\,\mathbf{a} \times \dot{\mathbf{a}} + \beta\,\mathbf{b} \times \dot{\mathbf{b}} + \gamma\,\mathbf{c} \times \dot{\mathbf{c}}\,.$$

16F Logic and Set Theory

(a) Define the *von Neumann hierarchy* of sets V_α. Show that each V_α is transitive, and explain why $V_\alpha \subset V_\beta$ whenever $\alpha \leqslant \beta$. Prove that every set x is a member of some V_α.

(b) What does it mean to say that a relation r on a set x is *well-founded* and *extensional*? State *Mostowski's Collapsing Theorem*. Give an example of a set x whose rank is greater than ω but for which the Mostowski collapse of x (equipped with the relation $\in$) is equal to ω.

Which of the following statements are always true and which can be false? Give proofs or counterexamples as appropriate.

(i) If a relation r on a set x is isomorphic to the relation $\in$ on some transitive set y then r is well-founded and extensional.

(ii) If a relation r on a set x is isomorphic to the relation $\in$ on some (not necessarily transitive) set y then r is well-founded.

(iii) If a relation r on a set x is isomorphic to the relation $\in$ on some (not necessarily transitive) set y then r is extensional.

17F Graph Theory

Define the *binomial random graph* $G(n,p)$, where $n \in \mathbb{N}$ and $p \in [0,1]$.

Let $G_n \sim G(n,p)$ and let E_n be the event that $\delta(G_n) > 0$. Show that for every $\varepsilon > 0$, if $p = p(n)$ satisfies $p \geqslant (1+\varepsilon)n^{-1}\log n$ then $\mathbb{P}(E_n) \to 1$.

State *Chebyshev's inequality* and show that for every $\varepsilon > 0$, if p is such that $p \leqslant (1-\varepsilon)n^{-1}\log n$ then $\mathbb{P}(E_n) \to 0$.

For $G_n \sim G(n,p)$, let F_n be the event that G_n is connected. Prove that for every $\varepsilon > 0$, if $p \geqslant (1+\varepsilon)n^{-1}\log n$ then $\mathbb{P}(F_n) \to 1$ as $n \to \infty$ and if $p \leqslant (1-\varepsilon)n^{-1}\log n$ then $\mathbb{P}(F_n) \to 0$ as $n \to \infty$. [You may wish to consider separately the case when there is a component of size at most say $n\varepsilon/10$ and the case when there is not.]

[*You may use, without proof, the fact that* $1-x \leqslant e^{-x}$ *for all* $x \in [0,1]$, *and also that for any fixed* $\delta \in (0,1)$ *we have* $1-x \geqslant e^{-(1+2\delta)x}$ *for all* $x \in [0,\delta)$. *All logarithms in this question are natural logarithms.*]

18H Galois Theory

(a) Stating carefully all the theorems that you use, prove that for every integer $r > 1$ there is a Galois extension $L/\mathbb{Q}$ with Galois group $\mathbb{Z}/r\mathbb{Z}$.

(b) Suppose L_1 and L_2 are two extensions of a field K, and both L_1 and L_2 are subfields of some field M. Let L_1L_2 be the smallest subfield of M containing both L_1 and L_2. If $[L_i : K] = d_i$ and $\gcd(d_1,d_2) = 1$, show that $[L_1L_2 : K] = d_1d_2$.

(c) Let $p \geqslant 3$ be a prime number. Give examples of two non-isomorphic groups G, G' of order $p(p-1)$ containing normal subgroups N, N' of order p such that $G/N \cong G'/N'$.

Fix $p = 3$. For the groups G, G' above, give explicit examples of Galois extensions $L/\mathbb{Q}$ and $L'/\mathbb{Q}$ with $\mathrm{Aut}(L/\mathbb{Q}) \cong G$ and $\mathrm{Aut}(L'/\mathbb{Q}) \cong G'$. Identify the fixed fields L^N and $(L')^{N'}$. Justify your answer.

Now suppose $p > 3$ is an arbitrary prime. Prove that there are extensions L and L' of $\mathbb{Q}$ with $\mathrm{Aut}(L/\mathbb{Q}) \cong G$ and $\mathrm{Aut}(L'/\mathbb{Q}) \cong G'$.

19H Representation Theory

Suppose that H is a subgroup of a group G and χ is a complex character of H.

State *Mackey's restriction formula* and *Frobenius reciprocity* for characters. Use them to deduce Mackey's irreducibility criterion for an induced representation.

Suppose that k is a finite field of order $q \geqslant 4$, $G = SL_2(k)$ and

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \, a, b \in k, a \neq 0 \right\}.$$

Describe the degree 1 complex characters χ of B and explain, with justification, for which of them $\mathrm{Ind}_B^G \chi$ is irreducible.

20H Number Fields

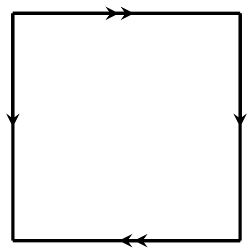
Let K be a number field. What is an *ideal class* of K? Show that the set of ideal classes of K forms an abelian group. [You may use any results about ideals in number fields provided you state them clearly.]

Assuming that there exists a constant c_K such that every nonzero ideal I of $\mathcal{O}_K$ contains a nonzero element α with $\left|N_{K/\mathbb{Q}}(\alpha)\right| \leqslant c_K\, N(I)$, show that the ideal class group of K is finite.

Compute the ideal class group of $\mathbb{Q}(\sqrt{-33})$. [You may assume that the Minkowski constant c_K of an imaginary quadratic field is $\dfrac{2}{\pi}\,|d_K|^{1/2}$.]

21I Algebraic Topology

Let K be the Klein bottle obtained by identifying the sides of the unit square as shown in the figure, and let $k_0 \in K$ be the image of the corners of the square.



Show that K is the union of two Möbius bands with their boundaries identified. Deduce that $\pi_1(K, k_0)$ has a presentation

$$\pi_1(K, k_0) = \langle a, b \,|\, a^2 b^{-2} \rangle.$$

Show that there is a degree two covering map $p : (T^2, x_0) \to (K, k_0)$. Describe generators α, β for $\pi_1(T^2, x_0)$ and express $p_*(\alpha)$ and $p_*(\beta)$ in terms of a and b.

Let $Y = T^2 \times [0,1)/\sim$, where $\sim$ is the smallest equivalence relation with $(x,0) \sim (x',0)$ whenever $p(x) = p(x')$. What is $\pi_1(Y, y_0)$, where y_0 is the image of $(x_0, 0)$ in Y?

Suppose X is a path-connected Hausdorff space, that $U \subset X$ is an open subset, and that U is homeomorphic to Y. Can X be simply connected? Justify your answer.

22G Linear Analysis

(a) Define what it means for a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ to be *equi-continuous* on $[0,1]$. State the *Arzelà–Ascoli theorem.*

(b) Given a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$, we can inductively define functions $f_n : \mathbb{R} \to \mathbb{R}$ for $n \geqslant 0$ by $f_{n+1}(t) = \int_0^t \varphi(f_n(s))\,\mathrm{d}s$, and $f_0(t) = 0$ for all $t \in \mathbb{R}$. Show that there exists $T_1 > 0$ so that the sequence $(f_n)_{n\geqslant 1}$ is equi-bounded and equi-continuous on $[0, T_1]$.

(c) Deduce the existence of $T_2 \in (0, T_1]$ and a continuously differentiable function $f : [0, T_2] \to \mathbb{R}$ such that $f(0) = 0$ and $f'(t) = \varphi(f(t))$ on $[0, T_2]$. [*Hint: Prove that if* $T_2 \in (0, T_1]$ *is small enough,* $R_n(t) = f_{n+1}(t) - f_n(t) \to 0$ *uniformly on* $[0, T_2]$.]

23G Analysis of Functions

For $s \in \mathbb{R}$, define the *Sobolev space* $H^s(\mathbb{R}^n)$. Show that for any multi-index α, the map $u \mapsto D^\alpha u$ is a bounded linear map from $H^s(\mathbb{R}^n)$ to $H^{s-|\alpha|}(\mathbb{R}^n)$.

Given $f \in H^s(\mathbb{R}^n)$, show that the PDE

$$-\Delta u + u = f$$

admits a unique solution with $u \in H^{s+2}(\mathbb{R}^n)$. Show that the map taking f to u is a linear isomorphism of $H^s(\mathbb{R}^n)$ onto $H^{s+2}(\mathbb{R}^n)$.

Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Consider a sequence of functions $(u_j)_{j=1}^\infty$ with $u_j \in C^\infty(\mathbb{R}^n)$, supported in Ω, such that

$$\|\Delta u_j\|_{L^2(\Omega)} + \|u_j\|_{L^2(\Omega)} \leqslant K\,,$$

for some constant K independent of j. Show that there exists a subsequence $(u_{j_k})_{k=1}^\infty$ which converges strongly in $H^1(\mathbb{R}^n)$.

24H Algebraic Geometry

What is the *degree* of a divisor on a smooth projective algebraic curve? What is a *principal divisor* on a smooth projective algebraic curve?

Let $D = \sum a_i p_i$ be a divisor of degree 0 on $\mathbb{P}^1$. Construct a rational function f such that $\operatorname{div}(f)$ is D. Deduce that if E and E' are divisors of the same degree on $\mathbb{P}^1$ then E is linearly equivalent to E'.

Let X_0, X_1 be the usual homogenous coordinates on $\mathbb{P}^1$, and let t be the rational function X_0/X_1. Calculate the divisor associated to the rational differential dt on $\mathbb{P}^1$.

Fix an integer m and let D be a divisor equivalent to $mK_{\mathbb{P}^1}$, where $K_{\mathbb{P}^1}$ is the canonical divisor computed above. Without appealing to the Riemann–Roch theorem, calculate the dimension of the vector space $L(D)$ of rational functions with poles bounded by D.

Let C be a smooth projective curve of genus at least 1. Prove that for distinct points p and q in C, the divisor $p - q$ is not principal.

25I Differential Geometry

(a) State *Wirtinger's inequality*. State and prove the *isoperimetric inequality* for domains $\Omega \subset \mathbb{R}^2$ with compact closure and C^1 boundary $\partial\Omega$.

(b) Let $Q \subset \mathbb{R}^2$ be a *cyclic* quadrilateral, meaning that there is a circle through its four vertices. Say its edges have lengths a, b, c and d (in cyclic order). Assume $Q' \subset \mathbb{R}^2$ is another quadrilateral with edges of lengths a, b, c and d (in the same order). Show that $\text{Area}(Q) \geqslant \text{Area}(Q')$. Explain briefly for which Q' equality holds.

26G Probability and Measure

Denote by L^1 the space of real-valued functions on $\mathbb{R}$ that are integrable with respect to Lebesgue measure. For $f \in L^1$ and g_t the probability density function of a normal $N(0,t)$ random variable with variance $t > 0$, show that their convolution

$$f * g_t(x) = \int_{\mathbb{R}} f(x-y) g_t(y) dy\,, \ \ x \in \mathbb{R},$$

defines another element of L^1. Show carefully that the Fourier inversion theorem holds for $f * g_t$.

Now suppose that the Fourier transform of f is also in L^1. Show that $f * g_t(x) \to f(x)$ for almost every $x \in \mathbb{R}$ as $t \to 0$.

[*You may use Fubini's theorem and the translation invariance of Lebesgue measure without proof.*]

27J Applied Probability

(a) Let $X = (X_t)$ be the queue length process of an $M/M/1$ queue with arrival rate $\lambda > 0$ and service rate $\mu > 0$. Suppose $\rho = \lambda/\mu < 1$. Show that X is positive recurrent and derive its invariant distribution π.

(b) Now suppose that each arriving customer observes the current queue length $X_t = n$, and either decides to join the queue with probability $p(n)$ or to leave the system with probability $1 - p(n)$, independently of all other customers.

(i) Find the invariant distribution π of X if $p(n) = 1/(n+1)$, $n \geqslant 0$.

(ii) Find the invariant distribution π of X if $p(n) = 2^{-n}$, $n \geqslant 0$, and show that, in equilibrium, an arriving customer joins the queue with probability $\mu(1-\pi_0)/\lambda$.

28K Principles of Statistics

(a) Suppose it is possible to generate samples from a Uniform$[0, 1]$ distribution. Describe a method for generating samples from an exponential distribution with rate parameter 1, and prove that the method is valid.

(b) Recall that the accept/reject algorithm, which operates on two pdfs f and h satisfying $f \leqslant Mh$, proceeds as follows:

1. Generate $X \sim h$ and $U \sim$ Uniform$[0, 1]$.
2. If $U \leqslant \frac{f(X)}{Mh(X)}$, take $Y = X$. Otherwise, return to Step 1.

Prove that the output Y has pdf f.

(c) Suppose the pdf f is given by

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}, \quad \text{for all } x \geqslant 0.$$

Let h be the pdf of an exponential distribution with rate parameter 1. Explain how to apply the accept/reject algorithm in this special case. Identify an appropriate value for M.

(d) Compute the expected number of steps required to generate one sample from the pdf f in part (c) using the accept/reject algorithm.

(e) Let Y be a random variable generated according to the algorithm in (c). Now suppose we generate a random variable X using the following additional steps:

1. Generate $V \sim$ Uniform$[0, 1]$.
2. If $V \leqslant \frac{1}{2}$, take $Z = Y$. Otherwise, take $Z = -Y$.

What is the distribution of Z?

(f) Suppose the final goal is to generate samples from the distribution of Z in part (e). Following the steps outlined in parts (a)–(e), could the efficiency of the algorithm be improved by choosing X to be an exponential random variable with rate parameter $\lambda \neq 1$?

29K Stochastic Financial Models

Consider a discrete-time market with constant interest rate r and a stock with time-n price S_n for $0 \leqslant n \leqslant N$.

(a) Suppose a self-financing investor holds θ_n shares of the stock between times $n-1$ and n for $1 \leqslant n \leqslant N$. Explain why the investor's wealth process $(X_n)_{0 \leqslant n \leqslant N}$ evolves as

$$X_n = (1+r)X_{n-1} + \theta_n[S_n - (1+r)S_{n-1}] \quad \text{for } 1 \leqslant n \leqslant N.$$

For the rest of the question, suppose $S_n = S_{n-1}\xi_n$ where

$$\mathbb{P}(\xi_n = 1+b) = p$$
$$\mathbb{P}(\xi_n = 1+a) = 1-p$$

for all $n \geqslant 1$, for given constants $0 < p < 1$ and $a < r < b$.

(b) Show that

$$\mathbb{Q}\Big(S_N = S_0(1+b)^i(1+a)^{N-i}\Big) = \binom{N}{i} q^i (1-q)^{N-i}$$

for all $0 \leqslant i \leqslant N$, where $\mathbb{Q}$ is the unique risk-neutral measure and q is a constant which you should find.

(c) Now introduce a European contingent claim into this market with time-N payout $g(S_N)$ for a given function g. Find, with proof, the constant x and the previsible process $\theta = (\theta_n)_{1 \leqslant n \leqslant N}$ such that if an investor has time-0 wealth $X_0 = x$ and employs the trading strategy θ then the time-N wealth is $X_N = g(S_N)$ almost surely. Express your answer in terms of the function V defined by

$$V(n,s) = (1+r)^{-(N-n)}\mathbb{E}^{\mathbb{Q}}[g(S_N)|S_n = s] \quad \text{for } 0 \leqslant n \leqslant N, s > 0.$$

(d) Suppose the claim in part (c) is a European call option with strike K. Show that the corresponding initial cost x of the claim is of the form

$$S_0\widehat{\mathbb{Q}}(S_N > K) - K(1+r)^{-N}\mathbb{Q}(S_N > K)\,,$$

where $\widehat{\mathbb{Q}}$ is a probability measure such that

$$\widehat{\mathbb{Q}}\Big(S_N = S_0(1+b)^i(1+a)^{N-i}\Big) = \binom{N}{i} \hat{q}^i (1-\hat{q})^{N-i}$$

for $0 \leqslant i \leqslant N$ and a constant $\hat{q}$ which you should find.

30J Mathematics of Machine Learning

Throughout this question, you may assume that the optimum is achieved in any relevant optimisation problems, so for instance in part (a) you may assume $\hat{f}$ is well-defined.

Suppose $(X_1, Y_1), \ldots, (X_n, Y_n) \in \mathcal{X} \times \{-1, 1\}$ are i.i.d. input–output pairs. Let $\mathcal{B}$ be a set of classifiers $h : \mathcal{X} \to \{-1, 1\}$ such that $h \in \mathcal{B} \Rightarrow -h \in \mathcal{B}$.

(a) Write down the *Adaboost* algorithm using $\mathcal{B}$ as the base set of classifiers with tuning parameter M, which produces $\hat{f} : \mathcal{X} \to \mathbb{R}$ of the form $\hat{f} = \sum_{m=1}^{M} \hat{\beta}_m \hat{h}_m$ where $\hat{\beta}_m \geqslant 0$ and $\hat{h}_m \in \mathcal{B}$ for $m = 1, \ldots, M$. [You need not derive explicit expressions for $\hat{\beta}_m$ or $\hat{h}_m$.]

(b) For a set $S \subseteq \mathbb{R}^d$, what is meant by the *convex hull*, $\operatorname{conv} S$? What does it mean for a vector $v \in \mathbb{R}^d$ to be a *convex combination* of vectors $v_1, \ldots, v_m \in \mathbb{R}^d$? State a result relating convex hulls and convex combinations.

(c) Let ϕ denote the exponential loss. What is meant by the *ϕ-risk* $R_\phi(f)$ of $f : \mathcal{X} \to \mathbb{R}$? What is the corresponding *empirical ϕ-risk* $\hat{R}_\phi(f)$? Let $x_{1:n} \in \mathcal{X}^n$. What is meant by the *empirical Rademacher complexity* $\hat{\mathcal{R}}(\mathcal{B}(x_{1:n}))$?

(d) Consider a modification of the Adaboost algorithm where, if at any iteration $m \leqslant M$ we have $\sum_{k=1}^{m} \hat{\beta}_k > 1$, we terminate the algorithm and output $\hat{f} := \sum_{k=1}^{m-1} \hat{\beta}_k \hat{h}_k$, or the zero function if $m = 1$; otherwise we output $\hat{f} = \sum_{k=1}^{M} \hat{\beta}_k \hat{h}_k$ as usual. Let $r_{\mathcal{B}} = \sup_{x_{1:n} \in \mathcal{X}^n} \hat{\mathcal{R}}(\mathcal{B}(x_{1:n}))$. Show that

$$\mathbb{E} R_\phi(\hat{f}) \leqslant \mathbb{E} \hat{R}_\phi(\hat{f}) + 2\exp(1) r_{\mathcal{B}}.$$

[*Hint: Introduce*

$$\mathcal{H} := \left\{ \sum_{m=1}^{M} \beta_m h_m : \sum_{m=1}^{M} \beta_m \leqslant 1,\ \beta_m \geqslant 0,\ h_m \in \mathcal{B} \textit{ for } m = 1, \ldots, M \right\}.$$

You may use any results from the course without proof, but should state or name any result you use.]

31E Asymptotic Methods

Consider the differential equation

$$x^2y'' + xy' - \frac{1}{x^2}y = 0\,. \qquad (*)$$

(i) What type of regular or singular point does equation $(*)$ have at $x = 0$?

(ii) For $x > 0$, find a transformation that maps equation $(*)$ to an equation of the form

$$u'' + q(x)u = 0 \qquad (\dagger)$$

and compute $q(x)$.

(iii) Determine the leading asymptotic behaviour of the solution u of equation $(\dagger)$, as $x \to 0^+$, using the Liouville-Green method and justifying your assumptions at each stage.

(iv) Conclude from the above an asymptotic expansion of two linearly independent solutions of equation $(*)$, as $x \to 0^+$.

32B Dynamical Systems

Consider the dynamical system

$$\begin{aligned} \dot{x} &= x(y - k - 3x + x^2) \\ \dot{y} &= y(y - 1 - x)\,, \end{aligned}$$

where k is a constant.

(a) Find all the fixed points of this system. By considering the existence and location of the fixed points, determine the values of k for which bifurcations occur. For each of these, what types of bifurcation are suggested from this approach?

(b) For the fixed points whose positions are independent of k, determine their linear stability. Verify that these results are consistent with the bifurcations suggested above.

(c) Focusing only on the bifurcations which occur for $0 \leqslant k \leqslant \frac{1}{2}$, use centre manifold theory to analyse these bifurcations. In particular, for each bifurcation derive an equation for the dynamics on the extended centre manifold and hence classify the bifurcation. [*Hint: There are two bifurcations in this range.*]

33A Principles of Quantum Mechanics

A particle travels in one dimension subject to the Hamiltonian

$$H_0 = \frac{P^2}{2m} - U\,\delta(x)\,,$$

where U is a positive constant. Let $|0\rangle$ be the unique bound state of this potential and E_0 its energy. Further let $|k, \pm\rangle$ be unbound H_0 eigenstates of even/odd parity, each with energy E_k, chosen so that $\langle k', +|k, +\rangle = \langle k', -|k, -\rangle = \delta(k' - k)$.

(a) At times $t \leqslant 0$ the particle is trapped in the well. From $t = 0$ it is disturbed by a time-dependent potential $v(x,t) = -Fx\, e^{-i\omega t}$ and subsequently its state may be expressed as

$$|\psi(t)\rangle = a(t)\, e^{-iE_0 t/\hbar}|0\rangle + \int_0^\infty \Big(b_k(t)|k, +\rangle + c_k(t)|k, -\rangle \Big)\, e^{-iE_k t/\hbar}\; dk\,.$$

Show that

$$\dot{a}(t)\, e^{-iE_0 t/\hbar}|0\rangle + \int_0^\infty e^{-iE_k t/\hbar} \Big(\dot{b}_k(t)|k, +\rangle + \dot{c}_k(t)|k, -\rangle \Big)\; dk = \frac{iF}{\hbar} e^{-i\omega t}\, x|\psi(t)\rangle$$

for all $t > 0$.

(b) Working to first order in F, hence show that $b_k(t) = 0$ and that

$$c_k(t) = \frac{iF}{\hbar}\langle k, -|x|0\rangle\, e^{i\Omega_k t/2}\, \frac{\sin(\Omega_k t/2)}{\Omega_k/2}\,,$$

where $\Omega_k = (E_k - E_0 - \hbar\omega)/\hbar$.

(c) The original bound state has position space wavefunction $\langle x|0\rangle = \sqrt{K}\, e^{-K|x|}$ where $K = mU/\hbar^2$, while the position space wavefunction of the odd parity unbound state is $\langle x|k, -\rangle = \sin(kx)\,/\sqrt{\pi}$ and its energy $E_k = \hbar^2 k^2/2m$. Show that at late times the probability that the particle escapes from the original potential well is

$$P_{\text{free}}(t) = \frac{8\hbar F^2 t}{m E_0^2} \frac{\sqrt{E_f/|E_0|}}{(1 + E_f/|E_0|)^4}$$

to lowest order in F, where $E_f > 0$ is the final energy. [*You may assume that as* $t \to \infty$, *the function* $\sin^2(\lambda t)/(\lambda^2 t) \to \pi\,\delta(\lambda)$.]

34D Applications of Quantum Mechanics

A particle of mass m and charge e moves in a constant homogeneous magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ with vector potential

$$\mathbf{A}(\mathbf{x}) = \frac{B}{2}\,(-y, x, 0)\,,$$

where $\mathbf{x} = (x, y, z)$ are Cartesian coordinates on $\mathbb{R}^3$.

(a) Write down the Hamiltonian $\hat{H}$ for the particle as a differential operator in Cartesian coordinates. Find a corresponding expression for $\hat{H}$ in cylindrical polar coordinates (r, θ, z), where $x = r\cos\theta$ and $y = r\sin\theta$.

[*You may use without proof the relations*

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \qquad \text{and} \qquad x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = \frac{\partial}{\partial \theta}\,. \quad]$$

(b) Consider wavefunctions of the form

$$\psi_{k_z,n}\,(r, \theta, z) = \exp(ik_z z)\,\exp(in\theta)\,\phi_n(r)\,.$$

What is the physical interpretation of the quantum numbers $k_z \in \mathbb{R}$ and $n \in \mathbb{Z}$? For $n \geqslant 0$, show that $\psi_{k_z,n}$ is an eigenstate of $\hat{H}$ provided that

$$\phi_n(r) = r^{\alpha}\,\exp\left(-\beta\frac{r^2}{2}\right),$$

where α and β are (possibly n-dependent) constants which you should determine. Find the corresponding energy eigenvalue E.

(c) By noting that $\phi_n(r)$ is sharply peaked at a particular value of r, work out the total degeneracy of this energy level when the particle is confined to lie inside a large circle of radius R. Determine the number of states per unit area.

35A Statistical Physics

(a) State *Carnot's theorem.* Show how it can be used to define a thermodynamic temperature.

(b) Consider a solid body with heat capacity at constant volume C_V. Assume that the solid's volume remains constant throughout the following three scenarios:

(i) If the temperature changes from T_i to T_f, show that the entropy change is $\Delta S = S_f - S_i = C_V \ln{(T_f/T_i)}$.

(ii) Two identical such bodies (both with heat capacity C_V) with initial temperatures T_1 and T_2 are brought into equilibrium in a reversible process. What are the final temperatures of the bodies?

(iii) Now suppose that the two bodies are instead brought directly into thermal contact (irreversibly). What are the final temperatures of the bodies? Compute the entropy change and show that it is positive.

(c) The Gibbs free energy is given by $G = E + pV - TS$, where E is energy, p is pressure, V is volume and S is entropy. Explain why $G = \mu(T, p)N$, where μ is the chemical potential and N is the number of particles.

(d) What is a *first-order phase transition*?

(e) Consider a system at constant pressure where phase I is stable for $T > T_0$, phase II is stable for $T < T_0$, and there is a first-order phase transition at $T = T_0$. Show that in a transition from phase II to phase I, $S_I - S_{II} > 0$, where S_I is the entropy in phase I and S_{II} is the entropy in phase II. [*Hint: Consider* $S = -\left(\frac{\partial G}{\partial T}\right)_{p,N}$ *for each phase.*]

36B Electrodynamics

(a) Explain what is meant by a *dielectric material.*

(b) Define the *polarisation* of, and the *bound charge* in, a dielectric material. Explain the reason for the distinction between the electric field $\mathbf{E}$ and the electric displacement $\mathbf{D}$ in a dielectric material.

Consider a sphere of a dielectric material of radius R and permittivity ε_1 embedded in another dielectric material of infinite extent and permittivity ε_2. A point charge q is placed at the centre of the sphere. Determine the bound charge on the surface of the sphere.

(c) Define the *magnetisation* of, and the *bound current* in, a dielectric material. Explain the reason for making a distinction between the magnetic flux density $\mathbf{B}$ and the magnetic intensity $\mathbf{H}$ in a dielectric material.

Consider a cylinder of dielectric material of infinite length, radius R and permeability μ_1 embedded in another dielectric material of infinite extent and permeability μ_2. A line current I is placed on the axis of the cylinder. Determine the magnitude and direction of the bound current density on the surface of the cylinder.

37D General Relativity

(a) Determine whether each of the following spaces is, or is not, a manifold. Justify your answers.

(i) $\mathbb{R}^3$ with points identified if they are related by the transformation $(x, y, z) \to (-x, -y, -z)$.

(ii) $\mathbb{R}^3$, except that the closed ball of all points with $x^2+y^2+z^2 \leqslant 1$ is removed.

(b) Let a tensor $\mathbf{S}$ at point $p \in \mathcal{M}$ be defined as a linear map

$$\mathbf{S} : T_p^*(\mathcal{M}) \to T_p(\mathcal{M}) \times T_p(\mathcal{M}) \,,$$

where T_p is tangent space and T_p^* is cotangent space.

(i) What is the rank of $\mathbf{S}$? Use $\binom{r}{s}$ notation.

(ii) What is the rank of $\mathbf{S} \otimes \nabla\mathbf{S}$, where $\otimes$ is an outer product and ∇ is the covariant derivative?

Consider a spacelike geodesic which goes from point p to point q. As a geodesic, this curve minimizes the action

$$\mathcal{S} = \int_0^1 \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\, d\lambda \ ,$$

where $x = x(\lambda)$ with $x(0) = p, x(1) = q$ and $\dot{x}^\mu = dx^\mu/d\lambda$. Show using the Euler-Lagrange equations that

$$\frac{d^2x^\beta}{ds^2} + \Gamma^\beta_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \,,$$

where s is the proper distance along the geodesic and $\Gamma^\beta_{\mu\nu}$ is the Levi-Civita connection.

38C Fluid Dynamics II

A thin layer of fluid is flowing down an inclined plane due to the action of gravity. The gravitational acceleration is g, the viscosity of the fluid is μ and the density of the fluid is ρ. The angle between the plane and the horizontal is denoted by α. Cartesian coordinates are defined with x along the plane in the downward direction and y perpendicular to the plane. All quantities may be assumed to be constant in the in-plane direction perpendicular to the slope. The thickness of the fluid layer is denoted by $h(x,t)$.

(a) Assume that the dynamics of the layer is described by the lubrication equations and hence estimate the order of magnitude for the flow speed u in the film. Deduce the two conditions involving h, $\partial h/\partial x$ and the other parameters of the problem that are required for the assumption of the lubrication limit to be self-consistent.

(b) State the momentum equations in the (x,y) coordinates under the lubrication-limit assumption. What are the boundary conditions for the velocity and the pressure?

(c) Solve for the pressure in the fluid and deduce the flow velocity along the plane.

(d) Applying conservation of mass, deduce the partial differential equation satisfied by $h(x,t)$.

(e) Seek a travelling-wave solution $h(x,t) = f(x-ct)$ and hence derive a first-order ODE (containing an unknown constant of integration) satisfied by the function f.

39C Waves

Consider finite amplitude, one-dimensional sound waves in a perfect gas with ratio of specific heats γ.

(a) Show that the fluid speed u and local sound speed c satisfy

$$\left(\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right) R_\pm = 0\,,$$

where the *Riemann invariants* $R_\pm(x,t)$ should be defined carefully. Write down parametric equations for the paths on which these quantities are actually invariant.

(b) At time $t=0$ the gas occupies the region $x>0$. It is at rest and has uniform density ρ_0, pressure p_0 and sound speed c_0. A piston initially at $x=0$ starts moving backwards at time $t=0$ with displacement $x=-\varepsilon t(1-t)$, where $\varepsilon>0$ is constant.

(i) Show that prior to any shock forming $c = c_0 + \frac{1}{2}(\gamma-1)u$.

(ii) For small ε, derive an expression for the relative pressure fluctuation $\delta p/p_0 = p/p_0 - 1$ to second order in the relative sound speed fluctuation $\delta c/c_0 = c/c_0 - 1$.

(iii) Calculate the time average over the interval $0 \leqslant t \leqslant 1$ of the relative pressure fluctuation, measured on the piston, and briefly discuss your result physically.

40C Numerical Analysis

(a) State and prove the *Gershgorin circle theorem.*

(b) Consider the diffusion equation on the square $[0,1]^2$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a(x,y)\frac{\partial}{\partial x}u(x,y)\right) + \frac{\partial}{\partial y}\left(a(x,y)\frac{\partial}{\partial y}u(x,y)\right),$$

where $0 < a(x,y) < a_{\max}$ for all $(x,y) \in [0,1]^2$ is the diffusion coefficient, and with Dirichlet boundary conditions $u(x,y,t) = 0$ for (x,y) on the boundary of $[0,1]^2$.

Consider a uniform grid of size $M \times M$ with step $h = 1/(M+1)$ and let $u_{i,j} = u(ih, jh)$ for $1 \leqslant i \leqslant M$ and $1 \leqslant j \leqslant M$.

(i) Using finite differences, show that the right-hand side of the diffusion equation can be discretised by an expression of the form

$$\frac{1}{h^2}(\alpha u_{i-1,j} + \beta u_{i+1,j} + \gamma u_{i,j-1} + \delta u_{i,j+1} - (\alpha+\beta+\gamma+\delta)u_{i,j})$$

for some $\alpha, \beta, \gamma, \delta$ which you should specify, and which depend on i, j and the diffusion coefficient. Show that the error of this discretisation is $O(h^2)$.

(ii) The time derivative is discretised using a forward Euler scheme with a time step $\Delta t = k$. Use Gershgorin's theorem, clearly justifying all your steps, to show that the resulting scheme is stable when $0 < \mu \leqslant 1/(4a_{\max})$, where $\mu = k/h^2$ is the Courant number.

END OF PAPER