UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Thursday, 09 June, 2022 9:00am to 12:00pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most six questions*** *from Section I and from any number of questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet ***for each question*** *that you have attempted, and place it at the front of your answer to that question.*

Complete a green master cover sheet listing ***all the questions*** *that you have attempted.*

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into ***a single bundle****, with the master cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.*

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1I Number Theory

State *Lagrange's theorem* on the possible number of solutions of a polynomial congruence. State and prove the *Chinese remainder theorem.*

Find the smallest positive integer x satisfying $x^3 + 1 \equiv 0 \pmod{1729}$. Hence, or otherwise, determine the number of solutions of this congruence with $1 \leqslant x \leqslant 1729$.

2G Topics in Analysis

Let Ω be a non-empty bounded open subset of $\mathbb{R}^2$ with closure $\mathrm{Cl}\,\Omega$ and boundary $\partial\Omega$. We take $\phi : \mathrm{Cl}\,\Omega \to \mathbb{R}$ to be a continuous function which is twice differentiable on Ω.

If $\nabla^2\phi > 0$ on Ω show that ϕ attains a maximum on $\partial\Omega$.

By giving proofs or counterexamples establish which of the following are true and which are false.

(i) If $\nabla^2\phi = 0$ on Ω, then ϕ attains a maximum on $\partial\Omega$.

(ii) If $\nabla^2\phi = 0$ on Ω, then ϕ attains a minimum on $\partial\Omega$.

(iii) If $\nabla^2\phi = f$ on Ω for some continuous function $f : \mathrm{Cl}\,\Omega \to \mathbb{R}$, then ϕ attains a maximum on $\partial\Omega$.

3K Coding and Cryptography

(a) Let C_1 and C_2 be (binary) linear codes with $C_2 \subseteq C_1$. Define their *bar product* $C_1|C_2$.

(b) (i) Let $d \geqslant 1$. Identify the Reed–Muller codes $\mathrm{RM}(d, 0)$ and $\mathrm{RM}(d, d)$ as well-known codes of a certain length. [Proofs are not required.]

For $0 < r < d$, identify the Reed–Muller code $\mathrm{RM}(d, r)$ as a bar product of certain Reed–Muller codes. [Proofs are not required.] Use this to compute the rank of $\mathrm{RM}(d, r)$.

(ii) By considering the original definition of Reed-Muller codes, show that every codeword in $\mathrm{RM}(d, d-1)$ has even weight. Deduce that $\mathrm{RM}(d, r)$ has dual code $\mathrm{RM}(d, d-r-1)$.

4I Automata and Formal Languages

Define a *context-free grammar* (CFG) and a *context-free language* (CFL).

State the *pumping lemma for CFLs.*

Which of the following languages over the alphabet $\{a,b,c\}$ are CFLs? Justify your answers.

(i) $\{a^n b^{2n} c^n \mid n \geqslant 0\}$.

(ii) $\{a^n b^{2i} c^n \mid n, i \geqslant 0\}$.

5J Statistical Modelling

The density function of the Laplace distribution Laplace(μ, σ) with mean μ and scale parameter σ is given by

$$f(y;\mu,\sigma) = (2\sigma)^{-1} \exp\left\{-\frac{|y-\mu|}{\sigma}\right\}.$$

Briefly comment on why the Laplace distribution cannot be written in exponential dispersion family form.

Consider the linear model where $(X_i, Y_i), i = 1, \ldots, n$ are assumed independent and

$$Y_i \mid X_i \sim \text{Laplace}(X_i^T \beta, \sigma).$$

Show that the maximum likelihood estimator $\hat{\beta}$ of β is obtained by minimising

$$S(\beta) = \sum_{i=1}^{n} |Y_i - X_i^T \beta|.$$

Obtain the maximum likelihood estimator of σ in terms of $S(\hat{\beta})$.

6C Mathematical Biology

A biological population contains n individuals. The population increases or decreases according to the transition rates

$$n \xrightarrow{\lambda} n+1 \qquad\qquad n \xrightarrow{\beta n^2} n-2\,.$$

(a) Derive the master equation for $P(n,t)$, the probability that the population contains n individuals at time t, and a corresponding equation for $\langle n\rangle$. What condition does the latter imply on the steady state?

(b) The Fokker-Planck equation has the form:

$$\frac{\partial}{\partial t}P(n,t) = -\frac{\partial}{\partial n}\big[A(n)P(n,t)\big] + \tfrac{1}{2}\frac{\partial^2}{\partial n^2}\big[B(n)P(n,t)\big]\,. \tag{1}$$

Derive the Fokker-Planck equation from your master equation. Deduce the forms of $A(n)$ and $B(n)$ for this system.

(c) Give brief arguments why in the steady state (1) has the approximate solution $(2\pi\sigma^2)^{-1/2}\exp(-(n-\mu)^2/2\sigma^2)$ and derive the corresponding values of σ and μ.

(d) Comment on the relation to the steady-state condition you have derived in (a). Under what conditions on β and λ is the Fokker-Planck equation likely to give an accurate description of the steady state?

7E Further Complex Methods

Consider the partial differential equation

$$\frac{\partial T}{\partial t} = \kappa\frac{\partial^2 T}{\partial x^2}$$

in $x>0$ subject to the initial condition $T(x,0)=0$ for all $x>0$ and the boundary condition $T(0,t)=\sin\omega t$ for $t>0$.

Show that the Laplace transform of $T(x,t)$ takes the form

$$\tilde{T}(x,p) = \tilde{T}_0(p)\exp(-(p/\kappa)^{1/2}x)$$

and determine the function $\tilde{T}_0(p)$.

Consider $I(t)=\int_0^\infty T(x,t)\,dx$. Write down an expression for $\tilde{I}(p)$.

Applying the Bromwich contour inversion expression for Laplace transforms gives the result that for $t>0$

$$I(t) = A\cos(\omega t) + B\sin(\omega t) + \frac{1}{\pi}\int_0^\infty \frac{\omega\kappa^{1/2}}{(s^2+\omega^2)}\frac{e^{-st}}{s^{1/2}}\,ds\,,$$

where A and B are independent of t. Draw a diagram showing the Bromwich contour and explain clearly how the terms appearing in the above expression arise.

8B Classical Dynamics

The Lagrangian of the Lagrange top can be written as

$$L = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi} \cos \theta \right)^2 - Mgl \cos \theta \, .$$

Define the *generalized momenta* p_ϕ and p_ψ, and describe how they evolve in time.

Show that the nutation of the top is governed by the equation

$$\frac{1}{2} I_1 \dot{\theta}^2 + V_{\text{eff}}(\theta) = \text{constant} \, ,$$

where $V_{\text{eff}}(\theta)$ is an effective potential energy that you should define.

Explain why p_ϕ and p_ψ must be equal in order for the top to reach the vertical position $\theta = 0$. In this case, show that $\theta = 0$ is a stable equilibrium provided that the top spins sufficiently fast.

9A Cosmology

Combining the Friedmann and continuity equations

$$H^2 = \frac{8\pi G}{3c^2} \left(\rho - \frac{k\, c^2}{R^2\, a^2} \right) , \qquad \dot{\rho} + 3\, H \, (\rho + P) = 0 \, ,$$

derive the *Raychaudhuri equation* (also known as the *acceleration equation*), which expresses $\ddot{a}/a$ in terms of the energy density ρ and pressure P.

Assume that the strong energy condition $\rho + 3P \geqslant 0$ holds. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(H^{-1} \right) \geqslant 1.$$

Deduce that $H \to +\infty$ and $a \to 0$ at a finite time in the past or in the future. What property of H distinguishes the two cases? In one sentence, describe the implications for the evolution of this model universe.

10D Quantum Information and Computation

Let $\mathbf{x} = x_0 x_1 \ldots x_{N-1}$ be an N-bit string with $N = 2K$ being even. Let $\mathcal{H}_M$ denote a state space of dimension M with orthonormal basis $\{|\,k\rangle \,:\, k \in \mathbb{Z}_M\}$. A quantum oracle $O_{\mathbf{x}}$ for $\mathbf{x}$ is a unitary operation on $\mathcal{H}_N \otimes \mathcal{H}_2$ whose action is defined by $O_{\mathbf{x}}\,|\,i\rangle\,|\,y\rangle = |\,i\rangle\,|\,y \oplus x_i\rangle$, where $y \in \{0,1\}$ and $\oplus$ denotes addition modulo 2.

Consider the following oracle problem, called Problem A:
Input: an oracle $O_{\mathbf{x}}$ for some N-bit string $\mathbf{x}$.
Promise: $\mathbf{x}$ is either a constant string, or a balanced string (the latter meaning that $\mathbf{x}$ contains exactly K 0's and K 1's).
Problem: decide if $\mathbf{x}$ is balanced.

(a) Suppose we have a universal set of quantum gates available and any desired unitary operation that is independent of $\mathbf{x}$ may be exactly implemented. Also, we may perform measurements in the basis $\{|\,i\rangle : i \in \mathbb{Z}_N\}$ of an N-dimensional register.

Show that Problem A can be solved with certainty by a quantum algorithm that makes only one query to the oracle $O_{\mathbf{x}}$. The algorithm should begin with each register initially in the state $|\,0\rangle$ (in the appropriate state space).

(b) Suppose now that in addition to $O_{\mathbf{x}}$ and measurements in the basis $\{|\,i\rangle : i \in \mathbb{Z}_N\}$, we can implement only the Pauli Z gate on a qubit register and gates F and F^{-1} on an N-dimensional register, where F has the property that $F\,|\,0\rangle = \frac{1}{\sqrt{N}} \sum_{i \in \mathbb{Z}_N} |\,i\rangle$.

By considering the action of Z on a qubit register $|\,y\rangle$, or otherwise, show that with the restricted set of operations, Problem A can be solved with certainty by a quantum algorithm that makes two queries to the oracle $O_{\mathbf{x}}$, and as before, with each register starting in the state $|\,0\rangle$ (in the appropriate state space).

SECTION II

11I Number Theory

(a) Define what it means for an integer to be a *primitive root* mod n.

(b) Let p be an odd prime, and b a primitive root mod p. Prove the following are equivalent.

(i) b is a primitive root mod p^2.

(ii) b is a primitive root mod p^m for all $m \geqslant 2$.

(iii) No pseudoprime to the base b is divisible by p^2.

(c) Find the three smallest positive integers b with the property that b is a primitive root mod 5^m for all $m \geqslant 1$.

(d) Let $P(n)$ be the number of primitive roots mod n. Show that for each $k \geqslant 1$ there are only finitely many integers n with $P(n) = k$.

12I Automata and Formal Languages

Let $D = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite-state automaton (DFA).

What does it mean to say that $q \in Q$ is an *accessible* state? What does it mean to say that $p, q \in Q$ are *equivalent* states?

Explain the construction of the *minimal DFA* $D/\!\sim$ and show that the languages of D and of $D/\!\sim$ are the same. Show also that no two distinct states of $D/\!\sim$ are equivalent.

Now let Σ be the single-letter alphabet $\{1\}$. Suppose that D is a DFA with no inaccessible states and exactly one accept state. Justifying your answer, describe the corresponding minimal DFA $D/\!\sim$ in the form of a transition diagram or otherwise. [Remember that you need only consider accessible states.]

13C Mathematical Biology

A chemical species of concentration $C(\mathbf{x}, t)$ diffuses in a two-dimensional stationary medium with diffusivity $D(C)$. Write down an expression for the diffusive flux $\mathbf{J}$ that enters Fick's law and then show that C obeys the partial differential equation

$$\frac{\partial C}{\partial t} = \boldsymbol{\nabla} \cdot (D(C) \boldsymbol{\nabla} C) . \tag{1}$$

Suppose that at time $t = 0$ an amount $2\pi M$ of the chemical is deposited at the origin and diffuses outward in a circularly symmetric manner, so that $C = C(r, t)$ for $r > 0, t > 0$, where r is the radial coordinate. Assume the diffusivity is $D = kC$ for some constant k. Show, by dimensional analysis or otherwise, that an appropriate similarity solution has the form

$$C = \frac{M^\alpha}{(kt)^\beta} F(\xi) , \qquad \xi = \frac{r}{(Mkt)^\gamma} \quad \text{and} \quad \int_0^\infty \xi F(\xi) \, d\xi = 1 ,$$

where the exponents α, β, γ are to be determined, and derive the ordinary differential equation satisfied by F.

Solve this ordinary differential equation, subject to appropriate boundary conditions, and deduce that the chemical occupies a finite circular region of radius

$$r_0(t) = (NMkt)^{1/4} ,$$

with N a constant which you should find.

Still assuming that $D = kC$, show that if a term αC is added to the right-hand side of (1), a solution of the form $C(r, t) = G(r, \tau) e^{\alpha t}$ can be found, where $\tau(t)$ is a time-like variable satisfying $\tau(0) = 0$. Show that a suitable choice of τ reduces the dynamics to

$$\frac{\partial G}{\partial \tau} = k \boldsymbol{\nabla} \cdot (G \boldsymbol{\nabla} G) ,$$

and that the previous analysis can be applied to find the concentration. Describe the evolution in the cases $\alpha = 0, \alpha > 0$, and $\alpha < 0$.

[*Hint: In plane polar coordinates*

$$\boldsymbol{\nabla} C(r, t) \equiv \left(\frac{\partial C}{\partial r}, 0, 0 \right) \quad \text{and} \quad \boldsymbol{\nabla} \cdot (V(r, t), 0, 0) \equiv \frac{1}{r} \frac{\partial}{\partial r} (rV) .\Big]$$

14A Cosmology

(a) What are the cosmological *flatness* and *horizon* problems? Explain what forms of time evolution of the cosmological scale factor $a(t)$ must occur during a period of inflationary expansion in a Friedmann-Robertson-Walker universe. How can inflation solve the flatness and horizon problems? [You may assume an equation of state where the pressure P is proportional to the energy density ρ.]

(b) Consider a universe with a Hubble expansion rate $H = \dot{a}/a$ containing a single inflaton field ϕ with a potential $V(\phi) \geqslant 0$. The density and pressure are given by

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,,$$
$$P = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,.$$

Show that the continuity equation

$$\dot{\rho} + 3H(\rho + P) = 0$$

demands

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0\,. \qquad (\dagger)$$

(c) Consider the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\rho\,, \qquad (\dagger\dagger)$$

and show that

$$\frac{\ddot{a}}{a} = \frac{8\pi G}{3c^2}\left[V(\phi) - \dot{\phi}^2\right]\,.$$

Under what conditions does an inflationary phase occur?

(d) What is *slow roll inflation*? Show that in slow roll inflation, the scalar equation ($\dagger$) and Friedmann equation ($\dagger\dagger$) reduce to

$$3H\dot{\phi} \approx -\frac{\mathrm{d}V}{\mathrm{d}\phi} \quad \text{and} \quad H^2 \approx \frac{8\pi G}{3c^2}V(\phi)\,. \qquad (\star)$$

(e) Using the slow roll equations ($\star$), determine $a(\phi)$ and $\phi(t)$ when $V(\phi) = \frac{1}{4}\lambda\phi^4$, with $\lambda > 0$.

15D Quantum Information and Computation

For any positive integer N, let QFT_N denote the quantum Fourier transform mod N.

(a) Consider an N-dimensional state space equipped with an orthonormal basis $\mathcal{B} = \{|\,k\rangle : k \in \mathbb{Z}_N\}$. You may assume that QFT_N, measurements in the basis $\mathcal{B}$, and the basic arithmetic operations of addition and multiplication modulo N may all be performed in time $O(\mathrm{poly}(\log N))$.

Consider the function $f : \mathbb{Z}_N \to \mathbb{Z}_N$ defined by $f(x) = a^x \mod N$, where we have fixed a choice of $a \in \mathbb{Z}_N$ with $a \neq 0$. It is promised that f is periodic with period r which divides N exactly, and f is one-to-one within each period.

Describe a quantum algorithm which runs in time $O(\mathrm{poly}(\log N))$ that will identify r with success probability at least $1/2$. The algorithm should start with each quantum register (of suitable dimension) being in state $|\,0\rangle$ and it should have the property that in any run, we also learn whether it has succeeded or not. For any step of your algorithm that is not one of the operations listed above, give a brief justification that it can be performed in time $O(\mathrm{poly}(\log N))$. [You may use without proof any results from classical number theory or classical probability theory but they must be stated clearly.]

(b) Consider an N-dimensional state space with orthonormal basis $\{|\,i\rangle : i \in \mathbb{Z}_N\}$. Let S be the operation defined by $S\,|\,i\rangle = |\,i+1\rangle$ for all $i \in \mathbb{Z}_N$ (and $+$ being addition modulo N). Show that the states $\mathrm{QFT}_N\,|\,k\rangle$ for $k \in \mathbb{Z}_N$ are eigenvectors of S.
Now let $N = 4$ and represent each basis state $|\,j\rangle$ with two qubits as $|\,x\rangle\,|\,y\rangle$ where the 2-bit string xy is j written in binary. Suppose we can implement only the gates QFT_4, its inverse and any 1-qubit phase gate $P(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$. Show how S may be implemented on any input 2-qubit state and sketch the circuit for S.

16F Logic and Set Theory

State and prove the *Compactness Theorem* for first-order predicate logic. State and prove the *Upward Löwenheim–Skolem Theorem.*

[*You may assume the Completeness Theorem for first-order predicate logic.*]

For each of the following theories, is the theory axiomatisable (in the language of posets, extended by some set of constants if necessary) or not? Justify your answers.

(i) The theory of posets having only finitely many maximal elements.

(ii) The theory of posets having uncountably many maximal elements.

(iii) The theory of posets having infinitely many maximal elements or infinitely many minimal elements (or both).

(iv) The theory of posets having infinitely many maximal elements or infinitely many minimal elements, but not both.

(v) The theory of the total orders that are isomorphic to a subset of the reals.

17F Graph Theory

(a) Let G be a graph. Show that G contains a subgraph H with $\chi(H) \leqslant 3$ and

$$e(H) = \lfloor (2/3)e(G) \rfloor.$$

Show that the constant $2/3$ is sharp, in the following sense: for any $\epsilon > 0$ there exists a graph G (with $e(G) > 0$) such that every subgraph H of G with $\chi(H) \leqslant 3$ has $e(H) \leqslant (2/3 + \epsilon)e(G)$.

(b) An *unfriendly partition* of a graph $G = (V, E)$ is a partition $V = A \cup B$, where every $v \in A$ has $|N(v) \cap B| \geqslant |N(v) \cap A|$ and every vertex $v \in B$ has $|N(v) \cap A| \geqslant |N(v) \cap B|$. Show that every finite graph G has an unfriendly partition. [*Hint: Consider a partition $A \cup B = V$ maximizing the number of edges with one end in A and one end in B.*]

(c) Let $G = (\mathbb{N}, E)$ be a countably infinite graph in which all the vertices have finite degree. Show that G has an unfriendly partition.

(d) Let $G = (\mathbb{N}, E)$ be a countably infinite graph in which all the vertices have *infinite* degree. Show that G has an unfriendly partition. (In other words, in this infinite degree case, we want each vertex $v \in A$ to have $N(v) \cap B$ infinite and each $v \in B$ to have $N(v) \cap A$ infinite.)

18H Galois Theory

(a) Let L/K be an extension of fields, and suppose that K contains a primitive nth root of unity ζ. Let $\sigma \in \mathrm{Aut}(L/K)$ be a K-automorphism of L of order n. Prove that there exists a nonzero element $\alpha \in L$ with $\sigma(\alpha) = \zeta\alpha$. What is the minimal polynomial of α over L^σ, the fixed field of σ?

(b) Define what it means for L to be an *algebraic closure* of K. Given that

$$\overline{\mathbb{Q}} = \{\alpha \in \mathbb{C} \mid \alpha \text{ is algebraic over } \mathbb{Q}\}$$

is a field, show that $\overline{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$. State carefully any results that you use.

(c) Let L be an algebraically closed field of characteristic zero, and $\sigma : L \to L$ a homomorphism of fields. Suppose $\sigma^d = 1$ for some $d > 0$, and let $K = L^\sigma$ be the fixed field of σ. If M/K is a finite extension, show that M/K is a Galois extension with cyclic Galois group. [*Hint: Show that there is a K-homomorphism from M to L.*] Give an example showing that the assumption that L is algebraically closed is necessary.

19H Representation Theory

Let $G = SU(2)$ and let V_n be the complex vector space of homogeneous polynomials of degree n in two variables x, y. Construct a continuous homomorphism $\rho_n \colon G \to GL(V_n)$ so that (ρ_n, V_n) is an irreducible representation of G. Prove that (ρ_n, V_n) is indeed irreducible.

What is the character of V_n? Show that every irreducible representation of $SU(2)$ is isomorphic to (ρ_n, V_n) for some $n \geqslant 0$.

Suppose that χ is the character of a representation V of G. State a formula for the character of $\Lambda^2 V$ in terms of χ. Use it to decompose $\Lambda^2 V_4$ as a direct sum of irreducible representations up to isomorphism.

Express the character of $\Lambda^3 V$ in terms of χ. Justify your answer. Decompose $\Lambda^3 V_4$ as a direct sum of irreducible representations up to isomorphism.

20I Algebraic Topology

Suppose $f : S^{n-1} \to X$ is a continuous map. Show that f extends to a continuous map $F : D^n \to X$ if and only if f is homotopic to a constant map.

Let X be a path-connected and locally path-connected topological space. Define what it means for a space $\widetilde{X}$ to be a *universal covering space* of X. State a suitable lifting property and use it to prove that any two universal covering spaces of X are homeomorphic.

Now suppose that $\widetilde{X}$ is a universal covering space of X, and that $\widetilde{X}$ is contractible. Let K be a path-connected simplicial complex with 1-skeleton K_1, and let $i : K_1 \to K$ be the inclusion. Given a continuous map $f : |K_1| \to X$, prove that f extends to a continuous map $F : |K| \to X$ if and only if there is a homomorphism $\Phi : \pi_1(|K|, v) \to \pi_1(X, f(v))$ with $f_* = \Phi \circ i_*$, where v is any vertex of K. [*Hint: Induct on the number of simplices in* $K \setminus K_1$.]

21G Linear Analysis

(a) Prove that any metric space (X, d) is normal for the induced topology.

(b) State the *Urysohn lemma* and the *Tietze extension theorem.*

(c) Prove that a metric space (X, d) is compact if and only if all continuous functions from X to $\mathbb{R}$ are bounded.

22G Analysis of Functions

State and prove the *Riemann–Lebesgue lemma*. State *Parseval's identity*, including any assumptions you make on the functions involved.

Suppose that $f : \mathbb{R}^n \to \mathbb{C}$ is given by

$$f(x) = \frac{|x|^a}{(1+|x|^2)^{\frac{b+a}{2}}}\,.$$

Show that if $2a > -n$ and $b > n$ then $\hat{f} \in L^p(\mathbb{R}^n)$ for all $2 \leqslant p \leqslant \infty$, where $\hat{f}$ is the Fourier transform of f.

23F Riemann Surfaces

(a) Consider a finite group H of conformal equivalences of the Riemann sphere $\mathbb{C}_\infty$ such that H fixes a point $p \in \mathbb{C}_\infty$. Prove that H is cyclic and that there is a neighbourhood U of p, invariant under H, so that the quotient $V = H\backslash U$ has the structure of a Riemann surface. Show furthermore that there are charts on U and V so that the quotient map takes the form $z \mapsto z^n$ for some $n \in \mathbb{N}$.

[*You may use without proof the fact that every Möbius transformation is conjugate to either a dilation $z \mapsto \lambda z$ or a translation $z \mapsto z + c$.*]

(b) Let G be a finite group of conformal automorphisms of $\mathbb{C}_\infty$. Prove that the quotient $R = G\backslash\mathbb{C}_\infty$ has a conformal structure such that the quotient map $\mathbb{C}_\infty \to R$ is holomorphic.

(c) For each positive integer $n \geqslant 2$, construct a faithful action of the dihedral group D_{2n} on $\mathbb{C}_\infty$. Furthermore, exhibit a rational function f such that z_1 and z_2 are in the same D_{2n}-orbit if and only if $f(z_1) = f(z_2)$.

24H Algebraic Geometry

What is a *singular point* on an irreducible algebraic variety? Let X be an irreducible affine variety. Prove that the set of nonsingular points in X is dense in the Zariski topology.

Find the set of singular points on the projective variety

$$\mathbb{V}(X_0^2 + \cdots + X_{n-1}^2) \subset \mathbb{P}^n,$$

where $X_0, \ldots, X_n$ are the homogeneous coordinates on $\mathbb{P}^n$.

Let X be an irreducible variety of dimension n and let $Z \subset X$ be the closed subvariety consisting of all singular points of X. Suppose the dimension of Z is k. If Y is smooth of dimension m, what is the dimension of the set of singular points of $X \times Y$? Justify your answer.

Given integers $n > k \geqslant 0$, give an example of an n-dimensional irreducible subvariety of projective space whose subvariety of singular points is nonempty and has dimension k.

Let C be an irreducible curve in $\mathbb{P}^2$. If C is birational to a smooth projective curve of genus 2, show that C contains a singular point.

25I Differential Geometry

Let $S \subset \mathbb{R}^3$ be a surface. Define the *first fundamental form* of S. If $R \subset \mathbb{R}^3$ is also a surface, we say that a smooth map $\phi : S \to R$ is a *local isometry* if $D\phi$ preserves the first fundamental form at each point.

(a) Let $\alpha : I \to S$ be a curve, and let V be a vector field along α. Define the *covariant derivative* of V. What does it mean for α to be *geodesic*? If $\phi : S \to R$ is a local isometry, show that for an arbitrary geodesic $\alpha : I \to S$, $\phi \circ \alpha$ is also a geodesic. [You may use without proof the fact that Christoffel symbols only depend on the first fundamental form.] Must the converse be true? Give a proof or counterexample.

(b) Define the *Gauss curvature* of S. Suppose $\phi : S \to R$ is a local isometry, and let K_S and K_R denote the Gauss curvatures of S and R respectively. Is it true that $K_R \circ \phi = K_S$? State any theorem you use.

(c) Let R be the surface of revolution defined by the curve $\gamma(u) = (e^u, 0, u)$, with $-\infty < u < \infty$. Let S be the surface of revolution defined by the curve $\delta(s) = (\cosh s, 0, s)$, with $0 < s < \infty$.

(i) Show that there is a diffeomorphism $\phi : S \to R$ such that $K_R \circ \phi = K_S$.

(ii) Does there exist a local isometry $\psi : S \to R$? Justify your answer.

[*Hint: You may use without proof that the surface of revolution defined by the curve* $(f, 0, g)$ *has Gauss curvature given by*

$$\frac{(f'g'' - f''g')g'}{((f')^2 + (g')^2)^2 f}\,.$$

Standard facts about surfaces of revolution may be used without proof if clearly stated.]

26G Probability and Measure

Suppose that as $n \to \infty$, a sequence of real random variables $X_n \to^d X$, i.e. X_n converges in distribution to some limiting random variable X. Suppose further that as $n \to \infty$ a sequence of real random variables $Y_n \to^P c$, i.e. Y_n converges in probability to some constant (non-random) limit $c > 0$. Show that $X_n Y_n \to^d cX$ as $n \to \infty$.

Now let $(Z_n : n \in \mathbb{N})$ be i.i.d. real random variables with $\mathbb{E}Z_i = 0$ and finite variance $\mathrm{Var}(Z_i) = 1$ for all i. Show that

$$\frac{\sqrt{n}\sum_{i=1}^n Z_i}{\sum_{i=1}^n Z_i^2} \to^d N(0,1)$$

as $n \to \infty$, where $N(0,1)$ denotes the standard normal distribution.

[*You may use the strong law of large numbers and the central limit theorem without proof, provided they are clearly stated. You may further use without proof the equivalence of weak convergence of laws of probability measures and convergence in distribution for real random variables.*]

27J Applied Probability

(a) Define what we mean by a *renewal process* associated with the independent and identically distributed sequence of nonnegative random variables $\{\xi_n\}$.

(b) Define the *size-biased* distribution corresponding to ξ_1.

(c) Define the *excess process* $E = (E(t))$ and state a result regarding its asymptotic behaviour, giving the required conditions carefully.

(d) Let $X = (X_t)$ be a Poisson process and $N = (N_t)$ be a renewal process with non-arithmetic inter-renewal times, independent of X. Suppose that $Y = (Y_t)$ defined by $Y_t = X_t + N_t$, $t \geqslant 0$ is also a renewal process. Show that the first event time of Y has an exponential distribution by deriving an integral equation for its distribution function that is satisfied by the exponential.

28K Principles of Statistics

Suppose T_n is an estimator computed from n i.i.d. observations $X_1, \ldots, X_n$. Recall that the jackknife bias-corrected estimate of T_n is given by $\widetilde{T}_{\mathrm{JACK}} = T_n - \widehat{B}_n$, where

$$\widehat{B}_n = (n-1)\left(\frac{1}{n}\sum_{i=1}^{n} T_{(-i)} - T_n\right).$$

(a) Suppose that as $n \to \infty$ the bias function $B_n(\theta) = \mathbb{E}_\theta[T_n] - \theta$ can be approximated as

$$B_n(\theta) = \frac{a}{n} + \frac{b}{n^2} + O\left(\frac{1}{n^3}\right),$$

for some $a, b \in \mathbb{R}$. Prove that

$$|\mathbb{E}[\widetilde{T}_{\mathrm{JACK}}] - \theta| = O\left(\frac{1}{n^2}\right).$$

For the remainder of this problem, suppose $X_i \overset{i.i.d.}{\sim} N(\mu, 1)$.

(b) Consider the estimator $T_n = (\bar{X}_n)^2$ for $\theta = \mu^2$, where $\bar{X}_n$ denotes the sample mean. Compute the biases of T_n and $\widetilde{T}_{\mathrm{JACK}}$.

(c) What is the asymptotic distribution of $\sqrt{n}(T_n - \mu^2)$?

(d) Show that $\sqrt{n}(\widetilde{T}_{\mathrm{JACK}} - \mu^2)$ has the same asymptotic distribution as $\sqrt{n}(T_n - \mu^2)$. [*Hint: Define $g(t) = t^2$ and define $\bar{X}_{n-1,i}$ to be the sample mean of the observations with X_i excluded. Note that*

$$\widetilde{T}_{\mathrm{JACK}} = T_n - \frac{n-1}{n}\sum_{i=1}^{n}\big(g(\bar{X}_{n-1,i}) - g(\bar{X}_n)\big)$$

and use the identities

$$\sum_{i=1}^{n}(\bar{X}_{n-1,i} - \bar{X}_n) = 0 \quad \textit{and} \quad \bar{X}_{n-1,i} - \bar{X}_n = \frac{1}{n-1}(\bar{X}_n - X_i).$$
]

29K Stochastic Financial Models

(a) Let W be a Brownian motion and c a constant. Let $M_t = e^{cW_t - c^2 t/2}$ for $t \geqslant 0$. Show that M is a martingale in the filtration generated by W.

For the rest of the question, consider the Black–Scholes model with constant interest rate r and time-t stock price $S_t = S_0 e^{\mu t + \sigma W_t}$ for $0 \leqslant t \leqslant T$, where μ, σ, T are constants with $\sigma > 0$.

(b) Show that there exists a risk-neutral measure for the Black–Scholes model. [You may use the Cameron–Martin theorem without proof if it is clearly stated.]

(c) Find the time-0 Black–Scholes price of a European claim with time-T payout $Y_0 = S_T^p$ where the exponent p is a constant.

(d) Consider two European claims with time-T payouts

$$Y_1 = \max_{0 \leqslant t \leqslant T} S_t \text{ and } Y_2 = \frac{S_0^{2-p} S_T^p}{\min_{0 \leqslant t \leqslant T} S_t}\,.$$

Find, with proof, the exponent p such that these two claims have the same time-0 Black–Scholes price.

30E Asymptotic Methods

(a) Derive the leading order term of the asymptotic expansion, as $x \to \infty$, for the integral

$$I(x) = \int_0^2 \ln t \; e^{x(t^3 - 2t^2 + t)}\, dt\,.$$

Justify your steps.

(b) The derivative of the Gamma function has the following integral representation

$$\Gamma'(z) = \int_0^\infty \frac{\ln t}{t} e^{z \ln t - t}\, dt \qquad \text{for} \quad \operatorname{Re} z > 0.$$

In what follows we assume $z \in \mathbb{R}$ and $z > 0$.

(i) Justify briefly why the integral converges. Explain why Laplace's method cannot be used directly to find the leading order behaviour of $\Gamma'(z)$ as $z \to \infty$.

(ii) Now perform the change of variables $t = zs$, then apply Laplace's method to show that

$$\Gamma'(z) \sim \sqrt{\frac{a}{z}} e^{z \ln z - z} \ln z \qquad \text{as} \quad z \to \infty\,,$$

for a real number a, which you should determine.

31B Dynamical Systems

Consider the system

$$\begin{aligned}\dot{x} &= -ax + 3y + x(x^2 + y^2) \\ \dot{y} &= -x - ay + y(x^2 + y^2)\,,\end{aligned}$$

where $a > 0$ is a real constant. Throughout this question, you should state carefully any theorems or standard results used.

(a) Show that the origin is asymptotically stable.

(b) Define the term *Lyapunov function.* For the system above, for what values of k is $V(x, y) = x^2 + ky^2$ a valid Lyapunov function in some neighbourhood of the origin? Give your answer in the form $k_1(a) < k < k_2(a)$ where $k_1(a)$ and $k_2(a)$ should be given explicitly.

(c) By considering $V(x, y)$ for $k = 1$, what can be deduced about the domain of stability (for values of a for which $V(x, y)$ is a valid Lyapunov function)?

(d) State the *Poincaré-Bendixson theorem.* Show that the system above has a periodic orbit.

32E Integrable Systems

Explain what it means for a vector field $V = V_1(x, u)\partial_x + \phi(x, u)\partial_u$ to generate a *Lie symmetry* for a differential equation $\Delta(x, u, \partial_x u, \ldots, \partial_x^n u) = 0$. State a condition for this to hold in terms of the n^{th} *prolongation* of V, $\mathrm{pr}^{(n)}V$, giving also a definition of this latter concept.

Calculate the second prolongation of the vector field V, and hence show that if V generates an infinitesimal Lie symmetry for the equation

$$u'' = \frac{(u')^2}{u} - u^2 \tag{1}$$

then V_1 must be of the form

$$V_1(x, u) = F(x) \ln |u| + G(x)$$

for some functions F, G.

Show that if c and d are arbitrary real numbers then

$$V = (cx + d)\partial_x - 2cu\partial_u$$

is an infinitesimal Lie symmetry for equation (1), and give the form of the group of symmetries that it generates.

[*Assume $u > 0$ throughout.*]

33A Principles of Quantum Mechanics

(a) Show that $[L_z, z] = 0$ and hence that $\langle n', \ell', m'|z|n, \ell, m\rangle$ vanishes unless $m' = m$, where $|n, \ell, m\rangle$ is a simultaneous eigenstate of H, L^2 and L_z.

(b) Given that $[L^2, [L^2, z]] = 2\hbar^2(L^2 z + zL^2)$, show that $\langle n', \ell', m'|z|n, \ell, m\rangle$ vanishes unless $|\ell' - \ell| = 1$ or $\ell' = \ell = 0$. By considering parity, show that this matrix element also vanishes if $\ell' = \ell$.

(c) A hydrogen atom in its ground state $|n, \ell, m\rangle = |1, 0, 0\rangle$ is placed in a constant, uniform electric field $\mathbf{E}$. With reference to the atom's charge distribution, but without detailed calculation, give a physical explanation of why there is no correction of first-order (in $\mathbf{E}$) to the ground state energy, but higher-order corrections are possible.

(d) Show that the second-order correction to the energy of the ground state caused by the electric field is

$$\frac{e^2|\mathbf{E}|^2}{\mathcal{R}} \sum_{n=2}^{\infty} \frac{n^2}{1-n^2} \left|\langle n, 1, 0|z|1, 0, 0\rangle\right|^2 ,$$

where $-\mathcal{R}$ is the unperturbed energy of $|1, 0, 0\rangle$.

[*You may assume that, when a Hamiltonian is perturbed by* ΔH*, the second-order correction to the ground state energy is*

$$\sum_{\alpha} \frac{|\langle\alpha|\Delta H|\phi\rangle|^2}{E_\phi - E_\alpha} ,$$

where $\{|\alpha\rangle\}$ *is a complete set of unperturbed eigenstates states orthogonal to the unperturbed ground state* $|\phi\rangle$*, and* E_α*,* E_ϕ *are their unperturbed energies.*]

34D Applications of Quantum Mechanics

A two-dimensional Bravais lattice Λ has primitive basis vectors $\{\mathbf{a}_1, \mathbf{a}_2\}$, where

$$\mathbf{a}_1 = \hat{\mathbf{x}} , \qquad \mathbf{a}_2 = -\frac{1}{2}\hat{\mathbf{x}} + \frac{\sqrt{3}}{2}\hat{\mathbf{y}} ,$$

and $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$ is the standard Cartesian basis. Express a general primitive basis $\{\mathbf{a}'_1, \mathbf{a}'_2\}$ for Λ in terms of $\{\mathbf{a}_1, \mathbf{a}_2\}$.

Find the lattice Λ^* which is dual to Λ, giving a basis of primitive vectors dual to $\{\mathbf{a}_1, \mathbf{a}_2\}$. Sketch the region of the lattice Λ^* containing the origin, indicating all those points which are nearest neighbours of the origin. Determine the *Wigner-Seitz unit cell* of Λ^* as polygonal region of the plane, giving the coordinates of all vertices of this polygon. Determine the area of this unit cell.

A particle of mass m moves in a potential $V(\mathbf{x})$ which is invariant under shifts by vectors in Λ,

$$V(\mathbf{x} + \mathbf{l}) = V(\mathbf{x}) \qquad \forall\, \mathbf{l} \in \Lambda .$$

Define the n^{th} *Brillouin zone* of this system and briefly describe its physical significance. Draw a sketch showing the first and second Brillouin zones.

35A Statistical Physics

(a) What distinguishes bosons from fermions? What are the implications for the occupation number of states and for the ground state at low temperatures?

(b) Consider a gas of N non-interacting ultra-relativistic electrons in a large fixed 3-dimensional cubic volume V.

(i) Using the grand partition function, show that $pV = AE$, where p is the pressure, E is the average energy and A is a constant that you should determine.

(ii) Show that the Fermi energy, $E_F = D\,(N/V)^{1/3}$, where D is a constant that you should determine.

(iii) Show that at zero temperature $pV^a = K$, where a and K are constants that you should determine. How does this compare to an ultra-relativistic classical ideal gas?

(c) Now consider the same system as in part (b) with a magnetic field B, so the energy of an electron is $\pm\mu_B B$ depending on whether the spin is parallel or anti-parallel to the magnetic field, and μ_B is a constant. Assuming that $\mu_B B \ll E_F$, show that at zero temperature the total magnetic moment

$$M \approx \alpha \mu_B^\gamma B^\delta g(E_F)\,,$$

where $g(E_F)$ is the density of states at energy E_F and α, γ and δ are numerical constants that you should find. Then find the magnetic susceptibility χ of the gas at zero temperature. Comment on the result.

36B Electrodynamics

Consider a time-dependent localised electromagnetic field in vacuum with a four-current density J^μ and vector potential A^μ.

(a) Determine the differential equation that relates the four-current density to the vector potential in the gauge choice $\partial_\mu A^\mu = 0$.

(b) Show that the solution to the above differential equation can be expressed as

$$A^\mu(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int \frac{J^\mu(\mathbf{x}', t')}{|\mathbf{x} - \mathbf{x}'|}\, d^3x'$$

where you should specify the form of t'.

(c) Show that the time derivative of the dipole moment $\mathbf{p}$ satisfies

$$\dot{\mathbf{p}} = \int \mathbf{J}(\mathbf{x}, t)\, d^3x$$

where $\mathbf{J}$ is the current density.

(d) A small circular loop of radius r is centred at the origin. The unit vector normal to the plane of the loop is $\mathbf{n}$. A current $I(t) = \sum_{n=0}^{\infty} I_n \sin(n\omega t)$ flows in the loop. Find the three vector potential $\mathbf{A}(\mathbf{x}, t)$ to first order in $r/|\mathbf{x}|$.

37D General Relativity

Recall that the Schwarzschild metric is

$$ds^2 = -(1 - 2M/r)\,dt^2 + (1 - 2M/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\ ,$$

in units where $c = G = 1$. An advanced alien civilization builds a static, spherically-symmetrical space station surrounding a non-rotating black hole of mass M. The station itself has mass $M_{\rm st} \ll M$ and is located at a radius $r_{\rm st} > 2M$ (in Schwarzschild coordinates). It occupies a very thin shell of width $\delta r \ll r_{\rm st}$.

(a) Some sodium lamps, which emit photons at a characteristic wavelength λ, are attached to the space station. In terms of $r_{\rm st}$, what is the wavelength of these photons as seen by an observer at radius $r \gg r_{\rm st}$? What happens in the limit that $r_{\rm st}$ approaches the event horizon?

(b) What is the magnitude and direction of the proper acceleration of the space station (*i.e.* the acceleration in its own instantaneous rest frame)? Verify that in the limit $r_{\rm st} \to \infty$, the magnitude is equal to the acceleration due to Newtonian gravity.

Now suppose we wish to take into account the gravitational effects of the space station itself, even though $M_{\rm st} \ll M$. The space station has a mass per unit area of ρ as measured in its own local frame of reference. However, its effective gravitational energy is reduced by the fact that it is in a gravitational potential.

(c) What is an appropriate metric to use outside of the space station? Your answer should indicate how the metric depends on ρ. Why is this justified? [*Hint: You do not need to explicitly solve the Einstein equation in order to answer this problem.*]

38C Fluid Dynamics II

A uniform rod in the shape of an elongated cylinder falls through a viscous fluid under the action of gravity. The motion is sufficiently slow that the fluid flow is described by the Stokes equations.

(a) Show that when the long axis of the rod is initially aligned with the horizontal direction the rod falls vertically.

(b) Show that for *any* initial orientation of the rod the motion of the rod occurs with no rotation.

(c) Denoting by $\mathbf{F}$ the hydrodynamic force exerted on the rod and $\mathbf{U}$ its translation speed, explain why we expect a linear relationship of the form $\mathbf{F} = -\mathbf{R} \cdot \mathbf{U}$, where $\mathbf{R}$ is a matrix.

(d) State the reciprocal theorem of Stokes flows. Show that it implies that $\mathbf{R}$ is symmetric.

(e) Use the energy equation, as applied to this steady flow problem, to deduce that the matrix $\mathbf{R}$ is also positive definite.

(f) We denote by $\mathbf{t}$ the unit tangent vector along the rod and by θ the angle between $\mathbf{t}$ and the vertical. Writing $\mathbf{R} = c_1\mathbf{tt} + c_2(\mathbf{1} - \mathbf{tt})$ with $c_2 \geqslant c_1 > 0$, compute the value of $\cos\alpha$ where α is the angle between the vertical and the direction of motion of the rod. Check the case where $c_1 = c_2$ and comment.

39C Waves

Waves propagating in a slowly-varying medium satisfy the local dispersion relation $\omega = \Omega(\boldsymbol{k}; \boldsymbol{x}, t)$ in the standard notation.

(a) Derive the ray-tracing equations:

$$\frac{dx_i}{dt} = \frac{\partial \Omega}{\partial k_i}, \quad \frac{dk_i}{dt} = -\frac{\partial \Omega}{\partial x_i}, \quad \frac{d\omega}{dt} = \frac{\partial \Omega}{\partial t},$$

governing the evolution of a wave packet specified by

$$\phi(\boldsymbol{x}, t) = A(\boldsymbol{x}, t; \varepsilon) \exp\left(\frac{i\theta(\boldsymbol{x}, t)}{\varepsilon}\right),$$

where $0 < \varepsilon \ll 1$. A rigorous derivation is not required, but assumptions should be clearly stated and the meaning of the d/dt notation should be carefully explained.

(b) The dispersion relation for two-dimensional, small amplitude, internal gravity waves of wavenumber vector $\boldsymbol{k} = (k, 0, m)$, relative to Cartesian coordinates (x, y, z) with z vertical, propagating in an inviscid, incompressible, stratified fluid with a slowly-varying mean flow $\boldsymbol{U}$ is

$$\omega = \frac{Nk}{\sqrt{k^2 + m^2}} + \boldsymbol{k} \cdot \boldsymbol{U},$$

where N is the buoyancy frequency. Consider the specific flow $\boldsymbol{U} = \gamma(x, 0, -z)$. N and γ are positive constants.

(i) Calculate $k(t)$ and $m(t)$, applying the initial conditions $k(0) = k_0 > 0$, $m(0) = m_0$.

(ii) Consider a wave packet with initial wave vector $(k_0, 0, m_0)$, released from $(x_0, 0, z_0)$ where $x_0 > 0$ and $z_0 > 0$. Show that the wave packet can initially propagate upwards provided $z_0 < z_m$, where z_m is a function of k_0 and m_0.

(iii) Demonstrate that such a wave packet eventually approaches $z = 0$, but takes an infinite amount of time to do so. [*Hint: It is not essential to solve for an explicit expression for the position of the wave packet at arbitrary time t.*]

40C Numerical Analysis

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix with real eigenvalues $\lambda_1, \ldots, \lambda_n$ ordered by their magnitudes in nonincreasing order, $|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots \geqslant |\lambda_n|$.

(a) Define the *power method* to compute the leading eigenvalue of A. Show that, under suitable assumptions, the iterates $(\mathbf{x}_k)$ of the power method satisfy

$$r(\mathbf{x}_k) - \lambda_1 = O(|\lambda_2/\lambda_1|^{2k})$$

as $k \to \infty$, where $r(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}/\mathbf{x}^T\mathbf{x}$ is the Rayleigh quotient.

(b) Let

$$A = \begin{bmatrix} 5 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 5 \end{bmatrix}$$

to which we apply the power method with starting vector $\mathbf{x}_0 = (1/\sqrt{2}, -1/\sqrt{2}, 0)$. Compute $\mathbf{x}_k$ and $r(\mathbf{x}_k)$ explicitly, and find the limit value $\lim_{k\to\infty} r(\mathbf{x}_k)$. Compare with the result in (a) and comment. [*Hint: The eigenvalues of A are 9, 6 and 2.*]

(c) Define the *inverse iteration with shift*, and describe (without proof) the convergence of the method, clearly stating the assumptions.

END OF PAPER