UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Tuesday, 07 June, 2022 1:30pm to 4:30pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most six questions*** *from Section I and from any number of questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet ***for each question*** *that you have attempted, and place it at the front of your answer to that question.*

Complete a green master cover sheet listing ***all the questions*** *that you have attempted.*

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into ***a single bundle****, with the master cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.*

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1I Number Theory

Explain what it means for a positive definite binary quadratic form to be *reduced*, and what it means for two such forms to be *equivalent*. Prove that every positive definite binary quadratic form is equivalent to a reduced form. Show that any two equivalent forms represent the same set of integers.

Carefully quoting any further results you need, show that $f(x, y) = 6x^2 + 5xy + 2y^2$ and $g(x, y) = 9x^2 + 25xy + 18y^2$ represent the same integers, but are not equivalent.

2G Topics in Analysis

In this question you should work in $\mathbb{R}^n$ with the usual Euclidean distance.

Define a set of *first Baire category*.

For each of the following statements, say whether it is true or false and give an appropriate proof or counterexample.

(i) The countable union of sets of first category is of first category.

(ii) If A is of first category in $\mathbb{R}^2$ and $y \in \mathbb{R}$, then

$$C_y = \{x \,:\, (x, y) \in A\}$$

is of first category in $\mathbb{R}$.

(iii) If C is of first category in $\mathbb{R}$, then

$$A = \{(x, y) \,:\, x \in C,\, y \in \mathbb{R}\}$$

is of first category in $\mathbb{R}^2$.

(iv) If A and B are sets of first category in $\mathbb{R}^2$, then

$$A + B = \{\mathbf{a} + \mathbf{b} \,:\, \mathbf{a} \in A,\, \mathbf{b} \in B\}$$

is of first category.

[*You may use results about complete metric spaces provided you state them precisely.*]

3K Coding and Cryptography

What is a *discrete memoryless channel* (DMC)? State *Shannon's second coding theorem.*

Consider two DMCs of capacities C_1 and C_2, each having input alphabet $\mathcal{A}$ and output alphabet $\mathcal{B}$. The *product* of these channels is a channel whose input and output alphabets are $\mathcal{A} \times \mathcal{A}$ and $\mathcal{B} \times \mathcal{B}$, respectively, with channel probabilities given by

$$\mathbb{P}(y_1 y_2 | x_1 x_2) = \mathbb{P}_1(y_1|x_1)\, \mathbb{P}_2(y_2|x_2),$$

where $\mathbb{P}_i(y|x)$ is the probability that y is received when x is transmitted through the ith channel ($i = 1, 2$). Find the capacity of the product channel in terms of C_1 and C_2.

4I Automata and Formal Languages

State and prove the *pumping lemma for regular languages.*

Are the following languages over the alphabet $\Sigma = \{0, 1\}$ regular? Justify your answers.

(i) $\{0^n 1 \,|\, n \geqslant 0\}$.

(ii) $\{0^n 1^{n^2} \,|\, n \geqslant 0\}$.

(iii) The set of all words in Σ^* containing the same number of 0s and 1s.

5J Statistical Modelling

(a) Give the definition of an *exponential family* of probability distributions. [You may assume the natural parameter is one-dimensional.]

(b) Suppose $Y_1, \ldots, Y_n \overset{i.i.d.}{\sim} f(y; \theta)$ where $f(y; \theta)$ is the density function of an exponential family with natural parameter θ and sufficient statistic Y. Show that $\bar{Y} = \sum_{i=1}^n Y_i/n$ is a sufficient statistic for θ.

(c) In the setting above, show that the maximum likelihood estimator of θ is given by setting the theoretical mean $\mu(\theta) = \mathbb{E}_\theta(Y_1)$ to the empirical mean $\bar{Y}$.

6C Mathematical Biology

Two species with populations N_1 and N_2 compete according to the equations

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1} - b_{12}\frac{N_2}{K_1}\right)$$
$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - b_{21}\frac{N_1}{K_2}\right),$$

so that only species 1 has limited carrying capacity. Assume that the parameters $r_1, r_2, K_1, K_2, b_{12}$, and b_{21} are all strictly positive.

(a) Rescale the variables N_1, N_2 and t to leave three parameters, $\rho = r_1/r_2$, $\alpha = b_{12}K_2/K_1$ and $\beta = b_{21}K_1/K_2$ and determine the steady states.

(b) Assuming $\beta > 1$, investigate the stability of the biologically relevant steady states and sketch the phase plane trajectories.

(c) Assuming $\beta > 1$, show that irrespective of the size of the parameters the principle of competitive exclusion holds. Briefly describe under what ecological circumstances species 2 becomes extinct.

7E Further Complex Methods

A complex function $\mathrm{Arcsinh}(z)$ may be defined by

$$\mathrm{Arcsinh}(z) = \int_0^z \frac{1}{(1+t^2)^{1/2}}\,dt\,,$$

where the integrand $(1+t^2)^{-1/2}$ is equal to $1/\sqrt{2}$ at $t=1$ and has a branch cut along the imaginary axis between the points $\pm i$ (deformed very slightly to the left of the origin).

Explain how to choose the path of integration to ensure that $\mathrm{Arcsinh}(z)$ is analytic and single valued in $0 \leqslant \arg z < 2\pi$, except for z on the branch cut specified for $(1+t^2)^{-1/2}$.

Evaluate $\mathrm{Arcsinh}(-\sinh(u))$, where u is real and $u > 0$.

Deduce that if $\mathrm{arcsinh}(z)$ is an analytic continuation of $\mathrm{Arcsinh}(z)$ to the whole complex plane, omitting the branch cut, but without restriction on $\arg(z)$, then it is multivalued. What are the possible values of $\mathrm{arcsinh}(\sinh(u))$, with u real and $u > 0$?

8B Classical Dynamics

Show that Hamilton's equations for a system with n degrees of freedom can be written in the form

$$\dot{x}_a = \Omega_{ab} \frac{\partial H}{\partial x_b}\,,$$

where $a, b \in \{1, 2, \ldots, 2n\}$ and Ω is a matrix that you should define.

Using a similar notation, define the Poisson bracket $\{f, g\}$ of two functions $f(\mathbf{x}, t)$ and $g(\mathbf{x}, t)$. Evaluate the Poisson bracket $\{x_a, x_b\}$.

Show that the transformation $\mathbf{x} \mapsto \mathbf{X}(\mathbf{x})$ preserves the form of Hamilton's equations if and only if the Jacobian matrix

$$J_{ab} = \frac{\partial X_a}{\partial x_b}$$

satisfies

$$J\Omega J^T = \Omega\,.$$

Deduce that such a canonical transformation leaves the phase-space volume invariant.

9A Cosmology

Consider a ball centered on the origin which is initially of uniform energy density ρ and radius L. The ball expands outwards away from the origin. Additionally, take a particle of mass m at some position $\mathbf{x}$ with $r \equiv |\mathbf{x}| \ll L$. Assume that the particle only experiences gravity through Newton's inverse-square law.

Using the above model of the expanding universe, derive the *Friedmann equation* describing the evolution of the scale factor $a(t)$ appearing in the relation $\mathbf{x}(t) = a(t)\,\mathbf{x}_0$.

Describe the two main flaws in this derivation of the Friedmann equation.

10D Quantum Information and Computation

(a) Suppose that Alice and Bob are distantly separated in space and they can communicate classically publicly. They also have available a noiseless quantum channel on which there is no eavesdropping. Describe the steps of the BB84 protocol that results in Alice and Bob sharing a secret key of expected length $n/2$. [Note that the steps of information reconciliation and privacy amplification will not be needed in this idealised situation.]

(b) Suppose now that an eavesdropper Eve taps into the quantum channel. Eve also possesses a supply of ancilla qubits each in state $|\,0\rangle_E$. For each passing qubit $|\,\psi\rangle_A$ sent by Alice, Eve intercepts it and applies a CX operation to it and one of her ancilla qubits $|\,0\rangle_E$ with Alice's qubit being the control i.e. Eve applies CX_{AE}. After this action Eve sends Alice's qubit on to Bob while retaining her ancilla qubit.

(i) Show that for two choices of Alice's sent qubits, the qubit received by Bob will be entangled with Eve's corresponding ancilla qubit.

(ii) Calculate the bit error rate for Alice and Bob's final key in part (a) that results from Eve's action.

SECTION II

11G Topics in Analysis

Suppose $f : [0,1]^2 \to \mathbb{R}$ is continuous. Show, quoting carefully any theorems that you use, that

$$\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{n}{j}\binom{n}{k}f(j/n,k/n)t^j(1-t)^{n-j}s^k(1-s)^{n-k} \to f(t,s)$$

uniformly on $[0,1]^2$ as $n \to \infty$.

Deduce that

$$\int_0^1\left(\int_0^1 f(s,t)\,ds\right)dt = \int_0^1\left(\int_0^1 f(s,t)\,dt\right)ds$$

whenever $f : [0,1]^2 \to \mathbb{R}$ is continuous.

By giving proofs or counterexamples establish which of the following statements are true and which are false. You may not use the Stone–Weierstrass theorem without proof.

(i) If $f : [0,1]^2 \to \mathbb{R}$ is continuous and $\int_0^1\left(\int_0^1 s^n t^m f(s,t)\,ds\right)dt = 0$ for all integers $n, m \geqslant 0$, then $f = 0$.

(ii) Suppose $a < b$. If $f : [a,b]^2 \to \mathbb{R}$ is continuous and $\int_a^b\left(\int_a^b s^n t^m f(s,t)\,ds\right)dt = 0$ for all integers $n, m \geqslant 0$, then $f = 0$.

(iii) If $f : [-1,1]^2 \to \mathbb{R}$ is continuous and $\int_{-1}^1\left(\int_{-1}^1 s^{2n} t^{2m} f(s,t)\,ds\right)dt = 0$ for all integers $n, m \geqslant 0$, then $f = 0$.

(iv) If $f : [0,1]^2 \to \mathbb{R}$ is continuous and $\int_0^1\left(\int_0^1 s^{2n} t^{2m} f(s,t)\,ds\right)dt = 0$ for all integers $n, m \geqslant 0$, then $f = 0$.

12K Coding and Cryptography

(a) Consider two large distinct primes $p, q \equiv 3 \pmod 4$ and let $N = pq$. Briefly describe the *Rabin cipher* with modulus N.

I announce that I shall be using the Rabin cipher with modulus N. My friendly agent in Doxfor sends me a message m (with $1 \leqslant m \leqslant N-1$) encoded in the required form. Unfortunately, my cat eats the piece of paper on which the prime factors of N are recorded so I am unable to decipher it. I therefore find a new pair of primes and announce that I shall be using the Rabin code with modulus $N' > N$. My agent now re-encodes the message and sends it to me again.

The enemy agent Omicron intercepts both code messages. Show that Omicron can find m. Can Omicron decipher any other messages sent to me using only one of the coding schemes?

(b) Let p be a large prime and g a primitive root modulo p. What is the *discrete logarithm problem*? Explain what is meant by the *Diffie-Hellmann key exchange* and say briefly how an enemy can break the cipher if she can compute discrete logarithms efficiently.

Extend the Diffie–Hellman key exchange to cover three participants in a way that is likely to be as secure as the two-party system.

Extend the system further to n parties in such a way that they can compute their common secret key in at most $n^2 - n$ communications. (The numbers p and g of our original Diffie-Hellman system are known by everybody in advance.)

13E Further Complex Methods

Consider the differential equation

$$\frac{d^3w}{dz^3} - zw = 0\,.$$

Use Laplace's method to find solutions of the form

$$w(z) = \int_\gamma e^{zt}\, f(t)\, dt\,,$$

where γ is a contour in the complex t-plane. Determine the function $f(t)$ and state clearly the condition required for the contour γ.

Draw a sketch of the complex t-plane showing the possible choices of γ, relating these to the behaviour of $f(t)$.

Show that three different suitable contours $\gamma_i, i = 1, 2, 3$, may be formed from the positive real axis plus parts of the real axis or the imaginary axis, with each γ_i defining a function $w_i(z)$. Write down expressions for the values of $w_i(0)$, $w_i'(0)$ and $w_i''(0)$ $(i = 1, 2, 3)$ and evaluate them in terms of Gamma functions.

Give an expression for

$$\det\begin{pmatrix} w_1(0) & w_1'(0) & w_1''(0) \\ w_2(0) & w_2'(0) & w_2''(0) \\ w_3(0) & w_3'(0) & w_3''(0) \end{pmatrix}.$$

Deduce that the functions $w_i(z)$ $(i = 1, 2, 3)$ are linearly independent.

14B Classical Dynamics

(a) A homogeneous, solid ellipsoid of mass M occupies the region

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} < 1\,,$$

where a, b and c are positive constants. Calculate the inertia tensor of the ellipsoid.

(b) According to Poinsot's construction, the evolution of the angular velocity vector $\boldsymbol{\omega}(t)$ of a rigid body undergoing free rotational motion corresponds to the movement of an inertia ellipsoid on an invariable plane. Derive this construction, explaining why the inertia ellipsoid is tangent to the invariable plane and rolls on it.

(c) Describe qualitatively the general free rotational motion of the body considered in part (a) in an inertial frame of reference, in the special case $a = b < c$.

15D Quantum Information and Computation

(a) (i) Define the *Bell measurement* on two qubits.

(ii) In terms of the Bell measurement and the Bell state $|\phi^+\rangle$ give the steps of the quantum teleportation protocol. You need not give a derivation of the steps but you should clearly state all inputs and outputs of the protocol.

(iii) Suppose now that the $|\phi^+\rangle$ state used in the protocol is replaced by $|\xi\rangle = I \otimes U |\phi^+\rangle$, where U is any 1-qubit unitary and all steps of the protocol remain otherwise the same as in part (ii) above. State the outputs of this modified protocol and give a justification of your answer. [You may quote any statements from part (ii) above.]

(b) A *programmable 1-qubit gate* $\mathcal{G}$ is defined to be a device acting on two registers A and B, where A is a 1-qubit register called the input register and B is a K-qubit register (for some fixed $K \in \mathbb{N}$) called the program register. For any given state of AB the action of $\mathcal{G}$ is a fixed unitary operation G on the $K+1$ qubits, which is required to satisfy the following condition called (PROG):

For any 1-qubit unitary U there is a K-qubit state $|P_U\rangle$ such that for any 1-qubit state $|\alpha\rangle$ we have

$$|\alpha\rangle \otimes |P_U\rangle \longmapsto G(|\alpha\rangle \otimes |P_U\rangle) = (U|\alpha\rangle) \otimes \left|\tilde{P}_U\right\rangle.$$

Here $\left|\tilde{P}_U\right\rangle$ is some K-qubit state (which could generally depend on $|\alpha\rangle$ too). Thus $|P_U\rangle$ serves as a "program" for the application of U to any 1-qubit state $|\alpha\rangle$ via the fixed unitary action G.

(i) By considering suitable inner products or otherwise, show that if (PROG) holds then $\left|\tilde{P}_U\right\rangle$ must be independent of the state $|\alpha\rangle$.

(ii) Suppose that $|P_U\rangle$ and $|P_V\rangle$ implement 1-qubit unitaries U and V that have physically different actions i.e. $U \neq Ve^{i\theta}$ for any phase θ. Show that $|P_U\rangle$ and $|P_V\rangle$ must then be orthogonal if (PROG) holds. [*Hint: It may be helpful to show that for any unitary W, if $\langle\alpha|W|\alpha\rangle$ is independent of $|\alpha\rangle$ then W must be the identity gate (up to an overall phase).*]

(iii) Show that a programmable 1-qubit gate $\mathcal{G}$ satisfying (PROG) cannot exist.

(iv) Suppose now that (PROG) is extended to allow the action of $\mathcal{G}$ to involve quantum measurements as well as unitary operations and we require of the "program" $|P_U\rangle$ only that it succeeds in applying U to $|\alpha\rangle$ with at least some constant probability $0 < p < 1$ independent of U and $|\alpha\rangle$, *i.e.* the action of $\mathcal{G}$ on $|\alpha\rangle \otimes |P_U\rangle$ results in $U|\alpha\rangle$ in the first register with probability at least p for each U and $|\alpha\rangle$. Can such a probabilistic programmable 1-qubit gate exist? Give a reason for your answer.

16F Logic and Set Theory

(a) Give the inductive and synthetic definitions of ordinal addition, and prove that they are equivalent.

(b) Which of the following assertions about ordinals α, β and γ are always true, and which can be false? Give proofs or counterexamples as appropriate.

(i) $(\alpha+\beta)\gamma = \alpha\gamma + \beta\gamma$.

(ii) $\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$.

(iii) If α is a limit ordinal then $\alpha\omega = \omega\alpha$.

(iv) If $\alpha \geqslant \omega_1$ and $\beta < \omega_1$ then $\beta + \alpha = \alpha$.

(v) If $\alpha+\alpha+\beta$ and $\beta+\alpha+\alpha$ are equal then they are both equal to $\alpha+\beta+\alpha$.

17F Graph Theory

(a) For a graph H and a positive integer n, define $ex(n, H)$. Prove that $ex(n, K_3) \leqslant n^2/4$. [You may not assume Turan's theorem without proof.]

(b) For a fixed $\delta > 0$, suppose that G is a graph on n vertices with $e(G) > (1+\delta)n^2/4$. Prove that G must contain at least ϵn^3 triangles, where $\epsilon > 0$ is a constant that does not depend on n or G.

(c) Prove that $ex(n, K_{3,2}) < cn^{3/2}$, for some constant $c > 0$.

(d) Let $x_1, \ldots, x_n$ be distinct points in $\mathbb{R}^2$. Show that there exists a constant $c > 0$ such that at most $cn^{3/2}$ of the ordered pairs (x_i, x_j) can satisfy $|x_i - x_j| = 1$.

18H Galois Theory

(a) Let L be a finite field of order p^n. Suppose that $\gamma \in L$, and let $f \in \mathbb{F}_p[x]$ be the minimal polynomial of γ over $\mathbb{F}_p$. Show that $\deg f$ divides n. Prove that there is a $\gamma \in L$ for which $\deg f = n$.

Show that for every $r \geqslant 1$, there is an irreducible polynomial $g \in \mathbb{F}_p[x]$ of degree r.

[*You may assume the tower law and the existence of splitting fields, but should prove any results about finite fields that you use.*]

(b) Suppose that K is a field and that L is a finite extension of K. Define what it means for $\alpha \in L$ to be *separable* over K. If $f \in K[x]$ is the minimal polynomial of α and $\gcd(f, f') = 1$ show that α is separable over K.

Now suppose that $L = K(\beta)$ is a finite extension of K and that $\operatorname{char} K = p$. Show there exists a unique intermediate field M with $K \subseteq M \subseteq L$, such that the following conditions hold: M is a separable extension of K, $[L : M] = p^h$ for some h, and $\gamma^{p^h} \in M$ for all $\gamma \in L$. [*Hint: If β is not separable, what is its minimal polynomial?*]

19H Representation Theory

Suppose that G is a group of order 16. Let $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_r$ be the degrees of the irreducible characters of G. What are the possible values of r and $d_1, \ldots, d_r$? For each such collection $d_1, \ldots, d_r$ find a group of order 16 with these character degrees and construct the character table of the group. [You may assume any general results from the course provided that you state them clearly. You may restrict yourself to brief justifications of the values in each character table.]

20H Number Fields

Let K be a number field.

(a) Let $P_1, \ldots, P_k$ (where $k \geqslant 1$) be distinct nonzero prime ideals of $\mathcal{O}_K$ and let $m_1, \ldots, m_k$ be positive integers. Let I be the product $P_1^{m_1} \cdots P_k^{m_k}$. Explain why $I = P_1^{m_1} \cap \cdots \cap P_k^{m_k}$, and hence show that the map

$$\mathcal{O}_K/I \to \mathcal{O}_K/P_1^{m_1} \times \cdots \times \mathcal{O}_K/P_k^{m_k}$$

taking $\alpha + I$ to $(\alpha + P_1^{m_1}, \ldots, \alpha + P_k^{m_k})$ is an isomorphism of rings.

Deduce that there exists $\alpha \in I$ such that $\alpha \notin P_i I$ for all i. Show that there exists an ideal I' with $I + I' = \mathcal{O}_K$ such that II' is principal. Show also that any ideal of $\mathcal{O}_K$ can be generated by two elements.

(b) State *Dedekind's criterion* for the factorisation of rational primes in $\mathcal{O}_K$. Use it to compute the factorisation of any odd rational prime in $\mathcal{O}_K$ when $K = \mathbb{Q}(\sqrt{d})$ is a quadratic field.

Show that if $d > 0$ and K contains an element α with $N_{K/\mathbb{Q}}(\alpha) = -1$, then no prime $p \equiv 3 \pmod 4$ can ramify in K.

21I Algebraic Topology

State the *snake lemma* and derive the exactness of the Mayer–Vietoris sequence from it.

Suppose that K is a simplicial complex of dimension $n \geqslant 1$, that every $(n-1)$-simplex of K is a face of precisely two n-simplices, and that if σ and σ' are n-simplices of K then there is a sequence $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_k = \sigma'$ of n-simplices in K such that for all i, σ_i and σ_{i+1} have an $(n-1)$-simplex in common. Show that $H_n(K)$ is either trivial or isomorphic to $\mathbb{Z}$.

Now suppose that K is as above and that $H_n(K) \cong \mathbb{Z}$ is generated by $x \in H_n(K)$. If K is the union of subcomplexes L_1 and L_2 such that $L_1 \cap L_2$ has dimension less than n, describe ∂x, where ∂ is the boundary map in the Mayer–Vietoris sequence associated to the decomposition $K = L_1 \cup L_2$. Justify your answer. When is $\partial x \neq 0$?

Finally, suppose that K, L_1 and L_2 are as in the previous paragraph, that K is homeomorphic to S^3, that L_1 is homeomorphic to $S^1 \times D^2$, and that the image of $L_1 \cap L_2$ under this homeomorphism is $S^1 \times S^1 \subset S^1 \times D^2$. Compute $H_*(L_2)$.

22G Linear Analysis

(a) Given a complex Banach space $(V, \|\cdot\|)$, prove that the space of bounded linear maps $(\mathcal{B}(V,V), |||\cdot|||)$ endowed with the norm

$$|||T||| = \sup_{v\in V,\ \|v\|=1} \|Tv\|$$

is a Banach space.

(b) Assume $(V, \|\cdot\|)$ is a complex Hilbert space. State the definitions of a *compact operator* $T : V \to V$ and of a *Hilbertian basis*. Suppose $T \in \mathcal{B}(V,V)$ and V has a Hilbertian basis $(e_n)_{n\geqslant 1}$ such that $T(e_n) = \lambda_n e_n$ for complex numbers λ_n, $n \geqslant 1$. Prove that T is compact if and only if $\lambda_n \to 0$.

(c) Given a complex Hilbert space $(V, \|\cdot\|)$ and $(e_n)_{n\geqslant 1}$ a Hilbertian basis of V, consider $\mathcal{H}(V,V)$, the set of linear operators T such that $\sum_{n\geqslant 1} \|Te_n\|^2 < +\infty$. Prove that operators in $\mathcal{H}(V,V)$ are bounded and compact, and that $(\mathcal{H}(V,V), |||\cdot|||_*)$ with

$$|||T|||_* = \left(\sum_{n\geqslant 1} \|Te_n\|^2\right)^{1/2}$$

is a Hilbert space. Are $|||\cdot|||$ and $|||\cdot|||_*$ equivalent norms on $\mathcal{H}(V,V)$?

23G Analysis of Functions

Let X be a real vector space. State what it means for a functional $p : X \to \mathbb{R}$ to be *sublinear.*

Let $M \subsetneq X$ be a proper subspace. Suppose that $p : X \to \mathbb{R}$ is sublinear and the linear map $\ell : M \to \mathbb{R}$ satisfies $\ell(y) \leqslant p(y)$ for all $y \in M$. Fix $x \in X \setminus M$ and let $\widetilde{M} = \operatorname{span}\{M, x\}$. Show that there exists a linear map $\tilde{\ell} : \widetilde{M} \to \mathbb{R}$ such that $\tilde{\ell}(z) \leqslant p(z)$ for all $z \in \widetilde{M}$ and $\tilde{\ell}(y) = \ell(y)$ for all $y \in M$.

State the *Hahn–Banach theorem.*

Let $\{z_1, \ldots, z_n\}$ be a set of linearly independent elements of a real Banach space Z. Show that for each $j = 1, \ldots, n$ there exists $\ell_j \in Z'$ with $\ell_j(z_k) = \delta_{jk}$ for all $k = 1, \ldots, n$. Suppose $M \subset Z$ is a finite dimensional subspace. Show that there exists a closed subspace N such that $Z = M \oplus N$.

24F Riemann Surfaces

(a) Let $D = \{p_1, \ldots, p_n\}$ be a finite (possibly empty) subset of a Riemann surface R, and let $m_1, \ldots, m_n$ be strictly positive integers. Let V be the set of meromorphic functions f on R such that each pole of f is at some p_i, and the order of a pole at p_i is at most m_i. Prove that V is a vector space over $\mathbb{C}$.

(b) For any compact Riemann surface R, prove that

$$\dim_{\mathbb{C}} V \leqslant 1 + \sum_{i=1}^{n} m_i$$

by considering Laurent expansions about the p_i, or otherwise.

(c) Let $R = \mathbb{C}/\Lambda$ be a complex torus. For any meromorphic function f on R with poles $p_1, \ldots, p_n$, prove that

$$\sum_{i=1}^{n} \mathrm{res}_f(p_i) = 0\,.$$

Assuming that $n \geqslant 1$, deduce that $\dim_{\mathbb{C}} V = \sum_i m_i$.

25H Algebraic Geometry

State the *Riemann–Hurwitz theorem*. Show that, if C and C' are smooth projective connected curves over a characteristic zero field with $g(C) < g(C')$, then any morphism

$$C \to C'$$

is constant.

Let $C_d \subset \mathbb{P}^2$ be a smooth plane curve of degree d. Construct a morphism

$$\varphi : C_d \to \mathbb{P}^1$$

of degree $d-1$. Let $B \subset \mathbb{P}^1$ be the set of branch points for φ. Give an upper bound for the cardinality of B in terms of d.

Now let D be the divisor on C_d associated to a hyperplane section of C_d. Prove that if $d \geqslant 5$ then D is not linearly equivalent to the canonical divisor of C_d.

The *gonality* of a curve C is the minimum degree of a non-constant morphism $C \to \mathbb{P}^1$. Prove that a smooth plane curve of degree 4 has gonality equal to 3. What is the gonality of a smooth projective curve of genus 1?

26I Differential Geometry

Define a *k-dimensional smooth manifold*, and a *regular value* of a smooth map between smooth manifolds. State the *inverse function theorem*, and use it to prove the *preimage theorem.*

Suppose X and Y are smooth manifolds and $f : X \to Y$ is a smooth map. If X is compact, show that the set of regular values of f in Y is open.

Consider the space

$$X_a = \{x + y - z^2 - w^2 = a\} \cap \{x^2 + y^2 - z^4/2 = 0\},$$

where x, y, z, w are the standard coordinates on $\mathbb{R}^4$, and $a \in \mathbb{R}$ is a constant. Show that X_a is a 2-dimensional manifold whenever $a \neq 0$. Is X_0 a manifold? Justify your answer.

27G Probability and Measure

(a) State and prove the *monotone convergence theorem.*

(b) Let f_1 be a μ-integrable function and let f be a measurable function defined on some measure space $(E, \mathcal{E}, \mu)$. Suppose the sequence $(f_n : n \in \mathbb{N})$ of measurable functions on E is such that $f_n \uparrow f$ pointwise on E as $n \to \infty$. Show that $\mu(f_n) \uparrow \mu(f)$ as $n \to \infty$. Show that the conclusion may fail if f_1 is not integrable.

28J Applied Probability

(a) Let $X = (X_t)$ be a right-continuous process with values in a finite state space S, and let Q be a Q-matrix on S. State two different conditions that are equivalent to the statement that X is a continuous-time Markov chain with generator Q. Prove that these two conditions are equivalent.

(b) Let G be a finite connected graph and let A be a connected subgraph of G. Let X be a continuous time Markov chain that takes values in the vertices of A and evolves as follows: when at x it stays there for an exponential time of parameter 1 and then chooses a neighbour of x in G uniformly at random. If the neighbour is in A, then X jumps there, otherwise it waits for another independent exponential time of parameter 1 and proceeds as before. This continues until the first time that X chooses a neighbour of x in A and then jumps there. Find the Q-matrix and the invariant distribution of X. Justify your answer.

[*You may use the fact that, if N is a geometric random variable of parameter p and $(E_i)_{i \geqslant 1}$ is an i.i.d. sequence of exponential random variables of parameter* 1 *independent of N, then $\sum_{i=1}^{N} E_i$ is exponentially distributed with parameter p.*]

29K Principles of Statistics

Suppose $X_1, \dots, X_n$ are i.i.d. samples from a $N(\theta, 1)$ distribution. Consider an estimator $\widehat{\theta}_{a,b}$ of the form $a\bar{X}_n + b$, where $a, b \in \mathbb{R}$ and $\bar{X}_n$ denotes the sample mean. Throughout this question, we will consider risks computed with respect to the quadratic loss.

(a) Compute the risk of $\widehat{\theta}_{a,b}$ for estimating θ.

(b) Use the formula in part (a) to show that when $a > 1$, the estimator $\widehat{\theta}_{a,b}$ is inadmissible for estimating θ.

(c) Now use the formula in part (a) to show that when $a < 0$, the estimator $\widehat{\theta}_{a,b}$ is also inadmissible for estimating θ. [*Hint: Compare the estimator with the constant estimator* $\delta := \frac{-b}{a-1}$.]

(d) Prove that $\bar{X}_n$ is admissible for estimating θ. [*Hint: You may use, without proof, the general Cramér–Rao lower bound, and the facts that* $I(\theta) = 1$ *and* $\mathbb{E}_\theta[\delta(X)]$ *is differentiable for any estimator* δ *under the Gaussian model.*]

(e) Can any of the estimators considered in parts (b) and (c) be minimax for estimating θ?

30K Stochastic Financial Models

Consider a one-period market model with constant interest rate r and d risky assets. For $n \in \{0, 1\}$ let S_n denote the vector of time-n prices of the risky assets and let X_n be the time-n wealth of an investor. Let $\mu = \mathbb{E}(S_1)$ and $V = \operatorname{Cov}(S_1)$. Assume $\mu \neq (1+r)S_0$.

(a) Suppose V is non-singular. Find, with proof, the minimum of $\operatorname{Var}(X_1)$ subject to the constraints that $X_0 = x$ and $\mathbb{E}(X_1) = m$ for given constants x and m. Show that the optimal portfolio of risky assets is of the form $\theta^* = \lambda V^{-1}[\mu - (1+r)S_0]$ for a constant λ to be found. Now find the minimum of $\operatorname{Var}(X_1)$ subject to $X_0 = x$ and $\mathbb{E}(X_1) \geqslant m$.

(b) Again suppose V is non-singular. Find, with proof, the maximum of the quantity

$$\frac{\mathbb{E}(X_1) - (1+r)X_0}{\sqrt{\operatorname{Var}(X_1)}},$$

subject to $X_0 = x$. Show that all optimal portfolios are mean-variance efficient.

(c) Now suppose V is singular and that there exists *no* vector $\theta \in \mathbb{R}^d$ such that $V\theta = \mu - (1+r)S_0$. Show that for any m and x,

$$\min\{\operatorname{Var}(X_1) : \mathbb{E}(X_1) = m \text{ and } X_0 = x\} = 0.$$

Show that there exists an arbitrage in this market.

31J Mathematics of Machine Learning

(a)What does it mean for a function $f : \mathcal{Z}_1 \times \cdots \times \mathcal{Z}_n \to \mathbb{R}$ to have the *bounded differences property* with constants $L_1, \ldots, L_n$?

State the *bounded differences inequality.*

(b) Let $\mathcal{X}$ and $\mathcal{Y}$ be input and output spaces respectively. Let H be a machine learning algorithm taking as its argument a dataset $D \in (\mathcal{X} \times \mathcal{Y})^n$ to output a hypothesis $H_D : \mathcal{X} \to \mathbb{R}$. For $D = (x_i, y_i)_{i=1}^n \in (\mathcal{X} \times \mathcal{Y})^n$ and $(x, y) \in \mathcal{X} \times \mathcal{Y}$, for all $i = 1, \ldots, n$ we write

$$D_i(x,y) := ((x_1,y_1), \ldots, (x_{i-1}, y_{i-1}), (x,y), (x_{i+1}, y_{i+1}), \ldots, (x_n, y_n)) \, .$$

Let $\ell : \mathbb{R} \times \mathcal{Y} \to [0, M]$ be a bounded loss function. Suppose H has the following property: there exists $\beta \geqslant 0$ such that for all $i = 1, \ldots, n$ and for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, we have

$$\sup_{(\tilde{x},\tilde{y}) \in \mathcal{X} \times \mathcal{Y}} |\ell(H_{D_i(x,y)}(\tilde{x}), \tilde{y}) - \ell(H_D(\tilde{x}), \tilde{y})| \leqslant \beta.$$

Let $(X, Y) \in \mathcal{X} \times \mathcal{Y}$ be a random input–output pair. Show that $F : (\mathcal{X} \times \mathcal{Y})^n \to \mathbb{R}$ given by

$$F\left((x_1,y_1), \ldots, (x_n, y_n)\right) = \mathbb{E}\ell(H_D(X), Y) - \frac{1}{n}\sum_{i=1}^n \ell(H_D(x_i), y_i)$$

satisfies a bounded differences property with constants all equal to $2\beta + M/n$. [In the expectation above, the (x_i, y_i) are considered deterministic.]

(c) Now suppose $D = (X_i, Y_i)_{i=1}^n \in (\mathcal{X} \times \mathcal{Y})^n$ is a collection of i.i.d. input–output pairs independent of, and each having the same distribution as, (X, Y). Show that $\mathbb{E}F(D) \leqslant \beta$. [*Hint: Find an alternative expression for* $\mathbb{E}\ell(H_D(X), Y)$ *as a sum of expectations with the ith term involving* $H_{D_i(X,Y)}$.]

(d) Hence conclude that, given $0 < \delta \leqslant 1$,

$$\frac{1}{n}\sum_{i=1}^n \ell(H_D(X_i), Y_i) + \beta + (2n\beta + M)\sqrt{\frac{\log(1/\delta)}{2n}} \geqslant \mathbb{E}\ell(H_D(X), Y)$$

with probability at least $1 - \delta$.

32E Asymptotic Methods

(a) Let $n = 1, 2, \ldots$. Which of the following sequences are asymptotic and why?

(i) $\phi_n(x) = \ln(\cos(x^n))$ as $x \to 0$.

(ii) $\psi_n(x) = n^{1/x}$ as $x \to \infty$.

(iii) $\chi_n(x) = \sin(x^n)$ as $x \to \infty$.

(b) Let $\phi_n(x)$ and $\psi_n(x)$, for $n = 0, 1, 2, \ldots$, be two sequences of real positive functions defined on $\{x \in \mathbb{R} : 0 < |x - x_0| < 1\}$ which are asymptotic sequences as $x \to x_0$.

For $n = 0, 1, 2, \ldots$, show that the sequence

$$\chi_n(x) = \sum_{k=0}^{n} \phi_k(x)\psi_{n-k}(x)\,,$$

is an asymptotic sequence as $x \to x_0$.

33B Dynamical Systems

(a) Let $F : I \to I$ be a continuous one-dimensional map of an interval $I \in \mathbb{R}$. Define what it means for F to have a *horseshoe*.

Define what it means for F to be *chaotic*. [Glendinning's definition should be used throughout this question.]

Prove that if F has a 3-cycle then F^2 has a horseshoe. [You may assume corollaries of the Intermediate Value Theorem.]

(b) Suppose now that F has a 4-cycle, and consider each of these orderings of the points of the 4-cycle:

(i) $x_0 < x_1 < x_2 < x_3$

(ii) $x_0 < x_1 < x_3 < x_2$

(iii) $x_0 < x_2 < x_1 < x_3$

For each of these orderings, construct a suitable directed graph. Based on each of these directed graphs, determine if the corresponding F must be chaotic and also give the minimum number of distinct 3-cycles that F must have.

Give an explicit example of a continuous map $F : [0, 1] \to [0, 1]$ which has a 4-cycle and is not chaotic. [*Hint: choose a suitable ordering for the points on the 4-cycle, construct a function which is piece-wise linear between these points, and examine the dynamics of this map.*]

34E Integrable Systems

It is possible to obtain solutions of the partial differential equation

$$u_{XT} = \sin u\,, \tag{1}$$

at time T from certain discrete scattering data $\{\lambda_m(T), c_m(T)\}_{m=1}^N$ and corresponding eigenfunctions $\psi_m(X,T)$ for an associated linear problem by means of the formula

$$u_X(T,X) = -4\sum_m c_m \psi_m^{(1)}(X,T)e^{i\lambda_m X},$$

where $\psi_m = \begin{pmatrix}\psi_m^{(1)}\\ \psi_m^{(2)}\end{pmatrix}$ and $\tilde{\psi}_m = \begin{pmatrix}-\overline{\psi_m^{(2)}}\\ \overline{\psi_m^{(1)}}\end{pmatrix}$ solve

$$\tilde{\psi}_n(X,T)e^{i\overline{\lambda_n(T)}X} - \begin{pmatrix}0\\1\end{pmatrix} = \sum_m \frac{c_m(T)\psi_m(X,T)}{(\overline{\lambda_n(T)} - \lambda_m(T))}e^{i\lambda_m(T)X}\,.$$

Given the fact that the discrete scattering data $\{\lambda_m(T), c_m(T)\}_{m=1}^N$ evolve according to $\lambda_m(T) = \lambda_m(0) = \lambda_m$ and $c_m(T) = c_m(0)e^{-\frac{iT}{2\lambda_n}}$, obtain the solution in the case $N=1$ with $\lambda_1(T) = il$ purely imaginary and $c_1(0) = c = 2l > 0$. Show that there is a unique *positive* value of l for which the solution is of the form $F(X+T)$ for some function F, which you should give.

Show that

$$g^s : \begin{pmatrix}X\\T\\u\end{pmatrix} \mapsto \begin{pmatrix}e^s X\\ e^{-s}T\\ u\end{pmatrix} \tag{2}$$

defines a group of Lie point symmetries of (1). Show that all the solutions to (1) you obtained for $N=1$ transform under (2) into $F(X+T)$, with F as above.

In the case $N=2$ and $\lambda_1 = il + m$, $\lambda_2 = il - m$ with real $l > 0, m > 0$ there is a solution of (1) given by

$$u(T,X) = 4\arctan \frac{l\sin\Big(2mX - \frac{2mT}{4(l^2+m^2)}\Big)}{m\cosh\Big(\frac{2lT}{4(l^2+m^2)} + 2lX\Big)}\,. \tag{3}$$

Show that if $l^2+m^2 = \frac{1}{4}$ then this solution is periodic in $t = T - X$ for fixed $x = X+T$; find the period.

Show that for arbitrary l^2+m^2 the solutions (3) may be transformed by (2) into the case $l^2+m^2 = \frac{1}{4}$.

35A Principles of Quantum Mechanics

(a) Let $\{|\uparrow\rangle, |\downarrow\rangle\}$ be a basis of S_z eigenstates for a spin-$\frac{1}{2}$ particle. Find the eigenstates $|\uparrow_\theta\rangle$ and $|\downarrow_\theta\rangle$ of $\mathbf{n} \cdot \mathbf{S}$, where $\mathbf{n} = (\sin\theta, 0, \cos\theta)$, and give their corresponding eigenvalues.

(b) Two spin-$\frac{1}{2}$ particles are in the combined spin state

$$|\psi\rangle = \frac{|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle}{\sqrt{2}}\,.$$

Show that this state is unchanged under the substitution

$$(|\uparrow\rangle, |\downarrow\rangle) \mapsto (|\uparrow_\theta\rangle, |\downarrow_\theta\rangle).$$

Hence show that $|\psi\rangle$ is an eigenstate, with eigenvalue zero, of each Cartesian component of the combined spin operator $\mathbf{S} = \mathbf{S}^{(1)} + \mathbf{S}^{(2)}$, where $\mathbf{S}^{(i)}$ is the spin operator of the i^{th} particle.

(c) Two spin-$\frac{1}{2}$ particles are in the spin state

$$|\chi\rangle = \frac{|\uparrow\rangle|\downarrow_\theta\rangle - |\downarrow\rangle|\uparrow_\theta\rangle}{\sqrt{2}}\,.$$

A measurement of S_z for the first particle is carried out, followed by a measurement of S_z for the second particle. List the possible outcomes for this pair of measurements and find the total probability, in terms of θ, for each pair of outcomes to occur. For which of these outcomes is the system left in an eigenstate of the combined total spin operator $\mathbf{S} \cdot \mathbf{S}$, and what are the corresponding eigenvalues?

[*Hint: The Pauli sigma matrices are given by*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad]$$

36D Applications of Quantum Mechanics

A particle of mass m moves in one dimension in the periodic potential

$$V(x) = \sum_{n \in \mathbb{Z}} V_n \exp\left(\frac{2\pi i n x}{a}\right),$$

where $V_{-n} = (V_n)^*$. Treating the Hamiltonian $\hat{H} = \hat{H}_0 + V(x)$ as a small perturbation of the free Hamiltonian $\hat{H}_0$, show that the energy spectrum consists of continuous bands separated by gaps of width $2|V_n|$ that occur for each positive integer n.

What is meant by the *dispersion relation* of the particle? Determine an explicit form of the dispersion relation near each band gap.

Work out the locations and widths of the gaps in the energy spectrum for the potential

$$V(x) = \frac{8}{3} V_0 \cos^4\left(\frac{2\pi x}{a}\right).$$

Sketch the dispersion relation of a particle moving in this potential.

37A Statistical Physics

(a) What systems are described by a *microcanonical ensemble* and which by a *canonical ensemble*?

(b) Starting from the Gibbs formula for entropy, $S = -k_B \sum_n p(n) \ln p(n)$, where $p(n)$ is the probability of being in microstate n and k_B is the Boltzmann constant, show how maximising entropy subject to appropriate constraints leads to the correct forms of the probability distributions for (i) the microcanonical ensemble and (ii) the canonical ensemble.

(c) Derive an expression for the entropy in the canonical ensemble in terms of the partition function Z and temperature T.

(d) A system consists of N non-interacting particles fixed at points in a lattice in thermal contact with a reservoir at temperature T. Each particle has three possible states with energies $-\epsilon, 0, \epsilon$, where $\epsilon > 0$ is a constant. Compute the average energy E and the entropy S. Evaluate E and S in the limits $T \to \infty$ and $T \to 0$.

(e) For the system in part (d), describe a configuration that would have negative temperature. Justify your answer.

38D General Relativity

(a) Consider a 2-sphere with coordinates (θ, ϕ) and metric

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2 \ .$$

(i) Show that lines of constant longitude ($\phi =$ constant) are geodesics, and that the only line of constant latitude ($\theta =$ constant) that is a geodesic is the equator ($\theta = \pi/2$).

(ii) Take a vector with components $V^\mu = (1, 0)$ in these coordinates, and parallel transport it once around a circle of constant latitude. What are the components of the resulting vector, as functions of θ?

(b) In units where $8\pi G = 1$, the Einstein equation states that $T_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$. Solve for $R_{\alpha\beta}$ in terms of $T_{\alpha\beta}$ and $T = g^{\alpha\beta} T_{\alpha\beta}$, in general space-time dimension $n > 2$.

(c) Using the symmetries of the Riemann curvature tensor, show that in $n = 2$ dimensions, $R_{\alpha\beta} = \frac{1}{2} g_{\alpha\beta} R$. [*Hint: Since this is a tensor equation, it only needs to be proved in one particular coordinate system.*] Explain the implications of this if we try to define General Relativity in $n = 2$ space-time dimensions.

39C Fluid Dynamics II

(a) A fluid has kinematic viscosity $\nu > 0$. In flow over a stationary rigid boundary with length scale $\mathcal{L}$, the fluid velocity far from the boundary has typical magnitude $\mathcal{U}$. Define the *Reynolds number*. Explain why even if the Reynolds number is large the effects of viscosity cannot be neglected and explain briefly how boundary layer theory provides a useful approximate approach to including these effects.

(b) A steady high-Reynolds number flow is induced in a semi-infinite fluid otherwise at rest, in the region $y > 0$, by the in-plane motion of an extensible sheet lying along $x \geqslant 0, y = 0$. Points on the sheet move with velocity $\mathbf{V} = \alpha x \, \mathbf{e}_x$, where α is the prescribed constant rate of extension and $\mathbf{e}_x$ is the unit vector in the x-direction.

(i) What should be chosen for the typical flow speed $U(x)$ in the boundary layer? Give an estimate of the corresponding x-dependent Reynolds number and deduce that, for x sufficiently large, the flow is described by the boundary layer equations. Derive the fundamental boundary-layer scaling relating $U(x)$ and the thickness $\delta(x)$ of the boundary layer and deduce the scaling for $\delta(x)$ as a function of x.

(ii) State the two-dimensional boundary layer equations and their boundary conditions for this problem in terms of a streamfunction $\psi(x, y)$.

(iii) Seek a similarity solution to the boundary layer equations using

$$\psi(x, y) = U(x)\delta(x)f(\eta)\,,$$

where $\eta \equiv \frac{y}{\delta(x)}$. Derive the ODE and boundary conditions satisfied by $f(\eta)$.

(iv) Show that the ODE satisfied by f has a solution of the form $A + B\exp(-C\eta)$ and determine the values of the constants A, B and C.

(i) Comment on the behaviour of f as $\eta \to \infty$. What are the implications for the flow external to the boundary layer?

40C Waves

Infinitesimal displacements $\boldsymbol{u}(\boldsymbol{x},t)$ in a uniform, linear isotropic elastic solid with density ρ_0 and Lamé moduli λ and μ satisfy the linearised Cauchy momentum equation:

$$\rho_0 \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = (\lambda+\mu)\, \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) + \mu\nabla^2\boldsymbol{u}.$$

(a) Show that the dilatation $\boldsymbol{\nabla}\cdot\boldsymbol{u}$ and the rotation $\boldsymbol{\nabla}\times\boldsymbol{u}$ satisfy wave equations, and find the wave-speeds c_P and c_S.

(b) A plane harmonic P-wave with wavevector $\boldsymbol{k}$ lying in the (x,z) plane is incident from $z<0$ at an oblique angle on the planar horizontal interface $z=0$ between two elastic solids with different densities and elastic moduli. Show in a diagram the directions of all the reflected and transmitted waves, labelled with their polarisations, assuming that none of these waves is evanescent. State the boundary conditions on components of $\boldsymbol{u}$ and the stress tensor $\boldsymbol{\sigma}$ and explain why these are sufficient to determine the amplitudes. (You do not need to calculate the directions or amplitudes explicitly.)

(c) Now consider a plane harmonic P-wave of unit amplitude, with $\boldsymbol{k} = k(\sin\theta, 0, \cos\theta)$, incident from $z<0$ on the interface $z=0$ between two elastic (and inviscid) liquids with modulus λ, density ρ and wave-speed c_P in $z<0$ and modulus λ', density ρ' and wave-speed c_P' in $z>0$, with $\rho'\neq\rho$.

(i) Under what conditions is there a propagating transmitted wave in $z>0$?

(ii) Assume from here on that these conditions are met. Obtain solutions for the reflected and transmitted waves.

(iii) Show that the amplitude of the reflected wave is

$$R = \frac{\lambda'\sin 2\theta - \lambda\sin 2\theta'}{\lambda'\sin 2\theta + \lambda\sin 2\theta'},$$

where θ' is the angle the wave vector of the transmitted wave makes with the vertical.

(iv) Hence obtain an expression for θ in terms of the wave-speeds and densities of the two liquids that implies no reflection (i.e. $R=0$).

41C Numerical Analysis

(a) Consider a linear recurrence relation

$$\sum_{k=r}^{s} a_k u_{m+k}^{n+1} = \sum_{k=r}^{s} b_k u_{m+k}^{n} \qquad n \geqslant 0, \;\; m \in \mathbb{Z}\,,$$

where (a_k) and (b_k) are fixed coefficients.

(i) Show that if we define the Fourier transform of $\boldsymbol{u}^n = (u_m^n)_{m\in\mathbb{Z}}$ by $\widehat{u^n}(\theta) = \sum_{m\in\mathbb{Z}} e^{-im\theta} u_m^n$, then the linear recurrence relation takes the form

$$\widehat{u^{n+1}}(\theta) = H(\theta)\widehat{u^n}(\theta)\,,$$

where $H(\theta)$ is a function that you should specify.

(ii) Show that the sequence $(\boldsymbol{u}^n)_{n\geqslant 0}$ is bounded in the ℓ_2 norm, for all $\boldsymbol{u}^0$, if and only if $|H(\theta)| \leqslant 1$ for all $\theta \in [-\pi, \pi]$.

[*You may assume Parseval's identity:*

$$\|u\|_{\ell_2}^2 = \sum_{m\in\mathbb{Z}} |u_m|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\widehat{u}(\theta)|^2\, d\theta. \qquad]$$

(b) Consider the following three recurrence relations:

(i) $u_m^{n+1} = u_m^n + \mu(u_m^n - u_{m-1}^n)$

(ii) $u_m^{n+1} = \frac{1}{2}\mu(1+\mu)u_{m-1}^n + (1-\mu^2)u_m^n - \frac{1}{2}\mu(1-\mu)u_{m+1}^n$

(iii) $u_m^{n+1} - \frac{1}{2}(\mu-\alpha)(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}) = u_m^n + \frac{1}{2}(\mu+\alpha)(u_{m-1}^n - 2u_m^n + u_{m+1}^n)$

where $n \in \mathbb{N}$ is the time discretization index, $m \in \mathbb{Z}$ is the spatial discretization index, $\mu \geqslant 0$ is the Courant number, and, for (iii), $\alpha \geqslant 0$ is a parameter. In each case give an expression for the amplification factor $H(\theta)$, and deduce the set of values μ (and α for (iii)) for which we have stability.

END OF PAPER