UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Monday, 6 June, 2022 1:30pm to 4:30pm

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most six questions*** *from Section I and from any number of questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet ***for each question*** *that you have attempted, and place it at the front of your answer to that question.*

Complete a green master cover sheet listing ***all the questions*** *that you have attempted.*

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into ***a single bundle****, with the master cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.*

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1I Number Theory

A function $f : \mathbb{N} \to \mathbb{C}$ is *multiplicative* if $f(mn) = f(m)f(n)$ for all m, n coprime. Show that if f is multiplicative then so is $g(n) = \sum_{d|n} f(d)$. Define the *Möbius function* μ and *Euler function* ϕ. Establish the identities

$$\frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d} \qquad \text{and} \qquad \frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu(d)^2}{\phi(d)}\,.$$

2G Topics in Analysis

Show that if a, A, B, C, D are non-negative integers and $AD - BC = 1$, then

$$a + \frac{At + B}{Ct + D} = \frac{\alpha t + \beta}{\gamma t + \delta}$$

for some α, β, γ, δ non-negative integers with $\alpha\delta - \beta\gamma = 1$.

If $N, a_1, a_2, \ldots$ are strictly positive integers with $a_{N+k} = a_k$ for all k and

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ldots}}}}$$

show that x is a root of a quadratic (or linear) equation with integer coefficients.

Give the quadratic equation explicitly in the case when $N = 2$, $a_1 = a$, $a_2 = b$. Explain how you know which root gives the continued fraction.

3K Coding and Cryptography

(a) State *Kraft's inequality.*

Show that Kraft's inequality gives a necessary condition for the existence of a prefix-free code with given codeword lengths.

(b) A *comma code* is one where a special letter—the comma—occurs at the end of each codeword and nowhere else. Show that a comma code is prefix-free and give a direct argument to show that comma codes must satisfy Kraft's inequality.

Give an example of a non-decipherable code satisfying Kraft's inequality.

4I Automata and Formal Languages

What are the nth *register machine* P_n and the nth *recursively enumerable set* W_n?

Given subsets $A, B \subseteq \mathbb{N}$, define a *many–one reduction* $A \leqslant_m B$ of A to B.

State *Rice's theorem.*

Is there a total algorithm that, on input n in register 1 and m in register 2, terminates with 0 if $W_m = W_n$ and 1 if $W_m \neq W_n$? Is there a partial algorithm that, with the same inputs as above, terminates with 0 if $W_m = W_n$ and never halts if $W_m \neq W_n$? Justify your answers.

[*You may assume without proof that the halting set* $\mathbb{K}$ *is not recursive.*]

5J Statistical Modelling

Let Y_μ be the Poisson distribution with mean μ. Show that the transformation $g(y) = 2\sqrt{y}$ is "variance stabilising" for Y_μ in the sense that the variance of $g(Y_\mu)$ is approximately 1 when μ is large.

Suppose we fit a linear model to the transformed response $\sqrt{Y}$. How does this differ from using the square root link in the Poisson regression?

6C Mathematical Biology

Consider the discrete delay equation

$$x_{n+1} = x_n \exp\left[r\left(1 - x_{n-1}\right)\right],$$

with $r > 0$ a constant.

(a) Find the positive fixed point x^* of the model. Setting $x_n = x^* + u_n$, with $|u_n| \ll 1$, determine the linearised stability equation for u_n.

(b) Find the range of r for which the fixed point x^* is stable and for which perturbations decay monotonically in time.

(c) Find the range of r for which the decay of perturbations to x^* is oscillatory.

(d) Find the critical value r^* for x^* to become unstable, and show that at that value of r the system exhibits oscillations of period $p > 1$. Find p.

7E Further Complex Methods

Show that

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{s^{z-1}}{s-t}\, ds = \pi i t^{z-1},$$

where t is real and positive, $0 < \mathrm{Re}(z) < 1$ and the branch of s^z is chosen so that, for z real, s^z is real and positive for s real and positive and $s^z = (-s)^z e^{i\pi z}$ for s real and negative.

Deduce that for z real with $0 < z < 1$

$$\int_{0}^{\infty} \frac{s^{z-1}}{s+t}\, ds = \pi t^{z-1} \mathrm{cosec} \pi z$$

and

$$\mathcal{P}\int_{0}^{\infty} \frac{s^{z-1}}{s-t}\, ds = -\pi t^{z-1} \cot \pi z.$$

Why do these results actually hold for a large set of non-real z?

8B Classical Dynamics

(a) Show that the canonical transformation $(\mathbf{q},\mathbf{p}) \mapsto (\mathbf{Q},\mathbf{P})$ associated with a generating function $F_2(\mathbf{q},\mathbf{P})$ of type 2 satisfies

$$\mathbf{p} = \frac{\partial F_2}{\partial \mathbf{q}}\,, \qquad \mathbf{Q} = \frac{\partial F_2}{\partial \mathbf{P}}\,.$$

(b) A physical system with two degrees of freedom is described by the Hamiltonian

$$H(\mathbf{q},\mathbf{p}) = H_0(p_1,p_2) + H_1(p_1,p_2)\cos\theta\,,$$

where

$$\theta = n_1 q_1 + n_2 q_2$$

and n_1 and n_2 are non-zero integers.

Show that a certain linear combination of p_1 and p_2 is conserved, and that there is a (linear) canonical transformation $(\mathbf{q},\mathbf{p}) \mapsto (\mathbf{Q},\mathbf{P})$ such that $Q_1 = \theta$ and the transformed Hamiltonian does not depend on Q_2.

Explain why the system is integrable.

9A Cosmology

Consider the process where protons and electrons combine to form neutral hydrogen atoms at temperature T. Let n_H be the number density of hydrogen atoms, n_e the number density of electrons, m_e the mass of the electron and $E_{\rm bind}$ the binding energy of hydrogen. Derive *Saha's equation* which relates the ratio n_H/n_e^2 to m_e, $E_{\rm bind}$ and T. Clearly describe the steps required.

[*You may use without proof that at temperature T and chemical potential μ, the number density n of a non-relativistic particle species with mass $m \gg k_B T/c^2$ is given by*

$$n = g\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2} \exp\left[-\frac{(m\,c^2-\mu)}{k_B\,T}\right],$$

where g is the number of degrees of freedom of this particle species and k_B, $\hbar$ and c are the Boltzmann, Planck and speed of light constants, respectively.]

10D Quantum Information and Computation

Alice and Bob are separated in space and possess local quantum systems A and B respectively.

(a) State the *no-signalling theorem* for quantum states of the composite system AB.

(b) State and prove the *no-cloning theorem* (for unitary processes) for a set $\mathcal{S}$ of quantum states.

(c) Now let $\mathcal{S} = \{|\,0\rangle\,, |\,1\rangle\,, |\,+\rangle\,, |\,-\rangle\}$ where $|\,\pm\rangle = \frac{1}{\sqrt{2}}(|\,0\rangle \pm |\,1\rangle)$. Starting with a suitable state for a 2-qubit composite system AB, show how the no-cloning theorem for the set $\mathcal{S}$ can be seen as a consequence of the no-signalling theorem for AB.

SECTION II

11K Coding and Cryptography

(a) Let n be an odd integer. What does it mean to say that a code is a *cyclic code of length n with a defining set?* Define a *BCH code with design distance δ*. Show that a BCH code with design distance δ has minimum distance at least δ. [Properties of the Vandermonde determinant may be assumed.]

(b) Let $\alpha \in \mathbb{F}_{16}$ be a root of $X^4 + X + 1$. Let C be the BCH code of length 15 and design distance 5, with defining set the first few powers of α.

(i) Find the minimal polynomial for each element of the defining set, and hence find the generator polynomial of C.

(ii) Define the *error locator polynomial* $\sigma(X) \in \mathbb{F}_{16}[X]$ for any received word $r(X)$. [Properties of $\sigma(X)$ may be stated without proof.]

(iii) Suppose you receive the word $r(X) = 1 + X + X^7$. Find the error locator polynomial. Hence, either determine the error position or positions of $r(X)$, or explain why this is not possible.

12I Automata and Formal Languages

Give the definition of a *primitive recursive function* $f : \mathbb{N}^k \to \mathbb{N}$.

Show directly from the definition that, when $k = 2$, the functions

$$P(m,n) = m + n \text{ and } T(m,n) = mn$$

are both primitive recursive.

Show further that for $k \geqslant 2$ the function

$$T_k(n_1, \ldots, n_k) = n_1 \cdots n_k$$

is primitive recursive, as is $E_a : \mathbb{N} \to \mathbb{N}$ given by $E_a(n) = a^n$, where $a \geqslant 1$ is a fixed integer.

Suppose $F : \mathbb{N}^k \to \mathbb{N}^k$, where $F = (f_0, \ldots, f_{k-1})$ with each coordinate function f_i primitive recursive. Describe how F can be encoded as a primitive recursive function $\overline{F} : \mathbb{N} \to \mathbb{N}$.

Let the Fibonacci function $B : \mathbb{N} \to \mathbb{N}$ be defined by $B(0) = 0, B(1) = 1$ and $B(n+2) = B(n+1) + B(n)$ for $n \geqslant 0$. Is B primitive recursive? Justify your answer.

If $f : \mathbb{N} \to \mathbb{N}$ is a primitive recursive function, must there exist some $R > 0$ such that $f(n) \leqslant R^n$ for all $n \geqslant 1$? Justify your answer.

[*You may use without proof that for fixed $j \geqslant 2$ the maxpower function M_j is primitive recursive, where $M_j(n)$ is the exponent of the highest power of j that divides n. If you use any other results from the course, you should prove them.*]

13J Statistical Modelling

The following dataset contains information about some of the passengers on RMS *Titanic* when it sank on 15th April, 1912.

```
> head(titanic)
  Survived Pclass    Sex Age SibSp Parch    Fare Cabin Embarked
1        0      3   male  22     1     0  7.2500  <NA>        S
2        1      1 female  38     1     0 71.2833   C85        C
3        1      3 female  26     0     0  7.9250  <NA>        S
4        1      1 female  35     1     0 53.1000  C123        S
5        0      3   male  35     0     0  8.0500  <NA>        S
6        0      3   male  NA     0     0  8.4583  <NA>        Q
> nrow(titanic)
[1] 889
```

We would like to predict which passengers were more likely to survive (`Survived`, 0 = No, 1 = Yes) using the other covariates, including ticket class (`Pclass`, 1 = 1st, 2 = 2nd, 3 = 3rd), sex (`Sex`), age (`Age`), number of siblings/spouses aboard (`SibSp`), number of parents/children aboard (`Parch`), passenger fare (`Fare`), cabin number (`Cabin`), port of embarkation (`Embarked`, C = Cherbourg, Q = Queenstown, S = Southampton).

(a) Describe what the following chunk of `R` code does.

```
> apply(titanic, 2, function(x) sum(is.na(x)))
Survived   Pclass      Sex      Age    SibSp    Parch     Fare    Cabin 
       0        0        0      177        0        0        0      687 
Embarked 
       0 
> titanic$Cabin <- NULL
> titanic$Age[is.na(titanic$Age)] <- mean(titanic$Age, na.rm = TRUE)
```

(b) Write down the generalised linear model fitted (including the likelihood function maximised) by the code below. Define *Akaike's information criterion* (AIC) and explain, in words, how you can use the backward stepwise algorithm and AIC to select a model.

```
> summary(fit <- glm(Survived ~ ., family = binomial, data = titanic))

Deviance Residuals:
    Min       1Q   Median       3Q      Max  
-2.6445  -0.5907  -0.4227   0.6214   2.4432  
```

[QUESTION CONTINUES ON THE NEXT PAGE]

```
Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept)  5.284055   0.564696   9.357  < 2e-16 ***
Pclass      -1.100033   0.143530  -7.664 1.80e-14 ***
Sexmale     -2.718736   0.200779 -13.541  < 2e-16 ***
Age         -0.039885   0.007855  -5.078 3.82e-07 ***
SibSp       -0.325732   0.109368  -2.978   0.0029 **
Parch       -0.092470   0.118702  -0.779   0.4360
Fare         0.001919   0.002376   0.808   0.4192
EmbarkedQ   -0.035043   0.381920  -0.092   0.9269
EmbarkedS   -0.418564   0.236788  -1.768   0.0771 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 1182.82  on 888  degrees of freedom
Residual deviance:  784.21  on 880  degrees of freedom
AIC: 802.21

Number of Fisher Scoring iterations: 5
```

(c) The model summary above says "Dispersion parameter for binomial family taken to be 1". Do you think that is reasonable based on the model summary? Justify your answer. You might find the following information useful.

```
> qnorm(0.25) # 25th-percentile of the standard normal distribution
[1] -0.6744898
```

(d) Give an estimator of the dispersion parameter in this model when it is not fixed at 1.

14E Further Complex Methods

The polylogarithm function $\mathrm{Li_s}(z)$ is defined for complex values of z ($|z| < 1$) and s (all complex s) by

$$\mathrm{Li_s}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}.$$

(a) Briefly justify why the conditions given on z and s given above are appropriate.

Consider the integral

$$I(z,s) = \frac{\Gamma(1-s)}{2\pi i} \int_{-\infty}^{(0+)} \frac{z t^{s-1}}{e^{-t} - z}\, dt\,, \qquad (1)$$

where the integral is taken along a Hankel contour, as indicated by the limits.

(b) Show that $I(z,s)$ provides an analytic continuation of $\mathrm{Li_s}(z)$ for all $z \notin (1,\infty)$. [*Hint: You may assume where needed the Hankel representation of the Gamma function,* $\Gamma(z) = (2i\sin \pi z)^{-1} \int_{-\infty}^{(0+)} e^t t^{z-1}\, dt$, *and the result* $\Gamma(z)\Gamma(1-z) = \pi \mathrm{cosec}(\pi z)$.]

Include in your answer a sketch of the Hankel contour, with particular attention to the path of the contour relative to any singularities in the integrand when z is close to, but not on the part $(1,\infty)$ of the real axis.

(c) Describe how to evaluate $I(z,s)$ when s is a non-positive integer. Hence give explicit expressions for $\mathrm{Li_s}(z)$ for $s = 0$, $s = -1$ and $s = -2$.

(d) For $s > 0$ show that $I(z,s)$ can be expressed in the form

$$I(z,s) = \int_0^{\infty} K(z,s,t)\, dt,$$

where t is a real variable and $K(z,s,t)$ is to be determined. Comment on the required interpretation of the expression (1) when s is a positive integer.

Without detailed calculation, explain (for $s > 0$) why $I(z,s)$ jumps by the value $2\pi i(\log x)^{s-1}/\Gamma(s)$ when z moves from just below $(1,\infty)$ to just above $(1,\infty)$ at the point x ($x > 1$).

15A Cosmology

The continuity, Euler and Poisson equations governing how a non-relativistic fluid composed of particles with mass m, number density n, pressure P and velocity $\mathbf{v}$ propagate in an expanding universe take the form

$$\frac{\partial \rho}{\partial t} + 3H\rho + \frac{1}{a}\boldsymbol{\nabla}\cdot(\rho\,\mathbf{v}) = 0\,,$$

$$\rho a\left(\frac{\partial}{\partial t} + \frac{\mathbf{v}}{a}\cdot\boldsymbol{\nabla}\right)\mathbf{u} = -c^2\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\Phi\,,$$

$$\nabla^2\Phi = \frac{4\pi G}{c^2}\rho\, a^2\,,$$

where $\rho = mc^2 n$, $\mathbf{u} = \mathbf{v} + a\,H\,\mathbf{x}$, $H = \dot{a}/a$, Φ is the gravitational potential and $a(t)$ is the scale factor.

Consider small perturbations about a homogeneous and isotropic flow,

$$n = \bar{n}(t) + \epsilon\,\delta n\,, \quad \mathbf{v} = \epsilon\,\delta\mathbf{v}\,, \quad P = \bar{P}(t) + \epsilon\,\delta P \quad \text{and} \quad \Phi = \bar{\Phi}(t,\mathbf{x}) + \epsilon\,\delta\Phi\,,$$

with $\epsilon \ll 1$.

(a) Show that, to first order in ϵ, the continuity equation can be written as

$$\dot{\delta} + \frac{1}{a}\boldsymbol{\nabla}\cdot\delta\mathbf{v} = 0\,, \tag{$\dagger$}$$

where $\delta = \delta n/\bar{n}$ is the *density contrast.*

(b) Show that, to first order in ϵ, the Euler equation can be written as

$$m\,\bar{n}\,a\,(\dot{\delta\mathbf{v}} + H\,\delta\mathbf{v}) = -\boldsymbol{\nabla}\delta P - m\,\bar{n}\boldsymbol{\nabla}\delta\Phi\,. \tag{$\dagger\dagger$}$$

(c) Now assume that $\delta P = c_s^2\,m\,\delta n$. Using ($\dagger$), ($\dagger\dagger$) and the perturbed Poisson equation, show that the density contrast δ obeys

$$\ddot{\delta} + 2\,H\,\dot{\delta} - c_s^2\left(\frac{1}{a^2}\nabla^2 + k_{\mathrm{J}}^2\right)\delta = 0 \tag{$\star$}$$

and express k_{J} as a function of $\bar{n}$, m and c_s^2.

(d) Neglecting the bracketed terms in equation ($\star$), solve it to find the form of the growth of matter perturbations in a radiation-dominated universe.

16F Logic and Set Theory

State and prove the *Knaster–Tarski fixed-point theorem.*

A subset S of a poset X is called a *down-set* if whenever $x, y \in X$ satisfy $x \in S$ and $y \leqslant x$ then also $y \in S$. Show that the set P of down-sets of X, ordered by inclusion, is a complete poset.

Now let X and Y be totally ordered sets.

(i) Give an example to show that we may have X isomorphic to a down-set in Y, and Y isomorphic to a down-set in X, and yet X is not isomorphic to Y. [*Hint: Consider suitable subsets of the reals.*]

(ii) Show that if X is isomorphic to a down-set in Y, and Y is isomorphic to the complement of a down-set in X, then X is isomorphic to Y.

17F Graph Theory

(a) Define a *proper k-colouring* of a graph G. Define the *chromatic number* $\chi(G)$ of a graph G. Prove that $\chi(G) \leqslant \Delta(G) + 1$ for all graphs G. Do there exist graphs G for which $\chi(G) = \Delta(G) + 1$ for each $\Delta(G) = 0, 1, 2, \ldots$?

(b) What does it mean for a graph to be *k-connected*? If G is a non-complete 3-connected graph, show that $\chi(G) \leqslant \Delta(G)$.

(c) State *Euler's formula.* If G is a triangle-free planar graph, prove that $\chi(G) \leqslant 4$.

(d) Define the *edge-chromatic number* $\chi'(G)$ of a graph G. State *Hall's theorem.* If G is a 4-regular bipartite graph, determine $\chi'(G)$.

18H Galois Theory

(a) Let K be a field with $\operatorname{char} K \neq 2, 3$. If $f = x^3 + px + q \in K[x]$, define the *discriminant* of f, and compute it in terms of p and q.

Let L be the splitting field of f and let $G = \operatorname{Aut}(L/K)$ be the Galois group. Describe all possibilities for G. Justify your answer. [Do not assume that f is irreducible.]

Compute all subfields of L when $f = x^3 + 3x + 1 \in \mathbb{Q}[x]$. You may specify the subfields in terms of the roots; you do not need to determine the roots explicitly in terms of radicals.

(b) Let L/K be a Galois extension, and suppose $f \in L[x]$. Show that there exists a non-zero polynomial $g \in L[x]$ such that $fg \in K[x]$.

Now suppose only that L/K is a finite separable extension, and that $f \in L[x]$. Show that there exists a non-zero polynomial $g \in L[x]$ such that $fg \in K[x]$.

19H Representation Theory

Let G be a finite group.

State *Maschke's theorem* for complex representations of G. Deduce that every representation of G is isomorphic to a direct sum of irreducible representations.

Define the *character* χ_V of a complex representation V of G. Suppose that G acts on a finite set X. What is the *permutation representation* $\mathbb{C}X$? Describe its character $\chi_{\mathbb{C}X}$.

Show that if $V_1, \ldots, V_r$ are all the irreducible representations of G up to isomorphism then the regular representation decomposes as

$$\mathbb{C}G \cong \bigoplus_{i=1}^{r} (\dim V_i) V_i.$$

If V is a complex representation of G, let $\mathrm{Hom}_G(V, V)$ be the space of G-linear maps from V to V. If

$$V \cong \bigoplus_{i=1}^{r} n_i V_i,$$

what is the dimension of $\mathrm{Hom}_G(V, V)$? What is the dimension when $V = \mathbb{C}G$?

Now suppose V is a complex representation of G with character χ such that $\chi(g) = 0$ for all non-identity elements $g \in G$. Show that V is a direct sum of copies of the regular representation $\mathbb{C}G$.

Deduce that if W is any complex representation of G then

$$W \otimes \mathbb{C}G \cong \bigoplus_{i=1}^{\dim W} \mathbb{C}G.$$

[*You may assume that the irreducible complex characters of a finite group form an orthonormal basis of the space of class functions.*]

20H Number Fields

(a) Let K be a number field of degree n. Show that there are exactly n field embeddings $\sigma_1, \ldots, \sigma_n \colon K \hookrightarrow \mathbb{C}$. [You may assume that $K = \mathbb{Q}(\alpha)$ for some $\alpha \in K$.]

Define the *discriminant* d_K of K. Show that the sign of d_K is $(-1)^s$, where s is the number of pairs of complex conjugate embeddings $(\sigma_i, \bar{\sigma}_i \neq \sigma_i)$. [You may assume that d_K is nonzero.]

(b) If $L = \mathbb{Q}(\theta)$, where $\theta^3 + 2\theta^2 + 1 = 0$, show that $\mathcal{O}_L = \mathbb{Z}[\theta]$.

(c) Let K be as in part (a). Suppose that $\alpha \in K$ and that $|\sigma_j(\alpha)| = 1$ for some j.

(i) Prove that $\left|N_{K/\mathbb{Q}}(\alpha)\right| = 1$.

(ii) Deduce that if $\alpha \in \mathcal{O}_K$, then α is a unit.

(iii) Give an example of a number field K and an element $\alpha \in K \setminus \mathcal{O}_K$ for which $|\sigma_1(\alpha)| = \cdots = |\sigma_n(\alpha)| = 1$.

21I Algebraic Topology

Suppose $f, g : C_* \to C'_*$ are chain maps. Define what it means for f and g to be *chain homotopic*. Show that if f and g are chain homotopic then $f_* = g_*$.

Let $C_* = \widetilde{C}_*(\Delta^n)$ be the reduced chain complex of the n-dimensional simplex. Show that id_{C_*} is chain homotopic to 0_{C_*}. Hence compute $H_*(\Delta^n)$.

Now let $K = \Delta^6_2$ be the 2-skeleton of Δ^6. Compute $H_*(K)$. Let $f : K \to K$ be the simplicial map given by $f(e_i) = e_{\sigma(i)}$, where σ is the permutation given in cycle notation by $(0123)(456)$. Compute the trace of the linear map $f_* : H_2(K;\mathbb{Q}) \to H_2(K;\mathbb{Q})$.

22G Linear Analysis

Let ℓ^∞ denote the space of bounded real sequences and let ℓ^1 denote the space of summable real sequences. Suppose that $\varphi : \ell^\infty \to \mathbb{R}$ is linear and continuous, that φ is non-negative on non-negative sequences, that $\varphi((x_n)_{n\geqslant 1}) = \varphi((x_{n+1})_{n\geqslant 1})$, and that φ maps the constant sequence equal to one to one.

(a) Prove that $\liminf_{n\to\infty} x_n \leqslant \varphi((x_n)_{n\geqslant 1}) \leqslant \limsup_{n\to\infty} x_n$ for all $(x_n)_{n\geqslant 1} \in \ell^\infty$.

(b) Is there $(y_n)_{n\geqslant 1} \in \ell^1$ so that $\varphi((x_n)_{n\geqslant 1}) = \sum_{n\geqslant 1} x_n y_n$ for all $(x_n)_{n\geqslant 1} \in \ell^\infty$?

(c) Give an example of $(x_n)_{n\geqslant 1} \in \ell^\infty$ that does not converge but for which all φ defined as above give the same value.

(d) Let $y \in \mathbb{R}$. Assume $(x_n)_{n\geqslant 1} \in \ell^\infty$ satisfies $\dfrac{x_{n+1} + x_{n+2} + \cdots + x_{n+p}}{p} \to y$ as $p \to \infty$ uniformly in $n \geqslant 1$. Prove that $\varphi((x_n)_{n\geqslant 1}) = y$.

23G Analysis of Functions

In this question, $\mathcal{M}$ is the σ-algebra of Lebesgue measurable sets and λ is Lebesgue measure on $\mathbb{R}^n$.

State *Lebesgue's differentiation theorem* and the *Radon–Nikodym theorem*. For a set $A \in \mathcal{M}$, and a measure μ defined on $\mathcal{M}$, let the *μ-density* of A at $x \in \mathbb{R}^n$ be

$$\rho_{\mu,A}(x) = \lim_{r \searrow 0} \frac{\mu(A \cap B_r(x))}{\mu(B_r(x))},$$

whenever the limit exists, where $B_r(x) = \{y \in \mathbb{R}^n : |x-y| < r\}$ is the open ball of radius r centred at x.

For each $t \in [0,1]$, give an example of a set $B \subset \mathbb{R}^2$ and point $z \in \mathbb{R}^2$ for which $\rho_{\lambda,B}(z)$ exists and is equal to t.

Show that for λ-almost every $x \in \mathbb{R}^n$, $\rho_{\lambda,A}(x)$ exists and takes the value 0 or 1. Show that $\rho_{\lambda,A}$ vanishes λ-almost everywhere if and only if A has Lebesgue measure zero.

Let ν be a measure on $\mathcal{M}$ such that $\nu \ll \lambda$ and $\lambda \ll \nu$. Show that $\rho_{\nu,A}(x)$ exists and takes the value 0 or 1 at λ-almost every $x \in \mathbb{R}^n$.

24F Riemann Surfaces

(a) State the *Uniformisation theorem*, and deduce the Riemann mapping theorem.

(b) Let

$$E = \{x + iy \mid x, y \in \mathbb{R}, \, -\pi < x < \pi\}$$

be an infinite vertical strip in $\mathbb{C}$, and let $U \subseteq \mathbb{C}$ consist of $\mathbb{C}$ with the negative real axis (and zero) removed. A *Mercator projection* is a conformal equivalence $f : U \to E$ such that $\operatorname{Im} f(z) \to -\infty$ as $z \to 0$ and $\operatorname{Im} f(z) \to +\infty$ as $z \to \infty$. Exhibit an explicit Mercator projection.

(c) Consider a conformal equivalence $\phi : E \to E$ such that $\operatorname{Im} \phi(z) \to +\infty$ as $\operatorname{Im} z \to +\infty$ and $\operatorname{Im} \phi(z) \to -\infty$ as $\operatorname{Im} z \to -\infty$. Prove that ϕ is translation by an imaginary number, stating carefully any results that you use.

(d) Characterise all Mercator projections.

25H Algebraic Geometry

Define the *local ring* at a point p of an irreducible algebraic variety V. Define the *Zariski tangent space* to V at p.

Let $V \subset \mathbb{A}^2 \times \mathbb{P}^1$ be defined by the equation

$$XZ - WY = 0\,,$$

where X and Y are the coordinates on $\mathbb{A}^2$ and W and Z are the homogeneous coordinates on $\mathbb{P}^1$. Determine whether V is smooth.

Consider the projection morphism

$$\pi : V \to \mathbb{A}^2$$

obtained by restricting the projection from $\mathbb{A}^2 \times \mathbb{P}^1$ onto the first factor. Prove that π is birational but not an isomorphism. Use this to calculate the function field of V.

Let V' be an affine variety and $\varphi : V \to V'$ a morphism. Prove that φ is not injective. Deduce that V is not affine.

Assume the ground field is $\mathbb{C}$. Prove that if V is equipped with the Euclidean topology, then it is not homeomorphic to any projective variety.

26I Differential Geometry

Let $S \subset \mathbb{R}^3$ be an oriented surface. Define its *Gauss map* N. For each $p \in S$, show that the derivative of N defines a self-adjoint operator on T_pS, and define the *principal curvatures* of S at a point p. What does it mean for p to be an *umbilical point*? What does it mean for S to be a *minimal surface*?

(a) We say that a smooth map $f : S \to R$ between two surfaces in $\mathbb{R}^3$ is *conformal* if

$$\langle Df_p(u), Df_p(v)\rangle = \lambda(p)\langle u, v\rangle$$

for all $p \in S$ and $u, v \in T_pS$, where $\lambda(p) > 0$.

Show that, if S does not have any umbilical points, then S is a minimal surface if and only if its Gauss map is conformal.

(b) Now drop the assumption about umbilical points. If S is a minimal surface, must its Gauss map be conformal? If the Gauss map is conformal, must S be a minimal surface? Justify your answers.

(c) Suppose S is a connected minimal surface. Can the image of its Gauss map be a great circle in S^2?

27G Probability and Measure

(a) State and prove *Kolmogorov's zero-one law.*

(b) Consider the product space $E = \mathbb{R}^{\mathbb{N}}$ equipped with the σ-algebra $\sigma(\mathcal{C})$ generated by the cylinder sets

$$\mathcal{C} = \left\{ A = \times_{n=1}^{\infty} A_n \mid A_n \subseteq \mathbb{R}, A_n \text{ Borel for } n \leqslant N, A_n = \mathbb{R} \text{ for } n > N, \text{ some } N \in \mathbb{N} \right\}.$$

For m a probability measure on $\mathbb{R}$, show that there exists a unique product measure μ on $(E, \sigma(\mathcal{C}))$ for which $\mu(A) = \prod_{n=1}^{\infty} m(A_n)$ for all $A \in \mathcal{C}$. Show further that the shift map θ defined on E by $\theta((x_1, x_2, \dots)) = (x_2, x_3, \dots)$ is measure-preserving and ergodic for μ.

[*You may use without proof the existence of an infinite sequence of i.i.d. real random variables defined on any probability space.*]

28J Applied Probability

(a) Define what it means for a matrix Q to be a *Q-matrix* on a finite or countably infinite state space S.

Suppose S is a finite state space. Express the generator Q of a continuous-time Markov chain $X = (X_t)$ on S in terms of its transition semigroup $(P(t))_{t \geqslant 0}$, and conversely express the semigroup in terms of the generator. You do not need to prove the expressions you give.

Write down the forward and backward *Kolmogorov equations* for a chain X as above.

(b) Let $X = (X_t)$ be a continuous-time Markov chain on the state space $S = \{1, 2\}$, with generator

$$Q = \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix},$$

where $\lambda\mu > 0$.

(i) Compute the transition probabilities $p_{ij}(t)$, $i, j \in S$, $t > 0$.

(ii) Find Q^n for $n \geqslant 1$, and compute $\sum_{n=0}^{\infty} \frac{t^n}{n!} Q^n$ for $t > 0$. Compare the result with your answer in part (i).

(iii) Solve the equation $\pi Q = 0$ for a probability distribution π and identify the invariant distribution of X. Use your result in part (*i*) to verify that, indeed, the semigroup converges to the invariant distribution as $t \to \infty$.

(iv) Compute the probability $\mathbb{P}(X(t) = 2 | X(0) = 1, X(3t) = 1)$.

29K Principles of Statistics

(a) Suppose that Θ is an open subset of $\mathbb{R}^p$, that $\Phi : \Theta \to \mathbb{R}$ is continuously differentiable at some $\theta_0 \in \Theta$, and that $\{\widehat{\theta}_n\}_{n \geqslant 1}$ is a sequence of random vectors in $\mathbb{R}^p$ satisfying $\sqrt{n}(\widehat{\theta}_n - \theta_0) \xrightarrow{d} Z$, where $Z \in \mathbb{R}^p$. Prove that

$$\sqrt{n}(\Phi(\widehat{\theta}_n) - \Phi(\theta_0)) \xrightarrow{d} \nabla_\theta \Phi(\theta_0)^T Z\,.$$

For the remainder of this problem, consider the $N(0, \sigma^2)$ model, where $\sigma \in (0, \infty)$.

(b) Derive the maximum likelihood estimator $\widehat{\sigma}_{\text{MLE}}$ of σ based on an i.i.d. sample of size n from the model. What is the asymptotic distribution of $\sqrt{n}(\widehat{\sigma}_{\text{MLE}} - \sigma)$? [*Hint: You may use, without proof, the fact that* $\mathbb{E}[Z^4] = 3$ *when* $Z \sim N(0,1)$.]

(c) What is the Fisher information $I(\sigma)$ (for the sample size $n = 1$)?

(d) Now consider the alternative parametrization of the model in terms of $\rho = \sigma^2$, where $\rho \in (0, \infty)$. What is the maximum likelihood estimator $\widehat{\rho}_{\text{MLE}}$ of ρ?

30K Stochastic Financial Models

Fix a positive integer N and consider the problem of minimising

$$\mathbb{E}\left(X_N^2 + \sum_{n=1}^{N} u_n^2\right),$$

where X_0 is given and

$$X_n = X_{n-1} + u_n + \xi_n$$

for $1 \leqslant n \leqslant N$. Here $(\xi_n)_{1 \leqslant n \leqslant N}$ is an IID sequence of random variables with $\mathbb{E}(\xi_1) = 0$ and $\text{Var}(\xi_1) = \sigma^2$, and the controls $(u_n)_{1 \leqslant n \leqslant N}$ are previsible with respect to the filtration generated by $(\xi_n)_{1 \leqslant n \leqslant N}$.

(a) Write down the Bellman equation for this problem.

(b) Show that the value function can be expressed as

$$V(n, x) = A_n + B_n x + C_n x^2$$

for constants $(A_n, B_n, C_n)_{0 \leqslant n \leqslant N}$ to be found.

(c) Show that the optimal control is

$$u_n^* = -\frac{X_0}{N+1} - \frac{\xi_1}{N} - \frac{\xi_2}{N-1} - \cdots - \frac{\xi_{n-1}}{N-n+2}$$

for $1 \leqslant n \leqslant N$.

31J Mathematics of Machine Learning

(a) Let $\mathcal{F}$ be a family of functions $f : \mathcal{X} \to \{0, 1\}$ with $|\mathcal{F}| \geqslant 2$.

Define the *shattering coefficient* $s(\mathcal{F}, n)$ and the *VC dimension* $\mathrm{VC}(\mathcal{F})$ of $\mathcal{F}$.

State the *Sauer–Shelah lemma*.

(b) (i) Let

$$\mathcal{A}_1 = \left\{ \bigcup_{k=1}^{m} [a_k, b_k] : a_k, b_k \in \mathbb{R} \text{ for } k = 1, \ldots, m \right\}.$$

Show that $\mathcal{F}_1 := \{\mathbf{1}_A : A \in \mathcal{A}_1\}$ satisfies $\mathrm{VC}(\mathcal{F}_1) = 2m$.

(ii) Let $\mathcal{F}_2$ be a class of functions from $\mathbb{R}^p$ to $\{0, 1\}$ given by

$$\mathcal{F}_2 := \{x \mapsto \mathbf{1}_{(0,\infty)}(\mu + x^T\beta) : \beta \in \mathbb{R}^p,\ \mu \in \mathbb{R}\}.$$

Stating any result from the course you need, give an upper bound on $\mathrm{VC}(\mathcal{F}_2)$.

(c) (i) Let $\mathcal{G}$ be a family of functions $g : \mathcal{Z} \to \{0, 1\}$ with $|\mathcal{G}| \geqslant 2$ and define $\mathcal{H}$ to be the set of functions $h : \mathcal{X} \times \mathcal{Z} \to \{0, 1\}$ for which $h(x, z) = f(x)g(z)$ for some $f \in \mathcal{F}$ and $g \in \mathcal{G}$. Show that $s(\mathcal{H}, n) \leqslant s(\mathcal{F}, n)s(\mathcal{G}, n)$.

(ii) Now let $\mathcal{G}$ be a family of functions $g : \mathcal{X} \to \{0, 1\}$ with $|\mathcal{G}| \geqslant 2$ and define $\mathcal{H}$ to be the set of functions $h : \mathcal{X} \to \{0, 1\}$ for which $h(x) = f(x)g(x)$ for some $f \in \mathcal{F}$ and $g \in \mathcal{G}$. Show that $s(\mathcal{H}, n) \leqslant s(\mathcal{F}, n)s(\mathcal{G}, n)$.

(d) (i) Let

$$\mathcal{A}_3 = \left\{ \prod_{j=1}^{p} \Big(\bigcup_{k=1}^{m} [a_{jk}, b_{jk}] \Big) : a_{jk}, b_{jk} \in \mathbb{R} \text{ for } j = 1, \ldots, p,\ k = 1, \ldots, m \right\}.$$

Show that $\mathcal{F}_3 := \{\mathbf{1}_A : A \in \mathcal{A}_3\}$ satisfies $s(\mathcal{F}_3, n) \leqslant (n + 1)^{2mp}$.

(ii) For $m \geqslant 3$, let $\mathcal{A}_4$ be the set of all convex polygons in $\mathbb{R}^2$ with m sides, and set $\mathcal{F}_4 := \{\mathbf{1}_A : A \in \mathcal{A}_4\}$. Show that $s(\mathcal{F}_4, n) \leqslant (n + 1)^{3m}$.

32B Dynamical Systems

(a) Consider a dynamical system of the form

$$\begin{aligned} \dot{x} &= f(x,y)\,, \\ \dot{y} &= g(x,y) + \epsilon p(x,y)\,, \end{aligned}$$

which is Hamiltonian for $\epsilon = 0$. Explain the *energy balance method.* What does it tell us about periodic orbits of this system for small ϵ?

(b) (i) For $0 < \epsilon \ll 1$, use the energy balance method to seek leading-order approximations to periodic orbits of this system

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= -4x + \epsilon\left[(1-2x^2)ky - (1-3x^2)y^3\right]\,, \end{aligned}$$

where $k > 0$.

[*Hint:* $\int_0^{2\pi} \sin^4\theta d\theta = \frac{3}{4}\pi$ *and* $\int_0^{2\pi} \sin^6\theta d\theta = \frac{5}{8}\pi$.]

(ii) For the cases $0 < k < 6$ and for $k > 6$, deduce the stability of any periodic orbits.

(iii) What can we deduce from this approach about the existence of periodic orbits near $k = 6$?

33E Integrable Systems

(a) Show that if L is a symmetric $n \times n$ matrix ($L = L^T$) and B is a skew-symmetric $n \times n$ matrix ($B = -B^T$) then $[B, L] = BL - LB$ is symmetric. If L evolves in time according to

$$\frac{dL}{dt} = [B, L],$$

show that the eigenvalues of L are constant in time.

Write the harmonic oscillator equation $\ddot{q} + \omega^2 q = 0$ in Hamiltonian form. (The frequency ω is a fixed real number). Starting with the symmetric matrix

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}$$

find a Lax pair formulation for the harmonic oscillator and use this formulation to obtain the conservation of energy for the oscillator.

(b) Consider the Airy partial differential equation, given for $-\infty < x < \infty$ and $t \geqslant 0$ by

$$q_t + q_{xxx} = 0. \tag{1}$$

Show that this is a compatibility condition for the pair of linear equations

$$\psi_x - ik\psi = q \tag{2}$$

$$\psi_t - ik^3\psi = -q_{xx} - ikq_x + k^2 q \tag{3}$$

for a function $\psi = \psi(x, t, k) \in \mathbb{C}$. Show that for each t, equation (2) has a solution ψ_+ which is defined for $\operatorname{Im} k \geqslant 0$, analytic in k for $\operatorname{Im} k > 0$, and satisfies

$$\lim_{x \to +\infty} e^{-ikx}\psi_+(x, t, k) = \hat{q}(k, t) = \int_{-\infty}^{+\infty} e^{-ikx} q(x, t)\, dx\,.$$

Deduce from this and equation (3) that $\hat{q}(k, t)$ evolves in time according to

$$\hat{q}_t - ik^3\hat{q} = 0$$

and hence obtain a representation for the solution of the Airy equation (1).

[*You may assume that q is a smooth function whose derivatives are rapidly decreasing in x.*]

34A Principles of Quantum Mechanics

Let A and $A^\dagger$ respectively be the lowering and raising operator for a one-dimensional quantum harmonic oscillator, with $[A, A^\dagger] = 1$. Also let $|n\rangle$ be the n^{th} excited state of the oscillator, obeying $N|n\rangle = n|n\rangle$ where $N = A^\dagger A$ is the number operator.

(a) Show that $A|n\rangle \propto |n-1\rangle$ and find the constant of proportionality.

(b) For any $z \in \mathbb{C}$, define the coherent state $|z\rangle$ by

$$|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle \ .$$

Show that $\langle z|z\rangle = 1$ and that $A|z\rangle = z|z\rangle$.

(c) Calculate the expectation value $\langle N\rangle$ and uncertainty ΔN of the number operator in the state $|z\rangle$. Show that the relative uncertainty $\Delta N/\langle N\rangle \to 0$ as $\langle N\rangle \to \infty$.

(d) A harmonic oscillator is prepared to be in state $|z\rangle$ at time $t = 0$. Using the properties of the Hamiltonian of the one-dimensional harmonic oscillator, show that the state evolved to time $t > 0$ is still an eigenstate of A and find its eigenvalue. Calculate the probability that the oscillator is found to be in the original state $|z\rangle$ at time t, and show that this probability is 1 whenever $t = kT$, where $k \in \mathbb{N}$ and T is the classical period of the oscillator.

35D Applications of Quantum Mechanics

A particle of mass m and energy $E = \hbar^2 k^2/2m$, moving in one dimension, is incident on a localised potential barrier.

(a) Define *reflection* and *transmission coefficients*, r and t, for a right-moving particle incident from $x = -\infty$. Define corresponding coefficients r' and t' for a left-moving particle incident from $x = +\infty$. Prove that the S-matrix

$$\mathcal{S} \;=\; \begin{pmatrix} t' & r \\ r' & t \end{pmatrix}$$

is unitary. [You may use without proof the conservation of the probability current.]

(b) Explain what is meant by the *parity* of a wavefunction. Under what circumstances do energy eigenstates of the system described above have definite parity?

(c) Consider the potential barrier

$$V(x) = \begin{cases} V_0 & \text{for } |x| < a/2 \\ 0 & \text{for } |x| > a/2, \end{cases}$$

where $V_0 > 0$. Find an even parity wavefunction satisfying the Schrödinger equation for a particle of energy $E = \hbar^2 k^2/2m$ with $E < V_0$. Hence compute $r + t$.

36A Statistical Physics

(a) What systems are described by a *grand canonical ensemble*? If there are N_n particles in microstate n each with energy E_n, write down an expression for the *grand canonical partition function* $\mathcal{Z}$ in terms of the temperature T, the chemical potential μ and the Boltzmann constant k_B.

(b) Define the *grand canonical potential* Φ in terms of the average energy E, T, the entropy S, μ, and the average number of particles $\langle N \rangle$. Write down the relation between Φ and $\mathcal{Z}$.

(c) Using scaling arguments, express $\Phi(T, V, \mu)$ in terms of the pressure p and the volume V.

(d) Consider the grand canonical ensemble for a classical ideal gas of non-relativistic particles of mass m in a fixed 3-dimensional volume V.

(i) Compute $\mathcal{Z}$ and Φ.

(ii) Calculate $\langle N \rangle$ and $\Delta N / \langle N \rangle$, where $(\Delta N)^2 = \langle N^2 \rangle - \langle N \rangle^2$. Comment on the latter result.

(iii) Derive the equation of state for the gas.

[*You may assume that* $\int_{-\infty}^{\infty} e^{-a\,x^2} dx = \sqrt{\pi/a}$ *for* $a > 0$.]

(e) Using the grand canonical ensemble and your results from part (d), derive the equation of state for a classical ideal gas of relativistic particles with energies $\sqrt{|\mathbf{p}|^2\, c^2 + m^2 c^4}$. Compute $\Delta N / \langle N \rangle$.

37B Electrodynamics

Consider a localised electromagnetic field in vacuum with electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ respectively in the absence of charges and currents.

(a) Show that the energy density $\epsilon = \frac{\varepsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2$ obeys a local conservation law

$$\partial_t \epsilon + \boldsymbol{\nabla} \cdot \mathbf{N} = 0\,.$$

Hence obtain an expression for the vector $\mathbf{N}$ and remark on its physical significance. Here ε_0 and μ_0 are the electric and magnetic permeabilities of the vacuum.

(b) Show that the momentum density $\mathbf{g} = \varepsilon_0 \mathbf{E} \times \mathbf{B}$ obeys a local conservation law

$$\partial_t g_j + \nabla_i \sigma_{ij} = 0\,.$$

Hence obtain an expression for the second-rank tensor σ_{ij} and remark on its physical significance.

(c) Defining the tensor

$$T^{\mu\nu} = \begin{bmatrix} \epsilon & c g_j \\ N_i/c & \sigma_{ij} \end{bmatrix}$$

show that the results of (a) and (b) can be expressed as $\partial_\mu T^{\mu\nu} = 0$.

(d) Using the fact that the tensor σ_{ij} is symmetric, show that the integral over all space of the angular momentum density $\mathbf{L} = \mathbf{x} \times \mathbf{g}$ is independent of time. Here $\mathbf{x}$ is the position with respect to the origin of an inertial frame.

(e) Show that the symmetry of σ_{ij} in all inertial frames requires $\mu_0 \epsilon_0 = 1/c^2$.

38D General Relativity

A *Milne universe* is an isotropic, homogeneous model of cosmology which has negative spatial curvature, $k=-1$, and an expanding scale factor, $\dot{a}(t)>0$, even though there is no matter or radiation ($T_{\alpha\beta}=0$) and no cosmological constant ($\Lambda=0$).

(a) Write down the FLRW metric for this cosmological model. Calculate the scale factor $a(t)$ as an explicit function of the proper time t of a stationary observer.

(b) Verify that the singularity as $a\to 0$ is a coordinate singularity by calculating the Kretschmann scalar. [*Hint: You may find it useful to relate the Riemann tensor to the Ricci tensor.*]

(c) By constructing an appropriate coordinate transformation, show that the Milne universe is equivalent to the interior of the future light-cone of a point p in Minkowski space-time. What do the spatial isometries of the hyperbolic $t=$ const. slices correspond to in this Minkowski space-time?

[*Hint: You may wish to use the following formulae:*

$$3\frac{\dot{a}+k}{a^2}-\Lambda=8\pi\rho\,, \qquad \text{(Friedmann I)}$$

$$2a\ddot{a}+\dot{a}^2+ka^2-\Lambda=-8\pi P\,. \qquad \text{(Friedmann II)}$$

Riemann tensor in normal coordinates:

$$R_{\alpha\beta\mu\nu}=\frac{1}{2}(\partial_\beta\partial_\mu g_{\alpha\nu}+\partial_\alpha\partial_\nu g_{\beta\mu}-\partial_\alpha\partial_\mu g_{\beta\nu}-\partial_\beta\partial_\nu g_{\alpha\mu}).\,]$$

39C Fluid Dynamics II

A viscous fluid of viscosity μ and density ρ is located in the annulus confined between two long co-axial cylinders of radii R and αR with $\alpha < 1$. The ends of the annular space are open to the atmosphere. The axes of the cylinders are aligned in the vertical direction. We use cylindrical coordinates (r, θ, z) with unit vector $\mathbf{e}_z$ in the downward vertical direction. There is a gravitational force g per unit mass acting on the fluid in the downward direction. In the following you may consider the flow in the long central region of the annulus, far from the ends, and neglect any details of the flow near the ends.

The outer cylinder is fixed and stationary. The inner cylinder steadily translates along its axis with velocity $V\mathbf{e}_z$. The fluid flow between the two cylinders may be assumed to be steady and unidirectional.

(a) Explain why we expect the velocity $\mathbf{u}$ to be of the form $\mathbf{u} = u(r)\mathbf{e}_z$.

(b) Derive the equation satisfied by $u(r)$ and state the corresponding boundary conditions.

(c) Show that the pressure gradient in the z-direction is constant and compute its value.

(d) Solve for the flow $u(r)$ in the annular gap and sketch it for $V = 0$, and for two further values of V, one positive and one negative.

(e) Calculate the force per unit length acting on the inner cylinder and the corresponding force per unit length acting on the outer cylinder. Comment on the sum of these forces.

[*Hint: in cylindrical coordinates* (r, θ, z) *with velocity components* (u_r, u_θ, u_z) *we have*

$$\nabla^2 u_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}.$$

The rz*-component of the rate-of-strain tensor is* $e_{rz} = \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right)$.]

40C Waves

(a) Starting from the equations for mass and momentum conservation and a suitable equation of state, derive the linearised wave equation for perturbation pressure $\tilde{p}(\mathbf{x}, t)$ for 3-dimensional sound waves in a compressible gas with sound speed c_0 and density ρ_0.

(b) For a 1-dimensional wave of given frequency ω propagating in the x-direction, the perturbation pressure $\tilde{p}(x, t)$ may be written in the form $\Re(\hat{p}(x)e^{i\omega t})$. What is the form of $\hat{p}$ for a harmonic plane wave of frequency ω propagating in the positive x-direction? Express the perturbation fluid speed $\tilde{u}(x, t)$ in terms of $\tilde{p}(x, t)$.

(c) The gas occupies the region $x < L$, with a rigid boundary at $x = L$. A thin flexible membrane of mass m per unit area is located within the gas at equilibrium position $x = 0$. A plane wave of unit amplitude of the form specified in part (b) is incident from $x = -\infty$. The combined effects of the membrane and the rigid boundary result in a reflected wave of complex amplitude R, where R is the ratio between the individual complex amplitudes at $x = 0^-$ of the reflected and incident waves.

(i) Show that

$$R = \frac{\cos\beta + (\alpha - i)\sin\beta}{\cos\beta + (\alpha + i)\sin\beta} \quad \text{where } \alpha = \frac{\omega m}{\rho_0 c_0} \quad \text{and} \quad \beta = \frac{\omega L}{c_0}\,.$$

Deduce that $|R| = 1$ in general and briefly discuss this result physically.

(ii) Identify a condition on β so that the membrane is stationary and there is non-trivial pressure perturbation in $0 < x < L$. Briefly discuss this result physically.

(iii) Identify and interpret a limit for α in which the pressure perturbation in $0 < x < L$ becomes very small relative to that in $x < 0$.

41C Numerical Analysis

(a) Let $H \in \mathbb{R}^{n \times n}$ be diagonalisable. Show that the sequence defined by $\mathbf{z}^{(k+1)} = H\mathbf{z}^{(k)}$ converges to 0 for all initial vectors $\mathbf{z}^{(0)} \in \mathbb{C}^n$ if, and only if, $\rho(H) < 1$ where $\rho(H)$ is the spectral radius of H.

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix, and let $\mathbf{b} \in \mathbb{R}^n$.

(b) Prove that the solution to $A\mathbf{x} = \mathbf{b}$ is the unique minimiser of the function $f(\mathbf{x}) = (1/2)\mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x}$.

(c) The *steepest descent method* with constant step size α is defined by

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \alpha\nabla f(\mathbf{x}^{(k)}).$$

Applying the method to the function f given in (b), write down the iterations explicitly in terms of A and $\mathbf{b}$. Under what conditions on α does the sequence $\mathbf{x}^{(k)}$ converge to $A^{-1}\mathbf{b}$?

(d) Consider the steepest descent method with *exact line search*, where at each iteration k, the constant $\alpha = \alpha^{(k)}$ is chosen so that $f(\mathbf{x}^{(k+1)})$ is as small as possible. Give an explicit expression for the step size $\alpha^{(k)}$. Show that, in this case, the residuals $\mathbf{r}^{(k)} = \mathbf{b} - A\mathbf{x}^{(k)}$ satisfy $(\mathbf{r}^{(k)})^T\mathbf{r}^{(k+1)} = 0$ for all k.

END OF PAPER