UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday, 10 June, 2022 1:30pm to 4:30pm

PAPER 4

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most four*** *questions from Section I and* ***at most six*** *questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet ***for each question*** *that you have attempted, and place it at the front of your answer to that question.*

Complete a green master cover sheet listing ***all the questions*** *that you have attempted.*

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into ***a single bundle****, with the master cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.*

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Linear Algebra

What is a *Hermitian form* on a complex vector space V? If φ and ψ are two Hermitian forms and $\varphi(v,v)=\psi(v,v)$ for all $v\in V$, prove that $\varphi(v,w)=\psi(v,w)$ for all $v,w\in V$.

Determine whether the Hermitian form on $\mathbb{C}^2$ defined by the matrix

$$A=\begin{pmatrix}4 & 2i\\ -2i & 3\end{pmatrix}$$

is positive definite.

2G Analysis and Topology

Define the *closure* of a subspace Z of a topological space X, and what it means for Z to be *dense*. What does it mean for a topological space Y to be *Hausdorff*?

Assume that Y is Hausdorff, and that Z is a dense subspace of X. Show that if two continuous maps $f,g:X\to Y$ agree on Z, they must agree on the whole of X. Does this remain true if you drop the assumption that Y is Hausdorff?

3G Complex Analysis

Show that there is no bijective holomorphic map $f : D(0,1)\setminus\{0\}\to A$, where $D(0,1)$ is the disc $\{z\in\mathbb{C}: |z|<1\}$ and A is the annulus $\{z\in\mathbb{C}:\ 1<|z|<2\}$.

[*Hint: Consider an extension of f to the whole disc.*]

4B Quantum Mechanics

The radial wavefunction $g(r)$ for the hydrogen atom satisfies the equation

$$-\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left(r^2\frac{d}{dr}g(r)\right)-\frac{e^2}{4\pi\epsilon_0 r}g(r)+\frac{l(l+1)\hbar^2}{2mr^2}g(r)=Eg(r)\,. \qquad (\dagger)$$

(a) Explain the origin of each of the terms in ($\dagger$). What are the allowed values of l?

(b) For a given l, the lowest energy bound state solution of ($\dagger$) takes the form $r^a e^{-br}$. Find a, b, and the corresponding value of E, in terms of l.

(c) A hydrogen atom makes a transition between two such states, corresponding to $l+1$ and l. What is the frequency of the photon emitted?

5D Electromagnetism

(a) Use the Maxwell equations to show that, in the absence of electric charges and currents, the magnetic field obeys

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} = c^2 \nabla^2 \mathbf{B}$$

for some appropriate speed c that you should express in terms of ϵ_0 and μ_0.

(b) Show that

$$\mathbf{B} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \cos(kz - \omega t)$$

satisfies the Maxwell equations given appropriate conditions on the constants B_1, B_2, B_3, ω and k that you should find. What is the corresponding electric field $\mathbf{E}$?

(c) Compute and interpret the Poynting vector $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$.

6C Numerical Analysis

(a) Suppose that $w(x) > 0$ for all $x \in [a, b]$. The weights $b_1, \ldots, b_n$ and nodes $c_1, \ldots, c_n$ are chosen so that the Gaussian quadrature formula for a function $f \in C[a, b]$

$$\int_a^b w(x) f(x) dx \approx \sum_{k=1}^{n} b_k f(c_k)$$

is exact for every polynomial of degree $2n - 1$. Show that the b_i, $i = 1, \ldots, n$ are all positive.

(b) Evaluate the coefficients b_k and c_k of the Gaussian quadrature of the integral

$$\int_{-1}^{1} x^2 f(x) dx,$$

which uses two evaluations of the function $f(x)$ and is exact for all f that are polynomials of degree 3.

7H Markov Chains

Let X be an irreducible Markov chain with transition matrix P and values in the set S. For $i \in S$, let $T_i = \min\{n \geqslant 1 : X_n = i\}$ and $V_i = \sum_{n=0}^{\infty} \mathbf{1}(X_n = i)$.

(a) Suppose $X_0 = i$. Show that V_i has a geometric distribution.

(b) Suppose X is transient. Prove that for all $i, j \in S$, we have

$$P^n(i, j) \to 0 \quad \text{as } n \to \infty.$$

SECTION II

8F Linear Algebra

Let V and W be finite dimensional vector spaces, and α a linear map from V to W. Define the *rank* $r(\alpha)$ and *nullity* $n(\alpha)$ of α. State and prove the rank-nullity theorem.

Assume now that α and β are linear maps from V to itself, and let $n = \dim V$. Prove the following inequalities for the linear maps $\alpha + \beta$ and $\alpha\beta$:

$$|r(\alpha) - r(\beta)| \leqslant r(\alpha + \beta) \leqslant \min\{r(\alpha) + r(\beta), n\}$$

and

$$\max\{r(\alpha) + r(\beta) - n, 0\} \leqslant r(\alpha\beta) \leqslant \min\{r(\alpha), r(\beta)\}.$$

For arbitrary values of n and $0 \leqslant r(\alpha), r(\beta) \leqslant n$, show that each of the four bounds can be attained for some (α, β). Can both upper bounds always be attained simultaneously?

9E Groups, Rings and Modules

(a) Let R be a unique factorisation domain with field of fractions F. What does it mean for a polynomial $f \in R[X]$ to be *primitive*? Prove that the product of two primitive polynomials is primitive. Let $f, g \in R[X]$ be polynomials of positive degree. Show that if f and g are coprime in $R[X]$ then they are coprime in $F[X]$.

(b) Let $I \subset \mathbb{C}[X, Y]$ be an ideal generated by non-zero coprime polynomials f and g. By running Euclid's algorithm in a suitable ring, or otherwise, show that $I \cap \mathbb{C}[X] \neq \{0\}$ and $I \cap \mathbb{C}[Y] \neq \{0\}$. Deduce that $\mathbb{C}[X, Y]/I$ is a finite dimensional $\mathbb{C}$-vector space.

10G Analysis and Topology

Define what it means for a topological space to be *connected*. Describe without proof the connected subspaces of $\mathbb{R}$ with the standard topology. Define what it means for a topological space to be *path connected*, and show that path connectedness implies connectedness.

Given metric spaces A and B, let $C(A,B)$ be the space of continuous bounded functions from A to B with the topology induced by the uniform metric.

(a) For $n \in \mathbb{N}$, let $I_n \subset \mathbb{R}$ be

$$I_n = [1,2] \cup [3,4] \cup \ldots \cup [2n-1, 2n]$$

with the subspace topology. For fixed $m, n \in \mathbb{N}$, how many connected components does $C(I_n, I_m)$ have?

(b) (i) Give an example of a closed bounded subspace of $\mathbb{R}^2$ which is connected but not path connected, justifying your answer. Call your example S.

(ii) Show that $C([0,1], S)$ is not path connected.

(iii) Is $C([0,1], S)$ connected? Briefly justify your answer.

11E Geometry

(a) Write down the metric on the unit disc model $\mathbb{D}$ of the hyperbolic plane. Let C be the Euclidean circle centred at the origin with Euclidean radius r. Show that C is a hyperbolic circle and compute its hyperbolic radius.

(b) Let Δ be a hyperbolic triangle with angles α, β, γ, and side lengths (opposite the corresponding angles) a, b, c. State the hyperbolic sine formula. The hyperbolic cosine formula is $\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos \alpha$. Show that if $\gamma = \pi/2$ then

$$\tan \alpha = \frac{\sinh a}{\cosh a \sinh b} \qquad \text{and} \qquad \tan \alpha \tan \beta \cosh c = 1.$$

(c) Write down the Gauss–Bonnet formula for a hyperbolic triangle. Show that the hyperbolic polygon in $\mathbb{D}$ with vertices at $re^{2\pi ik/n}$ for $k = 0, 1, 2, \ldots, n-1$ has hyperbolic area

$$A_n(r) = 2n\left[\cot^{-1}\left(\frac{1-r^2}{1+r^2}\cot\left(\frac{\pi}{n}\right)\right) - \frac{\pi}{n}\right].$$

(d) Show that there exists a hyperbolic hexagon with all interior angles a right angle. Draw pictures illustrating how such hexagons may be used to construct a closed hyperbolic surface of any genus at least 2.

12A Complex Methods

The Laplace transform $F(s)$ of a function $f(t)$ is defined as

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

(a) For $f(t) = t^n$ for n a non-negative integer, show that

$$\begin{aligned} L\{f(t)\} &= F(s) = \frac{n!}{s^{n+1}}\,, \\ L\{e^{at} f(t)\} &= F(s-a) = \frac{n!}{(s-a)^{n+1}}\,. \end{aligned}$$

(b) Use contour integration to find the inverse Laplace transform of

$$F(s) = \frac{1}{s^2(s+2)^2}.$$

(c) Verify the result in part (b) by using the results in part (a) and the convolution theorem.

(d) Use Laplace transforms to solve the differential equation

$$\frac{d^4}{dt^4}[f(t)] + 4\frac{d^3}{dt^3}[f(t)] + 4\frac{d^2}{dt^2}[f(t)] = 0,$$

subject to the initial conditions

$$f(0) = \frac{d}{dt}f(0) = \frac{d^2}{dt^2}f(0) = 0 \text{ and } \frac{d^3}{dt^3}f(0) = 1.$$

13D Variational Principles

(a) Derive the Euler-Lagrange equation for the functional

$$\int_a^b f(y, y', y''; x)\, dx \ ,$$

where prime denotes differentiation with respect to x, and both y and y' are specified at $x = a, b$.

(b) If f does not depend explicitly on x show that, when evaluated on the extremum,

$$f - \left[\frac{\partial f}{\partial y'} - \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right)\right] y' - \frac{\partial f}{\partial y''} y'' = \text{constant} \ .$$

(c) Find $y(x)$ that extremises the integral

$$\int_0^{\pi/2} \left(-\frac{1}{2}y''^2 + y'^2 - \frac{1}{2}y^2\right) dx$$

subject to $y(0) = y'(0) = 0$ and $y(\pi/2) = \pi/2$ and $y'(\pi/2) = 1$.

14B Methods

(a) Let $h(x) = m'(x)$. Express the Fourier transform $\tilde{h}(k)$ of $h(x)$ in terms of the Fourier transform $\tilde{m}(k)$ of $m(x)$, given that $m \to 0$ as $|x| \to \infty$. [You need to show an explicit calculation.]

(b) Calculate the inverse Fourier transform of

$$\tilde{m}(k) = -i\pi \operatorname{sgn}(k) e^{-\alpha|k|} ,$$

with $\operatorname{Re}\alpha > 0$.

(c) The function $u(x,y)$ obeys Laplace's equation $\nabla^2 u = 0$ in the region defined by $-\infty < x < \infty$ and $0 < y < a$, with real positive a, where $u(x,0) = f(x)$, $u(x,a) = g(x)$ and $u \to 0$ as $|x| \to \infty$.

(i) By performing a suitable Fourier transform of Laplace's equation, determine the ordinary differential equation satisfied by $\tilde{u}(k,y)$. Hence express $\tilde{u}(k,y)$ in terms of the Fourier transforms $\tilde{f}(k)$, $\tilde{g}(k)$ of $f(x)$ and $g(x)$.

(ii) Find $\tilde{u}(k,y)$ for

$$f(x) = 0, \qquad g(x) = \frac{x}{x^2 + a^2} - \frac{x}{x^2 + 9a^2}.$$

Hence, determine $u(x,y)$.

[*The following convention is used in this question:*

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \quad \text{and} \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk \,.]$$

15B Quantum Mechanics

(a) Write down the time-dependent Schrödinger equation for the wavefunction $\psi(x,t)$ of a particle with Hamiltonian $\hat{H}$.

Suppose that A is an observable associated with the operator $\hat{A}$. Show that

$$i\hbar \frac{d\langle \hat{A}\rangle_\psi}{dt} = \langle [\hat{A},\hat{H}]\rangle_\psi + i\hbar \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle_\psi .$$

(b) Consider a particle of mass m subject to a constant gravitational field with potential energy $U(x) = mgx$.

[For the rest of the question you should assume that $\psi(x,t)$ is normalized.]

(i) Find the differential equation satisfied by the function $\Phi(x,t)$ defined by

$$\psi(x,t) = \Phi(x,t) \exp\left[-\frac{im}{\hbar} gt \left(x + \frac{1}{6} gt^2 \right) \right] .$$

(ii) Show that $\Theta(X,T) = \Phi(x,t)$, with $X = x + \frac{1}{2}gt^2$ and $T = t$, satisfies the free-particle Schrödinger equation

$$i\hbar \frac{\partial \Theta}{\partial T} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Theta}{\partial X^2} .$$

Hence, show that

$$\frac{d\langle \hat{X}\rangle_\Theta}{dT} = \frac{1}{m}\langle \hat{P}\rangle_\Theta , \qquad \frac{d\langle \hat{P}\rangle_\Theta}{dT} = 0 ,$$

where $\hat{P} = -i\hbar \dfrac{\partial}{\partial X}$.

(iii) Express $\langle \hat{X}\rangle_\Theta$ in terms of $\langle \hat{x}\rangle_\psi$. Deduce that

$$\langle \hat{x}\rangle_\psi = a + vt - \tfrac{1}{2}gt^2 ,$$

for some constants a and v. Briefly comment on the physical significance of this result.

16C Fluid Dynamics

A fluid of density ρ_1 occupies the region $z > 0$ and a second fluid of density ρ_2 occupies the region $z < 0$. The system is perturbed so that the subsequent motion is irrotational and the interface is at $z = \zeta(x, t)$. State the equations and nonlinear boundary conditions that are satisfied by the corresponding velocity potentials ϕ_1 and ϕ_2 and pressures p_1 and p_2.

Obtain a set of linearised equations and boundary conditions when the perturbations are small and proportional to $e^{i(kx-\omega t)}$. Hence derive the dispersion relation

$$\omega^2 = gk\, F\left(\frac{\rho_1}{\rho_2}\right),$$

where g is the gravitational acceleration and F is a function to be determined.

17H Statistics

Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. observations from a zero-inflated Poisson distribution with parameters $\pi \in [0,1]$ and $\lambda > 0$, which has probability mass function

$$f_{\pi,\lambda}(x) = \begin{cases} \pi + (1-\pi)e^{-\lambda} & \text{if } x = 0, \\ (1-\pi)\frac{\lambda^x e^{-\lambda}}{x!} & \text{if } x = 1, 2, \ldots. \end{cases}$$

Let $n_0 = \sum_{i=1}^n 1_{\{X_i=0\}}$ and $S = \sum_{i=1}^n X_i$.

(a) What is meant by *sufficient statistic* and *minimal sufficient statistic*? Show that $T = (n_0, S)$ is a sufficient statistic. Is it minimal sufficient?

(b) Suppose the parameter λ is known to be equal to some value λ_0. We wish to test the null hypothesis $H_0 : \pi = 0$ against the alternative $H_1 : \pi = 1/2$. Suppose there exists a likelihood ratio test of size α for H_0 against H_1. Specify the test statistic and the critical region. Is this test uniformly most powerful for the alternative $H_1 : \pi > 0$?

(c) Now suppose that both π and λ are unknown. We wish to test the null hypothesis $H_0 : \pi = 1/2$ against the alternative $H_1 : \pi \in [0,1]$. State the asymptotic null distribution of the generalised likelihood ratio statistic:

$$W = 2\log\frac{\max_{\lambda>0,\pi\in[0,1]} L(\lambda,\pi;X)}{\max_{\lambda>0} L(\lambda,1/2;X)},$$

where $L(\lambda, \pi; X)$ is the likelihood function. Describe a test of size α using this statistic.

[*You may quote any result from the lectures that you need without proof.*]

18H Optimisation

(a) Explain what is meant by a *two player zero-sum game*. What are *pure* and *mixed* strategies?

(b) Let $0 < a < b < c < d$, and let

$$A_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A_2 = \begin{pmatrix} a & b \\ d & c \end{pmatrix}, \quad \text{and } A_3 = \begin{pmatrix} a & c \\ d & b \end{pmatrix}.$$

Which of the three games with the payoff matrices given above admit optimal strategies that are pure?

(c) Consider the payoff matrix

$$A = \begin{pmatrix} 1 & 5 \\ 7 & 3 \end{pmatrix}.$$

Let $p = [p_1, p_2]^T$ be the strategy of player 1, and let v be the value of the game. Show that $v > 0$. Setting $x = [p_1/v, p_2/v]^T$, show that the optimal strategy for player 1 can be found by solving the problem

$$\begin{aligned} \text{minimize} \quad & e^T x \\ \text{subject to} \quad & A^T x \geqslant e \\ & x \geqslant 0, \end{aligned}$$

where $e = [1, 1]^T$.

(d) Find the dual of the linear program in part (c). Is the dual a linear program in standard form? Solve the dual using the simplex method and identify the optimal strategies for both players.

END OF PAPER