UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Thursday, 09 June, 2022 1:30pm to 4:30pm

PAPER 3

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most four*** *questions from Section I and* ***at most six*** *questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet ***for each question*** *that you have attempted, and place it at the front of your answer to that question.*

Complete a green master cover sheet listing ***all the questions*** *that you have attempted.*

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into ***a single bundle****, with the master cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.*

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Groups, Rings and Modules

State the first isomorphism theorem for rings.

Let R be a subring of a ring S, and let J be an ideal in S. Show that $R + J$ is a subring of S and that

$$\frac{R}{R \cap J} \cong \frac{R + J}{J}.$$

Compute the characteristics of the following rings, and determine which are fields.

$$\frac{\mathbb{Q}[X]}{(X+2)} \qquad \frac{\mathbb{Z}[X]}{(3, X^2 + X + 1)}$$

2F Geometry

Consider the space $S_{a,b} \subset \mathbb{R}^3$ defined by

$$x^2 + y^2 + z^3 + az + b = 0$$

for unknown real constants a, b with $(a, b) \neq (0, 0)$.

(a) Stating any result you use, show that $S_{a,b}$ is a smooth surface in $\mathbb{R}^3$ whenever $4a^3 + 27b^2 \neq 0$.

(b) What about the cases where $4a^3 + 27b^2 = 0$? Briefly justify your answer.

3A Complex Methods

The function $f(x)$ has Fourier transform

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx = \frac{-2ki}{p^2 + k^2},$$

where $p > 0$ is a real constant. Using contour integration, calculate $f(x)$ for $x > 0$.

[Jordan's lemma and the residue theorem may be used without proof.]

4D Variational Principles

Explain the method of Lagrange multipliers for finding the stationary values of a function $F(x, y, z)$ subject to the constraint $G(x, y, z) = 0$.

Use the method of Lagrange multipliers to find the minimum of $x^2 + y^2 + z^2$ subject to the constraint $z - xy = 1$.

Find the maximum of $z - xy$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

5A Methods

The Legendre polynomial $P_n(x)$ satisfies

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0, \quad n = 0, 1, \ldots, \text{ for } -1 \leqslant x \leqslant 1.$$

Show that $Q_n(x) = P_n'(x)$ satisfies an equation which can be recast in self-adjoint form with eigenvalue $(n-1)(n+2)$. Write down the orthogonality relation for $Q_n(x)$, $Q_m(x)$ for $n \neq m$.

6B Quantum Mechanics

(a) A beam of identical, free particles, each of mass m, moves in one dimension. There is no potential. Show that the wavefunction $\chi(x) = Ae^{ikx}$ is an energy eigenstate for any constants A and k.

What is the energy E and the momentum p in terms of k? What can you say about the sign of E?

(b) Write down expressions for the probability density ρ and the probability current J in terms of the wavefunction $\psi(x,t)$. Use the current conservation equation, i.e.

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0$$

to show that, for a stationary state of fixed energy E, the probability current J is independent of x.

(c) A beam of particles in a stationary state is incident from $x \to -\infty$ upon a potential $U(x)$ with $U(x) \to 0$ as $x \to \pm\infty$. Given the asymptotic behaviour of the form

$$\psi(x) = \begin{cases} e^{ikx} + Re^{-ikx}\,, & x \to -\infty\,, \\ \\ Te^{ikx}\,, & x \to \infty\,, \end{cases}$$

show that $|R|^2 + |T|^2 = 1$. Interpret this result.

7C Fluid Dynamics

A two-dimensional flow has velocity given by

$$\mathbf{u}(\mathbf{x}) = 2\,\frac{\mathbf{x}(\mathbf{d}\cdot\mathbf{x})}{r^4} - \frac{\mathbf{d}}{r^2}$$

as a function of the position vector $\mathbf{x}$, with $r = |\mathbf{x}|$, where $\mathbf{d}$ is a fixed vector.

(a) Show that this flow is incompressible for $r \neq 0$.

(b) Compute the stream function ψ for this flow in polar coordinates (r,θ) with $\theta = 0$ aligned with the vector $\mathbf{d}$.

[*Hint: in polar coordinates*

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta}$$

for a vector $\mathbf{F} = (F_r, F_\theta)$.]

8H Markov Chains

Let X be an irreducible, positive recurrent and reversible Markov chain taking values in S and let π be its invariant distribution. For $A \subseteq S$, we write

$$T_A = \min\{n \geqslant 0 : X_n \in A\} \quad \text{and} \quad T_A^+ = \min\{n \geqslant 1 : X_n \in A\}.$$

(a) Prove that for all $A \subseteq S$ and $z \in A$, we have

$$\mathbb{P}_\pi(X_{T_A} = z) = \pi(z)\mathbb{E}_z\left[T_A^+\right].$$

(b) Let π_A be the probability measure defined by $\pi_A(x) = \pi(x)/\pi(A)$ for $x \in A$. Prove that

$$\mathbb{E}_{\pi_A}\left[T_A^+\right] = \frac{1}{\pi(A)}.$$

SECTION II

9F Linear Algebra

Suppose that α is an endomorphism of an n-dimensional complex vector space. Define the *minimal polynomial* m_α of α. State the Cayley–Hamilton theorem, and explain why m_α exists and is unique.

(a) If α has minimal polynomial $m_\alpha(x) = x^m$, what is the minimal polynomial of α^3?

(b) If $\lambda \neq 0$ is an eigenvalue for α, show that λ^3 is an eigenvalue for α^3. Describe the λ^3–eigenspace of α^3 in terms of eigenspaces of α.

(c) Assume α is invertible with minimal polynomial $m_\alpha(x) = \prod_{i=1}^k (x-\lambda_i)^{c_i}$.

(i) Show that the minimal polynomial m_{α^3} of α^3 must divide $\prod_{i=1}^k (x-\lambda_i^3)^{c_i}$.

(ii) Prove that equality holds if in addition all λ_i are real (in other words, we have $m_{\alpha^3}(x) = \prod_{i=1}^k (x-\lambda_i^3)^{c_i}$).

10E Groups, Rings and Modules

Let R be a Euclidean domain. What does it mean for two matrices with entries in R to be *equivalent*? Prove that any such matrix is equivalent to a diagonal matrix. Under what further conditions is the diagonal matrix said to be in *Smith normal form*?

Let $M \leqslant \mathbb{Z}^n$ be the subgroup generated by the rows of an $n \times n$ matrix A. Show that $G = \mathbb{Z}^n/M$ is finite if and only if $\det A \neq 0$, and in that case the order of G is $|\det A|$.

Determine whether the groups G_1 and G_2 corresponding to the following matrices are isomorphic.

$$A_1 = \begin{pmatrix} 5 & 0 & 4 \\ 0 & 1 & 2 \\ 2 & 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 7 & 2 & -1 \\ 6 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

11G Analysis and Topology

Define a *contraction mapping* between two metric spaces. State and prove the contraction mapping theorem. Use this to show that the equation $x = \cos x$ has a unique real solution.

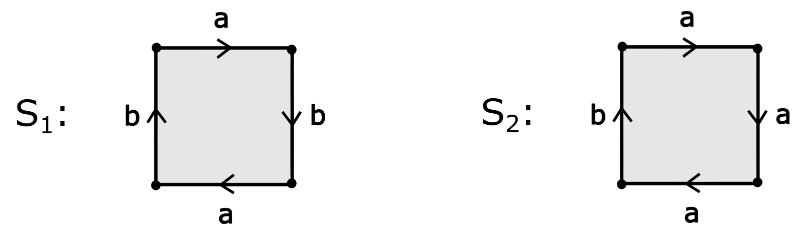
State the mean value inequality. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the map given by

$$f(x,y) = \left(\frac{\cos x + \cos y - 1}{2}, \cos x - \cos y \right).$$

Prove that f has a fixed point. [*Hint: Find a suitable subset of* $\mathbb{R}^2$ *on which* f *is a contraction mapping.*]

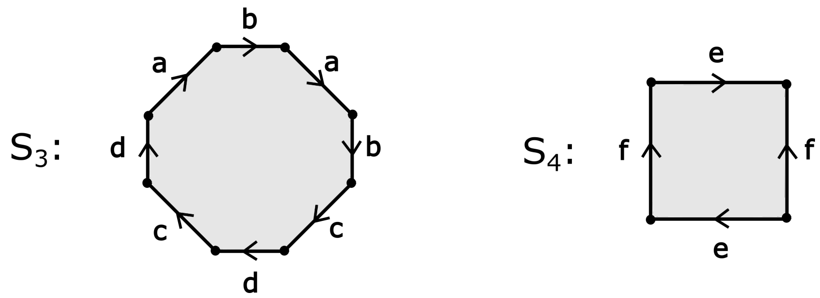
12F Geometry

(a) Define a *topological surface*. Consider the topological spaces S_1 and S_2 given by identifying the sides of a square as drawn. Show that S_1 is a topological surface. [*Hint: It may help to find a finite group G acting on the 2-sphere S^2 such that S^2/G is homeomorphic to S_1.*]



Is S_2 a topological surface? Briefly justify your answer.

(b) By cutting each along a suitable diagonal, show that the two topological surfaces S_3 and S_4 defined by gluing edges of polygons as shown are homeomorphic.



If you delete an open disc from S_4, can the resulting surface be embedded in $\mathbb{R}^3$? Briefly justify your answer. Can S_4 itself be embedded in $\mathbb{R}^3$? State any result you use.

13G Complex Analysis

Let $U \subset \mathbb{C}$ be a (non-empty) connected open set and let f_n be a sequence of holomorphic functions defined on U. Suppose that f_n converges uniformly to a function f on every compact subset of U. Show that f is holomorphic in U. Furthermore, show that f_n' converges uniformly to f' on every compact subset of U.

Suppose in addition that f is not identically zero and that for each n, there is a unique $c_n \in U$ such that $f_n(c_n) = 0$. Show that there is at most one $c \in U$ such that $f(c) = 0$. Find an example such that f has no zeros in U. Give a necessary and sufficient condition on the c_n for this to happen in general.

14A Methods

(a) Prove Green's third identity for functions $u(\boldsymbol{r})$ satisfying Laplace's equation in a volume V with surface S, namely

$$u(\boldsymbol{r}_0) = \int_S \left(u \frac{\partial G_{fs}}{\partial n} - \frac{\partial u}{\partial n} G_{fs} \right) dS,$$

where $G_{fs}(\boldsymbol{r};\boldsymbol{r}_0) = -1/(4\pi|\boldsymbol{r}-\boldsymbol{r}_0|)$ is the free space Green's function.

(b) A solution is sought to the Neumann problem for $\nabla^2 u = 0$ in the half-space $z > 0$ with boundary condition

$$\left.\frac{\partial u}{\partial z}\right|_{z=0} = p(x,y),$$

where both u and its spatial derivatives decay sufficiently rapidly as $|\boldsymbol{r}| \to \infty$.

(i) Explain why it is necessary to assume that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)dx\,dy \;= 0.$$

(ii) Using the method of images or otherwise, construct an appropriate Green's function $G(\boldsymbol{r};\boldsymbol{r}_0)$ satisfying $\partial G/\partial z = 0$ at $z = 0$.

(iii) Hence find the solution in the form

$$u(x_0,y_0,z_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y) f(x-x_0, y-y_0, z_0) dx\,dy\,,$$

where f is to be determined.

(iv) Now let

$$p(x,y) = \begin{cases} \sin(x) & \text{for } |x|,|y| < \frac{\pi}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

By expanding f in inverse powers of z_0, determine the leading order term for u (proportional to z_0^{-3}) as $z_0 \to \infty$.

15D Electromagnetism

(a) A Lorentz transformation is given by

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ,$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$. How does a 4-vector $X^{\mu} = (ct, x, y, z)$ transform?

(b) The electromagnetic field is an anti-symmetric tensor with components

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & B_3 & -B_2 \\ E_2/c & -B_3 & 0 & B_1 \\ E_3/c & B_2 & -B_1 & 0 \end{pmatrix} .$$

Determine how the components of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ transform under the Lorentz transformation given in part (a).

(c) An infinite, straight wire has uniform charge per unit length λ and carries no current. Determine the electric field and magnetic field. By applying a Lorentz boost, find the fields seen by an observer who travels with speed v in the direction parallel to the wire. Interpret your results using the appropriate Maxwell equation.

16C Fluid Dynamics

Consider an axisymmetric, two-dimensional, incompressible flow $\mathbf{u}(r) = (u_r, u_\theta)$ in polar coordinates (r, θ).

(a) Determine the behaviour of u_r if it is finite everywhere in space.

(b) Representing $u_\theta = \Omega(r)r$, express the vorticity of the flow $\boldsymbol{\omega}$ in terms of Ω.

(c) Starting from the Navier-Stokes equation

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = -\nabla p + \mu\nabla^2\mathbf{u}$$

derive the vorticity evolution equation

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}$$

for a general incompressible flow with kinematic viscosity $\nu = \mu/\rho$.

(d) Deduce the form of the evolution equation for the scalar vorticity $\omega = |\boldsymbol{\omega}|$ for the axisymmetric two-dimensional flow of part (a).

(e) Show that the equation derived in part (d) adopts a self-similar form $\omega(r,t) = \omega(\xi)$, where $\xi = r/\sqrt{\nu t}$ is the similarity variable.

[*You may use the fact that, in polar coordinates,*

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

and

$$\boldsymbol{\nabla}\times\mathbf{F} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial \theta}\right]\mathbf{e}_z$$

for a vector $\mathbf{F} = (F_r, F_\theta)$*, where* $\mathbf{e}_z$ *is a unit vector normal to the flow plane.*]

17C Numerical Analysis

(a) The equation $y' = f(t, y)$ is solved using the following multistep method with s steps,

$$\sum_{k=0}^{s} \rho_k y_{n+k} = h \sum_{k=0}^{s} \sigma_k f(t_{n+k}, y_{n+k}),$$

where h is the step size and ρ_k, σ_k are specified constants with $\rho_s = 1$. Prove that this method is of order p if and only if

$$\sum_{k=0}^{s} \rho_k P(t_{n+k}) = h \sum_{k=0}^{s} \sigma_k P'(t_{n+k}),$$

for all polynomials P of degree p.

(b) State the Dahlquist equivalence theorem regarding the convergence of a multistep method. Consider a multistep method

$$y_{n+3} + (2a - 3)(y_{n+2} - y_{n+1}) - y_n = ha(f_{n+2} + f_{n+1}),$$

where $a \neq 0$ is a real parameter. Determine the values of a for which this method is convergent, and find its order.

18H Statistics

Consider a linear model $Y = X\beta + \varepsilon$, where $X \in \mathbb{R}^{n\times p}$ is a fixed design matrix of rank $p < n/2$, $\beta \in \mathbb{R}^p$, and $\varepsilon \sim N(0, \sigma^2\Sigma_0)$, for some known positive definite matrix $\Sigma_0 \in \mathbb{R}^{n\times n}$ and an unknown scalar $\sigma^2 > 0$.

(a) Derive the maximum likelihood estimators $(\hat{\beta}, \hat{\sigma}^2)$ for the parameters (β, σ^2).

(b) Find the distribution of $\hat{\beta}$.

(c) Prove that $\hat{\beta}$ is the Best Linear Unbiased Estimator for β.

Now, suppose that $\varepsilon \sim N(0, \Sigma)$ where $\Sigma \in \mathbb{R}^{n\times n}$ is a diagonal matrix with

$$\Sigma_{ii} = \begin{cases} \sigma_1^2 & \text{if } i \leqslant n/2\,, \\ \sigma_2^2 & \text{if } i > n/2\,, \end{cases}$$

and where σ_1^2 and σ_2^2 are unknown parameters and n is even.

(d) Describe a test of size α for the null hypothesis $H_0 : \sigma_1^2 = \sigma_2^2$ against the alternative $H_1 : \sigma_1^2 < \sigma_2^2$, using the test statistic

$$T = \frac{\|Y_1 - X_1(X_1^T X_1)^{-1} X_1^T Y_1\|^2}{\|Y_2 - X_2(X_2^T X_2)^{-1} X_2^T Y_2\|^2}$$

where,

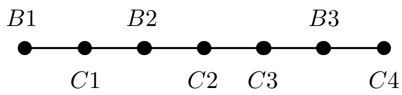
$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix},$$

with $Y_1, Y_2 \in \mathbb{R}^{n/2}$ and $X_1, X_2 \in \mathbb{R}^{n/2\times p}$. [You must specify the null distribution of T and the critical region, and you may quote any result from the lectures that you need without proof.]

19H Optimisation

Explain what is meant by a *transportation problem* with n suppliers and m consumers.

A straight road contains three bakeries, B1, B2, and B3, and four cafes, C1, C2, C3, and C4. They are arranged in the following order:



The distance between consecutive establishments is 1 mile: For example, the distance between B1 and C2 is 3 miles. Bakeries B1, B2, and B3 produce 6, 4, and 8 cakes daily, respectively. Cafes C1, C2, C3, and C4 consume 3, 5, 7, and 3 cakes daily, respectively. The cost of transporting one cake from a bakery to a cafe is equal to the distance between the two locations, measured in miles. Cakes may be cut into arbitrary pieces before transporting. The resulting cost matrix is

$$C = \begin{pmatrix} 1 & 3 & 4 & 6 \\ 1 & 1 & 2 & 4 \\ 4 & 2 & 1 & 1 \end{pmatrix}.$$

(a) Use the north-west corner rule to find a basic feasible solution. Is this solution degenerate? If not, find a degenerate basic feasible solution to this problem.

(b) Consider the following transportation plan:

- B1 delivers 3 cakes each to C1 and C3,
- B2 delivers 4 cakes to C2, and
- B3 delivers 1 cake to C2, 4 cakes to C3, and 3 cakes to C4.

Explain why this is a basic feasible solution. Calculate the complete transportation tableau for this solution. Is the solution optimal? If not, perform one step of the transportation algorithm. Is the solution optimal now?

END OF PAPER