UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Wednesday, 08 June, 2022 9:00am to 12:00pm

PAPER 2

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most four*** *questions from Section I and* ***at most six*** *questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet ***for each question*** *that you have attempted, and place it at the front of your answer to that question.*

Complete a green master cover sheet listing ***all the questions*** *that you have attempted.*

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into ***a single bundle****, with the master cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.*

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Groups, Rings and Modules

(a) Let R be an integral domain and M an R-module. Let $T \subset M$ be the subset of torsion elements, i.e., elements $m \in M$ such that $rm = 0$ for some $0 \neq r \in R$. Show that T is an R-submodule of M.

(b) Let $\phi : M_1 \to M_2$ be a homomorphism of R-modules. Let $T_1 \leqslant M_1$ and $T_2 \leqslant M_2$ be the torsion submodules. Show that there is a homomorphism of R-modules $\Phi : M_1/T_1 \to M_2/T_2$ satisfying $\Phi(m + T_1) = \phi(m) + T_2$ for all $m \in M_1$.

Does ϕ injective imply Φ injective?

Does Φ injective imply ϕ injective?

2G Analysis and Topology

Let $f : (M, d) \to (N, e)$ be a homeomorphism between metric spaces. Show that $d'(x, y) = e(f(x), f(y))$ defines a metric on M that is equivalent to d. Construct a metric on $\mathbb{R}$ which is equivalent to the standard metric but in which $\mathbb{R}$ is not complete.

3B Methods

The function $u(x, y)$ satisfies

$$x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x} = 0\,,$$

with boundary data $u(x, 0) = f(x^2)$. Find and sketch the characteristic curves. Hence determine $u(x, y)$.

4D Electromagnetism

A uniformly charged sphere of radius R has total charge Q. Find the electric field inside and outside the sphere.

A second uniformly charged sphere of radius R has total charge $-Q$. The centre of the second sphere is displaced from the centre of the first by the vector $\mathbf{d}$, where $|\mathbf{d}| < R$. Show that the electric field in the overlap region is constant and find its value.

5C Fluid Dynamics

An unsteady fluid flow has velocity field given in cartesian coordinates (x, y, z) by $\mathbf{u} = (2t, xt, 0)$, where $t > 0$ denotes time. Dye is continuously released into the fluid from the origin.

(a) Determine if this fluid flow is incompressible.

(b) Find the distance from the origin at time t of the dye particle that was released at time s, where $s < t$.

(c) Determine the equation of the curve formed by the dye streak in the (x, y)-plane.

6H Statistics

Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. random variables with probability density function

$$f_\theta(x) = \frac{1}{2} + \frac{1_{\{x<\theta\}}}{2\theta} \quad \text{for } x \in [0, 1],$$

with parameter $\theta \in (0, 1)$.

(a) Write down the likelihood function, and show that the maximum likelihood estimator coincides with one of the samples.

(b) Consider the estimator $\tilde{\theta} = 4\overline{X} - 1$ where $\overline{X} = n^{-1} \sum_{i=1}^{n} X_i$. Is $\tilde{\theta}$ unbiased? Construct an asymptotic $(1 - \alpha)$-confidence interval for θ around this estimator.

7H Optimisation

State the Lagrange sufficiency theorem. Using the Lagrange sufficiency theorem, solve the following optimisation problem:

$$\begin{aligned} \text{minimise} \quad & -x_1 - 3x_2 \\ \text{subject to} \quad & x_1^2 + x_2^2 \leqslant 25 \\ & -x_1 + 2x_2 \leqslant 5. \end{aligned}$$

SECTION II

8F Linear Algebra

Let V be a real vector space (not necessarily finite-dimensional). Define the *dual space* V^*. Prove that if f_1, $f_2 \in V^*$ are such that $f_1(v)f_2(v) = 0$ for all $v \in V$, then f_1 or f_2 is the zero element in V^*.

Now suppose that V is a finite-dimensional real vector space.

Let ϕ be a symmetric bilinear form on V. State Sylvester's law of inertia for ϕ.

Let q be a quadratic form on V, let r denote its rank and σ its signature. Show that q can be factorised as $q(v) = f_1(v)f_2(v)$ with $f_1, f_2 \in V^*$ for all $v \in V$ if and only if $r + |\sigma| \leqslant 2$.

A vector $v_0 \in V$ is called isotropic if $q(v_0) = 0$. Show that if there exist v_1 and v_2 in V such that $q(v_1) > 0$ and $q(v_2) < 0$, then one can construct a basis of V consisting of isotropic vectors.

9E Groups, Rings and Modules

Define a Sylow subgroup and state the Sylow theorems. Prove the third theorem, concerning the number of Sylow subgroups.

Quoting any general facts you need about alternating groups, show that A_n has no subgroup of index m if $1 < m < n$ and $n \geqslant 5$. Hence, or otherwise, show that there is no simple group of order 90.

10G Analysis and Topology

State the inverse function theorem for a function $F : \mathbb{R}^n \to \mathbb{R}^n$. Suppose F is a differentiable bijection with F^{-1} also differentiable. Show that the derivative of F at any point in $\mathbb{R}^n$ is a linear isomorphism.

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable map such that its derivative is invertible at any point in $\mathbb{R}^n$. Is $F(\mathbb{R}^n)$ open? Is $F(\mathbb{R}^n)$ closed? Justify your answers.

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$F(x, y, z) = (x + y + z, zy + zx + xy, xyz).$$

Determine the set C of points $p \in \mathbb{R}^3$ for which F fails to admit a differentiable local inverse around p. Is the set $\mathbb{R}^3 \setminus C$ connected? Justify your answer.

11F Geometry

Consider the surface $S \subset \mathbb{R}^3$ given by

$$(\sinh u \cos v, \sinh u \sin v, v) \qquad \text{for } u, v > 0.$$

Sketch S. Calculate its first fundamental form.

(a) Find a surface of revolution S' such that there is a local isometry between S and S'. Do they have the same Gauss curvature?

(b) Given an oriented surface $R \subset \mathbb{R}^3$, define the *Gauss map* of R. Describe the image of the Gauss map for S' equipped with the orientation associated to the outward-pointing normal. Use this to calculate the total Gaussian curvature of S'.

(c) By considering the total Gaussian curvature of S, or otherwise, show that there does not exist a global isometry between S and S'.

You should carefully state any result(s) you use.

12A Complex Analysis or Complex Methods

(a) Let $R = P/Q$ be a rational function, where $\deg Q \geqslant \deg P + 2$, and Q has no real zeros. Using the calculus of residues, write a general expression for

$$\int_{-\infty}^{\infty} R(x) e^{ix}\, dx$$

in terms of residues. Briefly justify your answer.

[You may assume that the polynomials P and Q do not have any common factors.]

(b) Explicitly evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{1 + x^4}\, dx\,.$$

13D Variational Principles

(a) A functional $I[z]$ of $z(x)$ is given by

$$I[z] = \int_a^b f(z, z'; x)\; dx$$

where $z' = dz/dx$. State the Euler-Lagrange equation that governs the extrema of I.

If f does not depend explicitly on x, construct a non-constant quantity that, when evaluated on the extrema of I, does not depend on x.

Explain how to determine the extrema of I subject to the further functional constraint that $J[z]$ is constant.

(b) A heavy, uniform rope of fixed length L is suspended between two points $(x_1, z_1) = (-a, 0)$ and $(x_2, z_2) = (+a, 0)$ with $L > 2a$. In a gravitational potential $\Phi(z)$, the potential energy is given by

$$V[z] = \rho \int_{-a}^{a} \Phi(z)\sqrt{1 + z'^2}\, dx \; .$$

where ρ is the mass per unit length.

(i) Show that, in a gravitational potential $\Phi(z) = gz$, the shape adopted by the rope is

$$z - z_0 = -B \cosh\left(\frac{x}{B}\right)$$

where z_0 and B are two constants. Find implicit expressions for z_0 and B in terms of a and L.

(ii) What is the gravitational potential $\Phi(z)$ if, for $L = \pi a$, the rope hangs in a semi-circle?

14A Methods

(a) Verify that $y = e^{-x}$ is a solution of the differential equation

$$(x + \lambda + 1)y'' + (x + \lambda)y' - y = 0,$$

where λ is a constant. Find a second solution of the form $y = ax + b$.

(b) Let $\mathcal{L}$ be the operator

$$\mathcal{L}[y] = y'' + \frac{(x + \lambda)}{(x + \lambda + 1)}y' - \frac{1}{(x + \lambda + 1)}y$$

acting on functions $y(x)$ satisfying

$$y(0) = \lambda y'(0) \quad \text{and} \quad \lim_{x \to \infty} y(x) = 0. \qquad (\star)$$

The Green's function $G(x; \xi)$ for $\mathcal{L}$ satisfies

$$\mathcal{L}[G] = \delta(x - \xi),$$

with $\xi > 0$. Show that

$$G(x; \xi) = -\frac{(x + \lambda)}{(\xi + \lambda + 1)}$$

for $0 \leqslant x < \xi$, and find $G(x; \xi)$ for $x > \xi$.

(c) Hence or otherwise find the solution when $\lambda = 2$ for the problem

$$\mathcal{L}[y] = -(x + 3)e^{-x},$$

for $x \geqslant 0$ and $y(x)$ satisfying the boundary conditions given in $(\star)$.

15B Quantum Mechanics

A particle of mass m is confined to the region $0 \leqslant x \leqslant a$ by a potential that is zero inside the region and infinite outside.

(a) Find the energy eigenvalues E_n and the corresponding normalised energy eigenstates $\chi_n(x)$.

(b) At time $t = 0$ the wavefunction $\psi(x,t)$ of the particle is given by

$$\psi(x,0) = f(x)\,,$$

where $f(x)$ is not an energy eigenstate and satisfies the boundary conditions $f(0) = f(a) = 0$.

(i) Express $\psi(x,t)$ in terms of $\chi_n(x)$ and E_n.

(ii) Show that $T = 2ma^2/\pi\hbar$ is the earliest time at which $\psi(a-x,T)$ and $\psi(x,0)$ correspond to physically equivalent states. Thus, determine $\psi(x,2T)$.

Show that if $\psi(x,0) = 0$ for $a/2 \leqslant x \leqslant a$, then the probability of finding the particle in $0 \leqslant x \leqslant a/2$ at $t = T$ is zero.

(iii) For

$$f(x) = \begin{cases} \dfrac{2}{\sqrt{a}} \sin \dfrac{2\pi x}{a}\,, & 0 \leqslant x \leqslant \dfrac{a}{2}\,, \\[2ex] 0\,, & \dfrac{a}{2} \leqslant x \leqslant a\,, \end{cases}$$

find the probability that a measurement of the energy of the particle at time $t = 0$ will yield a value $2\pi^2\hbar^2/ma^2$.

What is the probability if, instead, the same measurement is carried out at time $t = 2T$? What is the probability at $t = T$?

Suppose that the result of the measurement of the energy was indeed $2\pi^2\hbar^2/ma^2$. What is the probability that a subsequent measurement of energy will yield the same result?

16D Electromagnetism

(a) Starting from an appropriate Maxwell equation, derive Faraday's law of induction relating electromotive force to the change of flux for a static circuit.

(b) An infinite wire lies along the z-axis and carries current $I > 0$ in the positive z-direction.

(i) Use Ampère's law to calculate the magnetic field $\mathbf{B}$.

(ii) In addition to the infinite wire described above, a square loop of wire, with sides of length $2a$ and total resistance R, is restricted to lie in the $x = 0$ plane. The centre of the square initially sits at point $y = d > a$. The square loop is pulled away from the wire in the direction of increasing y at speed v. Calculate the current that flows in the loop and draw a diagram indicating the direction of the current.

(iii) The square loop is instead pulled in the z-direction, parallel to the infinite wire, at a speed u. Calculate the current in the loop.

17C Numerical Analysis

A scalar, autonomous, ordinary differential equation $y' = f(y)$ is solved using the Runge–Kutta method

$$\begin{aligned} k_1 &= f(y_n)\,, \\ k_2 &= f(y_n + (1-a)hk_1 + ahk_2)\,, \\ y_{n+1} &= y_n + \frac{h}{2}(k_1 + k_2)\,, \end{aligned}$$

where h is a step size and a is a real parameter.

(a) Determine the order of the method and its dependence on a.

(b) Find the range of values of a for which the method is A-stable.

18H Markov Chains

Let X be a random walk on $\mathbb{N} = \{0, 1, 2, \ldots\}$ with $X_0 = 0$ and transition matrix given by

$$P(i, i+1) = \frac{1}{3} = 1 - P(i, i-1), \quad \text{for } i \geqslant 1, \quad \text{and} \quad P(0,0) = \frac{2}{3} = 1 - P(0,1).$$

(a) Prove that X is positive recurrent.

(b) Let Y be an independent walk with matrix P and suppose that $Y_0 = 0$. Find the limit

$$\lim_{n\to\infty} \mathbb{P}(X_n = 0, Y_n = 1)\,,$$

stating clearly any theorems you use.

(c) Let $T = \min\{n \geqslant 1 : (X_n, Y_n) = (0,0)\}$. Find the expected number of times that Y visits 1 by time T.

END OF PAPER