UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Tuesday, 07 June, 2022 9:00am to 12:00pm

PAPER 1

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most four*** *questions from Section I and* ***at most six*** *questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet ***for each question*** *that you have attempted, and place it at the front of your answer to that question.*

Complete a green master cover sheet listing ***all the questions*** *that you have attempted.*

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into ***a single bundle****, with the master cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.*

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Linear Algebra

Define the determinant of a matrix $A \in M_n(\mathbb{C})$.

(a) Assume A is a block matrix of the form $\begin{pmatrix} M & X \\ 0 & N \end{pmatrix}$, where M and N are square matrices. Show that $\det A = \det M \det N$.

(b) Assume A is a block matrix of the form $\begin{pmatrix} 0 & M \\ N & 0 \end{pmatrix}$, where M and N are square matrices of sizes k and $n-k$. Express $\det A$ in terms of $\det M$ and $\det N$.

[You may assume properties of column operations if clearly stated.]

2E Geometry

Give a characterisation of the geodesics on a smooth embedded surface in $\mathbb{R}^3$.

Write down all the geodesics on the cylinder $x^2 + y^2 = 1$ passing through the point $(x, y, z) = (1, 0, 0)$. Verify that these satisfy your characterisation of a geodesic. Which of these geodesics are closed?

Can $\mathbb{R}^2 \setminus \{(0,0)\}$ be equipped with an abstract Riemannian metric such that every point lies on a unique closed geodesic? Briefly justify your answer.

3G Complex Analysis or Complex Methods

Show that $f(z) = \frac{z}{\sin z}$ has a removable singularity at $z = 0$. Find the radius of convergence of the power series of f at the origin.

4D Variational Principles

Write down the Euler-Lagrange equation for the functional

$$I[y] = \int_0^{\pi/2} \left[y'^2 - y^2 - 2y\sin(x) \right] \mathrm{d}x \ .$$

Solve it subject to the boundary conditions $y'(0) = y'(\pi/2) = 0$.

5C Numerical Analysis

Use the Gram–Schmidt algorithm to compute a reduced QR factorization of the matrix

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 0 & -4 \\ 2 & 2 & 2 \\ -2 & 0 & 2 \end{bmatrix},$$

i.e. find a matrix $Q \in \mathbb{R}^{4\times 3}$ with orthonormal columns and an upper triangular matrix $R \in \mathbb{R}^{3\times 3}$ such that $A = QR$.

6H Statistics

State the Rao-Blackwell theorem.

Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. Geometric(p) random variables; i.e., X_1 is distributed as the number of failures before the first success in a sequence of i.i.d. Bernoulli trials with probability of success p.

Let $\theta = p - p^2$, and consider the estimator $\hat{\theta} = 1_{\{X_1=1\}}$. Find an estimator for θ which is a function of the statistic $T = \sum_{i=1}^n X_i$ and which has variance strictly smaller than that of $\hat{\theta}$. [*Hint: Observe that T is a sufficient statistic for p.*]

7H Optimisation

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable convex function. Briefly describe the steps of the *gradient descent* method for minimizing f.

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a twice-differentiable function satisfying $\alpha I \preceq \nabla^2 f(x) \preceq \beta I$ for some $\alpha, \beta > 0$ and all $x \in \mathbb{R}^n$. Suppose the gradient descent method is run with step size $\eta = \frac{1}{\beta}$. How does the rate of convergence of the gradient descent method depend on the condition number $\frac{\beta}{\alpha}$?

Now let $f(x, y, z) = x^2 + 100y^2 + 10000z^2$. Compute a condition number for f. Find a linear transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f \circ A$ has a condition number of 1.

[*For two matrices $A, B \in \mathbb{R}^n$, we write $A \preceq B$ to denote the fact that $B - A$ is a positive semidefinite matrix.*]

SECTION II

8F Linear Algebra

(a) Let V be a finite dimensional complex inner product space, and let α be an endomorphism of V. Define its adjoint α^*.

Assume that α is normal, i.e. α commutes with its adjoint: $\alpha\alpha^* = \alpha^*\alpha$.

(i) Show that α and α^* have a common eigenvector $\mathbf{v}$. What is the relation between the corresponding eigenvalues?

(ii) Deduce that V has an orthonormal basis of eigenvectors of α.

(b) Now consider a real matrix $A \in \mathrm{Mat}_n(\mathbb{R})$ which is skew-symmetric, i.e. $A^T = -A$.

(i) Can A have a non-zero real eigenvalue?

(ii) Use the results of part (a) to show that there exists an orthogonal matrix $R \in O(n)$ such that $R^T AR$ is block-diagonal with the non-zero blocks of the form $\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$, $\lambda \in \mathbb{R}$.

9E Groups, Rings and Modules

Define a *Euclidean domain.* Briefly explain how $\mathbb{Z}[i]$ satisfies this definition.

Find all the units in $\mathbb{Z}[i]$. Working in this ring, write each of the elements 2, 5 and $1+3i$ in the form $u\,p_1^{\alpha_1} \ldots p_t^{\alpha_t}$ where u is a unit, and $p_1, \ldots, p_t$ are pairwise non-associate irreducibles.

Find all pairs of integers x and y satisfying $x^2 + 4 = y^3$.

10G Analysis and Topology

Let X and Y be metric spaces. Determine which of the following statements are always true and which may be false, giving a proof or a counterexample as appropriate.

(a) Let f_n and f be real-valued functions on X and let A, B be two subsets of X such that $X = A \cup B$. If f_n converges uniformly to f on both A and B, then f_n converges uniformly to f on X.

(b) If the sequences of real-valued functions f_n and g_n converge uniformly on X to f and g respectively, then $f_n g_n$ converges uniformly to fg on X.

(c) Let X be the rectangle $[1,2] \times [1,2] \subset \mathbb{R}^2$ and let $f_n : X \to \mathbb{R}$ be given by

$$f_n(x,y) = \frac{1+nx}{1+ny}.$$

Then f_n converges uniformly on X.

(d) Let A be a subset of X and x_0 a point such that any neighbourhood of x_0 contains a point of A different from x_0. Suppose the functions $f_n : A \to Y$ converge uniformly on A and, for each n, $\lim_{x \to x_0} f_n(x) = y_n$. If Y is complete, then the sequence y_n converges.

(e) Let f_n converge uniformly on X to a bounded function f and let $g : \mathbb{R} \to \mathbb{R}$ be continuous. Then the composition $g \circ f_n$ converges uniformly to $g \circ f$ on X.

11E Geometry

(a) Let $\mathbb{H}$ be the upper half plane model of the hyperbolic plane. Let G be the group of orientation preserving isometries of $\mathbb{H}$. Write down the general form of an element of G. Show that G acts transitively on (i) the points in $\mathbb{H}$, (ii) the boundary $\mathbb{R} \cup \{\infty\}$ of $\mathbb{H}$, and (iii) the set of hyperbolic lines in $\mathbb{H}$.

(b) Show that if $P \in \mathbb{H}$ then $\{g \in G \mid g(P) = P\}$ is isomorphic to $\mathrm{SO}(2)$.

(c) Show that for any two distinct points $P, Q \in \mathbb{H}$ there exists a unique $g \in G$ with $g(P) = Q$ and $g(Q) = P$.

(d) Show that if ℓ, m are hyperbolic lines meeting at $P \in \mathbb{H}$ with angle θ then the points of intersection of ℓ, m with the boundary of $\mathbb{H}$, when taken in a suitable order, have cross ratio $\cos^2(\theta/2)$.

12G Complex Analysis or Complex Methods

(a) Let $\Omega \subset \mathbb{C}$ be an open set such that there is $z_0 \in \Omega$ with the property that for any $z \in \Omega$, the line segment $[z_0, z]$ connecting z_0 to z is completely contained in Ω. Let $f : \Omega \to \mathbb{C}$ be a continuous function such that

$$\int_\Gamma f(z)\, dz = 0$$

for any closed curve Γ which is the boundary of a triangle contained in Ω. Given $w \in \Omega$, let

$$g(w) = \int_{[z_0,w]} f(z)\, dz.$$

Explain briefly why g is a holomorphic function such that $g'(w) = f(w)$ for all $w \in \Omega$.

(b) Fix $z_0 \in \mathbb{C}$ with $z_0 \neq 0$ and let $\mathcal{D} \subset \mathbb{C}$ be the set of points $z \in \mathbb{C}$ such that the line segment connecting z to z_0 does not pass through the origin. Show that there exists a holomorphic function $h : \mathcal{D} \to \mathbb{C}$ such that $h(z)^2 = z$ for all $z \in \mathcal{D}$. [You may assume that the integral of $1/z$ over the boundary of any triangle contained in $\mathcal{D}$ is zero.]

(c) Show that there exists a holomorphic function f defined in a neighbourhood U of the origin such that $f(z)^2 = \cos z$ for all $z \in U$. Is it possible to find a holomorphic function f defined on the disk $|z| < 2$ such that $f(z)^2 = \cos z$ for all z in the disk? Justify your answer.

13B Methods

A uniform string of length l and mass per unit length μ is stretched horizontally under tension $T = \mu c^2$ and fixed at both ends. The string is subject to the gravitational force μg per unit length and a resistive force with value

$$-2k\mu \frac{\partial y}{\partial t}$$

per unit length, where $y(x,t)$ is the transverse, vertical displacement of the string and k is a positive constant.

(a) Derive the equation of motion of the string assuming that $y(x,t)$ remains small.

[In the remaining parts of the question you should assume that gravity is negligible.]

(b) Find $y(x,t)$ for $t > 0$, given that

$$y(x,0) = 0, \quad \frac{\partial y}{\partial t}(x,0) = A \sin\left(\frac{\pi x}{l}\right) \qquad (\star)$$

with A constant, and $k = \pi c/l$.

(c) An extra transverse force

$$\alpha\mu \sin\left(\frac{3\pi x}{l}\right) \cos kt$$

per unit length is applied to the string, where α is a constant. With the initial conditions $(\star)$, find $y(x,t)$ for $t > 0$ and comment on the behaviour of the string as $t \to \infty$.

Compute the total energy E of the string as $t \to \infty$.

14B Quantum Mechanics

(a) Write down the time-dependent Schrödinger equation for a harmonic oscillator of mass m, frequency ω and coordinate x.

(b) Show that a wavefunction of the form

$$\psi(x,t) = N(t)\,\exp\Big(-F(t)x^2 + G(t)x\Big)\,,$$

where F, G and N are complex functions of time, is a solution to the Schrödinger equation, provided that F, G, N satisfy certain conditions which you should establish.

(c) Verify that

$$F(t) = A\tanh(a + i\omega t), \quad G(t) = \sqrt{\frac{m\omega}{\hbar}}\,\mathrm{sech}(a + i\omega t)\,,$$

where a is a real positive constant, satisfy the conditions you established in part (b). Hence determine the constant A. [You do not need to find the time-dependent normalization function $N(t)$.]

(d) By completing the square, or otherwise, show that $|\psi(x,t)|^2$ is peaked around a certain position $x = h(t)$ and express $h(t)$ in terms of F and G.

(e) Find $h(t)$ as a function of time and describe its behaviour.

(f) Sketch $|\psi(x,t)|^2$ for a fixed value of t. What is the value of $\langle \hat{x} \rangle_\psi$?

[*You may find the following identities useful:*

$$\cosh(\alpha + i\beta) = \cosh\alpha\cos\beta + i\sinh\alpha\sin\beta\,,$$

$$\sinh(\alpha + i\beta) = \sinh\alpha\cos\beta + i\cosh\alpha\sin\beta\,.]$$

15D Electromagnetism

(a) Use Gauss' law to compute the electric field $\mathbf{E}$ and electric potential ϕ due to an infinitely long, straight wire with charge per unit length $\lambda > 0$.

(b) Two infinitely long wires, both lying parallel to the z-axis, intersect the $z = 0$ plane at $(x, y) = (\pm a, 0)$. They carry charge per unit length $\pm\lambda$ respectively. Show that the equipotentials on the $z = 0$ plane form circles and determine the centres and radii of these circles as functions of a and

$$k = \frac{2\pi\epsilon_0\phi}{\lambda}\,,$$

where ϵ_0 is the permittivity of free space.

Sketch the equipotentials and the electric field. What happens in the case $\phi = 0$?

Find the electric field in the limit $a \to 0$ with $\lambda a = p$ fixed.

16C Fluid Dynamics

Consider a steady viscous flow (with viscosity μ) of constant density ρ through a long pipe of circular cross-section with radius R. The flow is driven by a constant pressure gradient $\partial p/\partial z$ along the pipe (z is the coordinate along the pipe).

The Navier-Stokes equation describing this flow is

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right]=-\nabla p+\mu\nabla^2\mathbf{u}.$$

(a) Using cylindrical coordinates (r,θ,z) aligned with the pipe, determine the velocity $\mathbf{u}=(0,0,w(r))$ of the flow.

[*Hint: in cylindrical coordinates*

$$\nabla^2=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\,.]$$

(b) The viscous stress exerted on the flow by the pipe boundaries is equal to

$$\mu\left.\left(\frac{\partial w}{\partial r}\right)\right|_{r=R}.$$

Demonstrate the overall force balance for the (cylindrical) volume of the fluid enclosed within the section of the pipe $z_0\leqslant z\leqslant z_0+L$.

(c) Compute the mass flux through the pipe.

17C Numerical Analysis

For a function $f\in C^3[-1,1]$ consider the following approximation of $f''(0)$:

$$f''(0)\approx\eta(f)=a_{-1}f(-1)+a_0f(0)+a_1f(1)\,,$$

with the error

$$e(f)=f''(0)-\eta(f).$$

We want to find the smallest constant c such that

$$|e(f)|\leqslant c\max_{x\in[-1,1]}\left|f'''(x)\right|. \qquad (\star)$$

(a) State the necessary conditions on the approximation scheme η for the inequality $(\star)$ to be valid with some $c<\infty$. Hence, determine the coefficients a_{-1}, a_0, a_1.

(b) State the Peano kernel theorem and use it to find the smallest constant c in the inequality $(\star)$.

(c) Explain briefly why this constant is sharp.

18H Statistics

A clinical study follows n patients being treated for a disease for T months. Suppose we observe $X_1, \ldots, X_n$, where $X_i = t$ if patient i recovers at month t, and $X_i = T + 1$ if the patient does not recover at any point in the observation period. For $t = 1, \ldots, T$, the parameter $q_t \in [0, 1]$ is the probability that a patient recovers at month t, given that they have not already recovered.

We select a prior distribution which makes the parameters $q_1, \ldots, q_T$ i.i.d. and distributed as $\text{Beta}(T, 1)$.

(a) Write down the likelihood function. Compute the posterior distribution of $(q_1, \ldots, q_T)$.

(b) The parameter γ is the probability that a patient recovers at or before month M. Write down γ in terms of $q_1, \ldots, q_T$. Compute the Bayes estimator for γ under the quadratic loss.

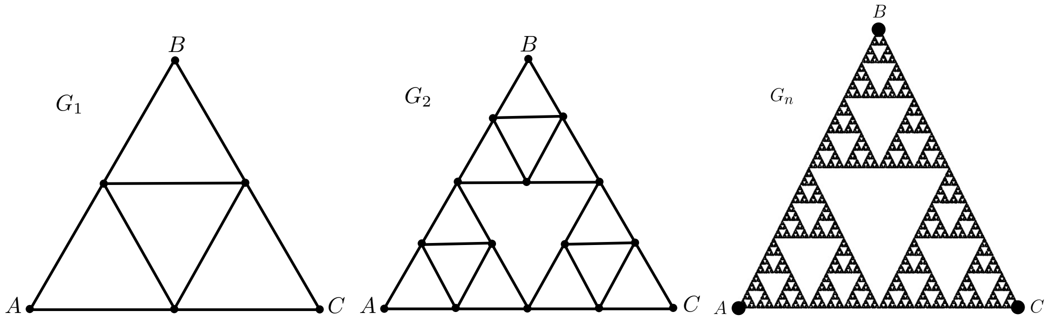
(c) Suppose we wish to estimate γ, but our loss function is asymmetric; i.e., we prefer to underestimate rather than overestimate the parameter. In particular, the loss function is given by

$$L(\delta, \gamma) = \begin{cases} 2|\gamma - \delta| & \text{if } \delta \geqslant \gamma \\ |\gamma - \delta| & \text{if } \delta < \gamma. \end{cases}$$

Find an expression for the Bayes estimator of γ under this loss function, in terms of the posterior distribution function F of γ. [You need not derive F.]

19H Markov Chains

The n-th iteration of the Sierpinski triangle is constructed as follows: start with an equilateral triangle, subdivide it into 4 congruent equilateral triangles, and remove the central one. Repeat the same procedure $n-1$ times on each smaller triangle that is not removed. We call G_n the graph whose vertices are the corners of the triangles and edges the segments joining them, as shown in the figure:



Let A, B, and C be the corners of the original triangle. Let X be a simple random walk on G_n, i.e., from every vertex, it jumps to a neighbour chosen uniformly at random. Let

$$T_{BC} = \min\{i \geqslant 0 : X_i \in \{B, C\}\}.$$

(a) Suppose $n = 1$. Show that $\mathbb{E}_A[T_{BC}] = 5$.

(b) Suppose $n = 2$. Show that $\mathbb{E}_A[T_{BC}] = 5^2$.

(c) Show that $\mathbb{E}_A[T_{BC}] = 5^n$ when X is a simple random walk on G_n, for $n \in \mathbb{N}$.

END OF PAPER