UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Wednesday, 08 June, 2022 1:30pm to 4:30pm

PAPER 4

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. Of the Section II questions, no more than three may be on the same course.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet ***for each question*** *that you have attempted, and place it at the front of your answer to that question.*

Complete a green master cover sheet listing ***all the questions*** *that you have attempted.*

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into ***a single bundle****, with the master cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.*

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Numbers and Sets

By considering numbers of the form $3p_1 \dots p_k - 1$, show that there are infinitely many primes of the form $3n + 2$ with $n \in \mathbb{N}$.

For which primes p is the number $2p^2 + 1$ also prime? Justify your answer.

2D Numbers and Sets

Prove that $\sqrt[3]{2} + \sqrt[3]{3}$ is irrational.

Using the fact that the number $e - e^{-1}$ can be represented by a convergent series $2\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$, prove that $e - e^{-1}$ is irrational.

What is a *transcendental* number? Given that e is transcendental, show that $ae + be^{-1}$ is also transcendental for any integers a, b that are not both zero.

3C Dynamics and Relativity

A particle of mass m, charge q, and position vector $\mathbf{x}$ moves in a constant non-zero electric field $\mathbf{E}$ and a constant non-zero magnetic field $\mathbf{B}$, with $\mathbf{E}$ perpendicular to $\mathbf{B}$. The particle's motion is described by $m\ddot{\mathbf{x}} = q(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B})$. At time $t = 0$ the particle is located at $\mathbf{x} = \mathbf{x}_0$ and has velocity $\dot{\mathbf{x}} = \mathbf{v}$, where $\mathbf{v}$ is perpendicular to both $\mathbf{E}$ and $\mathbf{B}$.

(a) Using vector methods, show that the motion lies in a plane and give the vector equation of that plane.

(b) Adopt a Cartesian coordinate system centred on $\mathbf{x}_0$ with the x-axis directed along $\mathbf{E}$ and the y-axis along $\mathbf{B}$. Assume $\mathbf{v} = \mathbf{0}$. Find an expression for $\mathbf{x}$ as a function of t.

4C Dynamics and Relativity

Consider space-time with only one spatial dimension, and two inertial frames S and S'. Frame S' moves relative to frame S with speed u, and their origins coincide when clocks in the two frames read $t = t' = 0$.

According to an observer at the origin of frame S, an event has coordinates (ct, x). According to an observer at the origin of frame S', its coordinates are (ct', x'), which are given by

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = A \begin{pmatrix} ct \\ x \end{pmatrix},$$

where c is the speed of light and A is a 2×2 matrix.

(a) Write down the matrix A in terms of $\beta = u/c$ when working in:

(i) Newtonian dynamics;

(ii) special relativity.

Show that the two transformations agree in an appropriate limit, assuming $|x| < c|t|$.

(b) Calculate the eigenvalues and eigenvectors of A in special relativity, and interpret the eigenvectors.

SECTION II

5E Numbers and Sets

State Bezout's theorem. Suppose that $p \in \mathbb{N}$ is prime and $a, b \in \mathbb{N}$. Show that if p divides ab then p divides a or p divides b.

Show that if $m, n \in \mathbb{N}$ are coprime then any pair of congruences of the form

$$x \equiv a \mod m \quad \text{and} \quad x \equiv b \mod n$$

has a unique simultaneous solution modulo mn.

Show that if p is an odd prime and $d \in \mathbb{N}$ then there are precisely 2 solutions of $x^2 \equiv 1$ modulo p^d. Deduce that if $n \geqslant 3$ is odd, then the number of solutions of $x^2 \equiv 1$ modulo n is equal to 2^k, where k denotes the number of distinct prime factors of n.

How many solutions of $x^2 \equiv 1$ modulo n are there when $n = 2^d$?

6D Numbers and Sets

(a) Define the binomial coefficient $\binom{n}{k}$ for $0 \leqslant k \leqslant n$. Show from your definition that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ holds when both sides are well-defined.

(b) Prove the following special case of the binomial theorem: $(1+t)^n = \sum_{k=0}^{n} \binom{n}{k} t^k$ for any real number t. By integrating this expression over a suitable range, or otherwise, evaluate $\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}$ and $\sum_{k=0}^{n} \frac{(-1)^k}{k+1}\binom{n}{k}$.

Deduce that $\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\binom{n}{k} = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$.

(c) The Fibonacci numbers are defined by

$$F_1 = 1, \qquad F_2 = 1, \qquad F_{n+2} = F_{n+1} + F_n \quad \text{for } n \geqslant 1.$$

By using induction, or otherwise, prove that

$$F_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}$$

for all $n \geqslant 1$, where $\lfloor \frac{n-1}{2} \rfloor$ denotes the largest integer less than or equal to $\frac{n-1}{2}$.

7F Numbers and Sets

For a given natural number $n \geqslant 2$, let S be the set of ordered real n-tuples $x = (x_1, \ldots, x_n)$ where $x_i \geqslant 0$ for $1 \leqslant i \leqslant n$. For $x \in S$, let

$$P(x) = \{i : x_i > 0\}.$$

Define the relation $\preceq$ by

$$x \preceq y \text{ if and only if } P(x) \subseteq P(y).$$

(a) Is the relation $\preceq$ reflexive? Is it transitive? Is it symmetric? Justify your answers.

(b) Show that $x \preceq y$ if and only if there exists $z \in S$ such that $x_i = y_i z_i$ for all $1 \leqslant i \leqslant n$.

(c) Define the relation $\sim$ by

$$x \sim y \text{ if and only if } x \preceq y \text{ and } y \preceq x.$$

Show that $\sim$ defines an equivalence relation on S. Into how many equivalence classes does $\sim$ partition S?

(d) Define the relation $\perp$ by

$$x \perp y \text{ if and only if } P(x) \cap P(y) = \emptyset.$$

Given $s \in S$, show that for every $x \in S$ there exist unique $y, z \in S$ such that $x = y + z$ where $y \preceq s$ and $z \perp s$.

8F Numbers and Sets

(a) What does it mean to say a set is *countable*?

(b) Show from first principles that the following sets are countable:

(i) the Cartesian product $\mathbb{N} \times \mathbb{N}$, where $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural numbers,

(ii) the rational numbers,

(iii) the points of discontinuity of an increasing function $F : \mathbb{R} \to \mathbb{R}$.

(c) Let $A_1, A_2, \ldots$ be a collection of non-empty countable sets and consider the Cartesian product

$$B = A_1 \times A_2 \times \cdots .$$

Show from first principles that B is countable if and only if there exists a natural number N such that $|A_n| = 1$ for all $n > N$.

9C Dynamics and Relativity

Consider two particles of masses m_1 and m_2, and locations $\mathbf{x}_1$ and $\mathbf{x}_2$, that exert forces $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ upon each other. There are no external forces. The particles' equations of motion are $m_1\ddot{\mathbf{x}}_1 = \mathbf{F}_{12}$, and $m_2\ddot{\mathbf{x}}_2 = \mathbf{F}_{21}$.

(a) Define the centre of mass $\mathbf{R}$. Prove that the centre of mass moves at a constant velocity. If $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$, show that $\mu\ddot{\mathbf{r}} = \mathbf{F}_{12}$, where you must give an expression for μ.

(b) For the remainder of the question, assume the force law

$$\mathbf{F}_{ij} = -km_i m_j \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^3},$$

with k a positive constant.

Let $m_1 = m_2 = m$. In a Cartesian coordinate system whose origin is at the centre of mass, verify that

$$\mathbf{x}_1 = a(\cos\omega t, \sin\omega t\,, 0), \qquad \mathbf{x}_2 = -a(\cos\omega t, \sin\omega t, 0), \tag{†}$$

is a solution to the equations of motion, where a is a fixed constant and ω is a frequency that you should find.

(c) A third particle is now placed upon the z-axis. Its mass m_3 is negligible compared to m, and its position vector $\mathbf{x}_3$ obeys $m_3\ddot{\mathbf{x}}_3 = \mathbf{F}_{31} + \mathbf{F}_{32}$, while the motion of particles 1 and 2 is given by (†).

(i) If the initial velocity of the third particle is parallel to the z-axis, show that it remains on that axis and that its location $z(t)$ obeys

$$\ddot{z} = -\frac{2mkz}{(z^2 + a^2)^{3/2}}.$$

(ii) What is the effective potential governing the particle's motion? Describe the different kinds of behaviour possible.

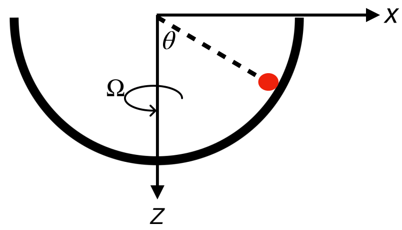
(iii) Assume that at $t = 0$, $z = z_0$, and $\dot{z} = 0$. If z_0 is very small, show that the motion is oscillatory and find the period of the oscillations.

(iv) Now assume that at $t = 0$, $z = 0$, and $\dot{z} = u$. What is the criterion for the particle to escape to infinity?

10C Dynamics and Relativity

Consider an infinitely long ramp with semi-circular cross-section of radius R, as shown in the figure. Adopt a Cartesian coordinate system with the y-axis directed along the ramp, pointing out of the page, and the z-axis directed vertically downwards. The ramp rotates about the z-axis with constant angular velocity $\mathbf{\Omega} = -\Omega\hat{\mathbf{z}}$ and the coordinate system rotates with the ramp.

A ball of mass m and negligible size slides along the surface of the ramp without any friction but experiences a constant gravitational acceleration $\mathbf{g} = g\hat{\mathbf{z}}$. A line from the ball to the origin projected on to the xz-plane makes an angle θ with the z-axis, as shown in the figure.



(a) If $\mathbf{x} = (x, y, z)$ is the ball's position vector, its equation of motion is

$$\ddot{\mathbf{x}} = -2\mathbf{\Omega} \times \dot{\mathbf{x}} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x}) + \mathbf{g} + \frac{1}{m}\mathbf{N},$$

where $\mathbf{N}$ is the normal force due to the ramp. What do the first two terms on the right side correspond to? Write down the equation's three Cartesian components.

(b) Using your results from part (a), or otherwise, show that

$$R\ddot{\theta} = -g\sin\theta + \Omega^2 R\cos\theta\sin\theta - 2\Omega\dot{y}\cos\theta,$$
$$\ddot{y} = \Omega^2 y + 2\Omega R\dot{\theta}\cos\theta.$$

Find all the solutions for which the ball is at rest in the rotating frame.

(c) Suppose that Ω is sufficiently small so that terms of order Ω^2 may be neglected and that at time $t = 0$, $\theta = \theta_0$, $\dot{\theta} = 0$, and $y = \dot{y} = 0$.

To linear order in small θ, show that the ball undergoes oscillations in θ and find their frequency. Determine the associated motion in y.

11C Dynamics and Relativity

(a) Define the moment of inertia of a rigid body V, of density ρ, rotating about a given axis.

A thin circular disc has radius r, thickness $\delta \ll r$, and uniform density ρ. Its centre of mass is at the origin of a Cartesian coordinate system whose z-axis is perpendicular to the disc's circular face. To leading order in small δ, find the disc's moment of inertia when it is rotating about:

(i) the x-axis,

(ii) a line of the form $y = 0$, $z = h$.

(b) Consider a cone with circular cross-section, base of radius R, height H, and uniform density ρ. The cone rotates about an axis that passes through its apex and which is perpendicular to its axis of symmetry.

(i) Using part (a)(ii), or otherwise, show that the cone's moment of inertia is $I = M(\alpha R^2 + \beta H^2)$, where M is the cone's mass, and α and β are constants you need to find.

[You may assume that the volume of the cone is $\frac{1}{3}\pi R^2 H$.]

(ii) If there is no friction and initially the cone has kinetic energy K_0, how long does it take to execute one full rotation?

(iii) Now suppose there is friction so that if the cone rotates by an angle $\Delta\theta$ the work done by the friction is equal to $W\Delta\theta$, where W is a constant.

If the cone initially has kinetic energy K_0, show that it comes to rest after a time

$$t = \sqrt{\frac{2K_0 I}{W^2}}.$$

12C Dynamics and Relativity

(a) State the definition of a four-vector U. Prove that $U \cdot U$ is the same in all inertial frames.

(b) Relative to an inertial reference frame S, a second inertial frame S' moves with constant three-velocity $\mathbf{V} = (V,\, 0,\, 0)$, and the two frames coincide when $t = t' = 0$.

A particle is travelling with a constant three-velocity $\mathbf{u} = (u_x,\, u_y,\, 0)$, as measured in frame S, and passes through the origin of S at $t = 0$.

(i) By considering the transformation of the particle's position vector in space-time, calculate $\mathbf{u}'$, the particle's three-velocity in S'.

(ii) Suppose that V/c is small. To leading order in V/c, show that

$$\mathbf{u}' = \mathbf{u} - \mathbf{V} + \frac{(\mathbf{V} \cdot \mathbf{u})}{c^2}\mathbf{u}.$$

(iii) A light source at the origin of frame S emits photons at an angle θ relative to the x-axis. According to an observer in frame S', the photons are emitted at an angle θ' relative to the x'-axis. Show

$$\theta' - \theta = \frac{V}{c}\sin\theta,$$

to leading order in small V/c.

(c) In the laboratory frame, a photon of wavelength λ collides with an electron of mass m, initially at rest. After the collision, the three-momenta of the photon and electron are collinear. Find the wavelength of the photon after the collision.

END OF PAPER