UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday, 6 June, 2022 9:00am to 12:00pm

PAPER 3

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. Of the Section II questions, no more than three may be on the same course.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet ***for each question*** *that you have attempted, and place it at the front of your answer to that question.*

Complete a green master cover sheet listing ***all the questions*** *that you have attempted.*

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into ***a single bundle****, with the master cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.*

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Groups

Let G and H be finite groups and $g \in G$.

Define the *order* of g.

Show that if $\phi \colon G \to H$ is a homomorphism then the order of $\phi(g)$ divides the order of g.

Show that if ϕ is surjective and H has an element of order m then G has an element of order m.

How many homomorphisms $C_9 \to S_4$ are there?

2E Groups

What does it mean to say a group is *abelian*? What does it mean to say a group is *cyclic*?

Show that every cyclic group is abelian. Show that not every abelian group is cyclic.

Recall that the *proper subgroups* of a group G are the subgroups of G not equal to G. If every proper subgroup of a group G is cyclic then must G be abelian? Justify your answer.

3A Vector Calculus

Let D be the region in the positive quadrant of the xy plane defined by

$$y \leqslant x \leqslant \alpha y\,, \qquad \frac{1}{y} \leqslant x \leqslant \frac{\alpha}{y}\,,$$

where $\alpha > 1$ is a constant. By using the change of variables $u = x/y$, $v = xy$, or otherwise, evaluate

$$\int_D x^2\, dx\, dy\ .$$

4A Vector Calculus

Consider the curve in $\mathbb{R}^3$ defined by $y = \log x$, $z = 0$. Using a parametrization of your choice, find an expression for the unit tangent vector $\mathbf{t}$ at a general point on the curve. Calculate the curvature κ as a function of your chosen parameter. Hence find the maximum value of κ and the point on the curve at which it is attained.

[*You may assume that* $\kappa = |\,\mathbf{t} \times (d\mathbf{t}/ds)\,|$ *where* s *is the arc-length.*]

SECTION II

5E Groups

What does it mean for a group G to *act* on a set X. Given such an action and $x \in X$ define the *orbit* and *stabiliser* of x. State and prove the orbit–stabiliser theorem for a finite group.

State and prove Cauchy's theorem.

Suppose that G is a group of order 33. By considering the conjugation action of a subgroup of G on G, show that G must be cyclic.

6E Groups

What is a *Möbius transformation*?

Show carefully that if (z_1, z_2, z_3) and (w_1, w_2, w_3) are two ordered subsets of the extended complex plane $\widehat{\mathbb{C}}$, each consisting of three distinct points, then there is a unique Möbius transformation f such that $f(z_i) = w_i$ for $i = 1, 2, 3$. [You may assume that the Möbius transformations form a group under composition.]

Define the *cross-ratio* $[z_1, z_2, z_3, z_4]$ of four distinct points $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$. Show that a bijection $f \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a Möbius transformation if and only if f preserves the cross-ratio of any four distinct points in $\widehat{\mathbb{C}}$; that is, if and only if

$$[z_1, z_2, z_3, z_4] = [f(z_1), f(z_2), f(z_3), f(z_4)]$$

for any four distinct points z_1, z_2, z_3, z_4 in $\widehat{\mathbb{C}}$.

Are there complex numbers a and b such that the map that sends z to $a\bar{z} + b$ for $z \in \mathbb{C}$ and fixes ∞ is Möbius? Justify your answer. [Here $\bar{z}$ denotes the complex conjugate of z.]

7E Groups

Suppose G is a group. What does it mean to say that a subset K of G is a *normal subgroup* of G? For N a normal subgroup of G explain how to define the *quotient group* G/N. Briefly explain why G/N is a group.

Define the *kernel* and the *image* of a group homomorphism. Show that a subset K of G is a normal subgroup of G if and only if there is a group H and a group homomorphism $\theta \colon G \to H$ such that K is the kernel of θ. Show moreover that in this case the image of θ is a subgroup of H and G/K is isomorphic to the image of θ.

By defining a suitable group homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{C}\backslash\{0\}, \cdot)$, show that $\mathbb{R}/\mathbb{Z}$ is isomorphic to the subgroup of $(\mathbb{C}\backslash\{0\}, \cdot)$ consisting of complex numbers of modulus 1. What characterises the elements of the image of $\mathbb{Q}/\mathbb{Z}$ under this isomorphism?

8E Groups

Show that the set $S(\mathbb{N})$ of invertible functions $\tau\colon \mathbb{N} \to \mathbb{N}$ is a group under composition. Show that the subset $S^{\text{fin}}(\mathbb{N})$ of invertible functions $\tau\colon \mathbb{N} \to \mathbb{N}$ such that there is some $n \geqslant 1$ with $\tau(m) = m$ for all $m > n$ is a subgroup of $S(\mathbb{N})$.

A *cycle* is a non-identity element σ of $S^{\text{fin}}(\mathbb{N})$ such that for every $m, n \in \mathbb{N}$ either $\sigma(m) = m$ or $\sigma(n) = n$ or there is an integer a such that $\sigma^a(m) = n$. Show that if σ is a cycle and $n \in \mathbb{N}$ such that $\sigma(n) \neq n$ then the order of σ is the least positive integer l such that $\sigma^l(n) = n$. Show in particular that the order of σ is always finite.

Show that every element τ of $S^{\text{fin}}(\mathbb{N})$ can be written as a product of cycles $\sigma_1 \cdots \sigma_k$ such that for every $1 \leqslant i < j \leqslant k$ and every $n \in \mathbb{N}$ either $\sigma_i(n) = n$ or $\sigma_j(n) = n$ (or both). Show moreover that $\sigma_i\sigma_j = \sigma_j\sigma_i$ for all $1 \leqslant i < j \leqslant k$. What is the relationship between the order of τ and the orders of $\sigma_1, \ldots, \sigma_k$? Justify your answer.

9A Vector Calculus

(a) Using Cartesian coordinates x_i in $\mathbb{R}^3$, write down an expression for $\partial r/\partial x_i$, where r is the radial coordinate ($r^2 = x_i x_i$), and deduce that

$$\nabla \cdot (\, g(r)\mathbf{x}\,) \;=\; r g'(r) + 3g(r)$$

for any differentiable function $g(r)$.

(b) For spherical polar coordinates r, θ, ϕ satisfying

$$x_1 = r \sin\theta \cos\phi\,, \quad x_2 = r\sin\theta\sin\phi\,, \quad x_3 = r\cos\theta\,,$$

find scalars h_r, h_θ, h_ϕ and unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ such that

$$d\mathbf{x} \;=\; h_r\,\mathbf{e}_r\,dr \;+\; h_\theta\,\mathbf{e}_\theta\,d\theta \;+\; h_\phi\,\mathbf{e}_\phi\,d\phi\;.$$

Hence, using the relation $df = d\mathbf{x}\cdot\nabla f$, find an expression for ∇f in spherical polars for any differentiable function $f(\mathbf{x})$.

(c) Consider the vector fields

$$\mathbf{A}^+ \;=\; \frac{1}{r}\tan\frac{\theta}{2}\,\mathbf{e}_\phi \quad (r \neq 0,\, \theta \neq \pi)\,, \qquad \mathbf{A}^- \;=\; -\frac{1}{r}\cot\frac{\theta}{2}\,\mathbf{e}_\phi \quad (r\neq 0,\, \theta \neq 0)\,.$$

Compute $\nabla\times\mathbf{A}^+$ and $\nabla\times\mathbf{A}^-$ and use the result in part (a) to check explicitly that your answers have zero divergence.

[*You may use without proof the formula* $\nabla\times\mathbf{A} \;=\; \frac{1}{h_r h_\theta h_\phi}\begin{vmatrix} h_r\mathbf{e}_r & h_\theta\mathbf{e}_\theta & h_\phi\mathbf{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ h_r A_r & h_\theta A_\theta & h_\phi A_\phi \end{vmatrix}$.]

(d) From your answers in part (c), explain briefly on general grounds why

$$\mathbf{A}^+ \;-\; \mathbf{A}^- \;=\; \nabla f$$

for some function $f(\mathbf{x})$. Find a solution for f that is defined on the region $x_1 > 0$.

10A Vector Calculus

Let H be the unbounded surface defined by $x^2 + y^2 = z^2 + 1$, and S the bounded surface defined as the subset of H with $1 \leqslant z \leqslant \sqrt{2}$. Calculate the vector area element $d\mathbf{S}$ on S in terms of ρ and ϕ, where $x = \rho\cos\phi$ and $y = \rho\sin\phi$. Sketch the surface and indicate the sense of the corresponding normal.

Compute directly

$$\int_S \nabla \times \mathbf{A} \cdot d\mathbf{S}$$

where $\mathbf{A} = (-yz^2, xz^2, 0)$. Now verify your answer using Stokes' Theorem.

What is the value of

$$\int_{S'} \nabla \times \mathbf{A} \cdot d\mathbf{S}$$

where S' is defined as the subset of H with $-1 \leqslant z \leqslant \sqrt{2}$? Justify your answer.

11A Vector Calculus

Let V be a region in $\mathbb{R}^3$ with boundary a closed surface S. Consider a function ϕ defined in V that satisfies

$$\nabla^2\phi - m^2\phi = 0$$

for some constant $m \geqslant 0$.

(i) If $\partial\phi/\partial n = g$ on S, for some given function g, show that ϕ is unique provided that $m > 0$. Does this conclusion change if $m = 0$?

[*Recall:* $\partial/\partial n = \mathbf{n} \cdot \nabla$, *where* $\mathbf{n}$ *is the outward pointing unit normal on* S.]

(ii) Now suppose instead that $\phi = f$ on S, for some given function f. Show that for any function ψ with $\psi = f$ on S,

$$\int_V (\,|\nabla\psi|^2 + m^2\psi^2\,)\,dV \;\geqslant\; \int_V (\,|\nabla\phi|^2 + m^2\phi^2\,)\,dV\ .$$

What is the condition for equality to be achieved, and is this result sufficient to deduce that ϕ is unique? Justify your answers, distinguishing carefully between the cases $m > 0$ and $m = 0$.

12A Vector Calculus

Consider a rigid body B of uniform density ρ and total mass M rotating with constant angular velocity $\boldsymbol{\omega}$ relative to a point $\mathbf{a}$. The angular momentum $\mathbf{L}$ about $\mathbf{a}$ is defined by

$$\mathbf{L} = \int_B (\mathbf{x} - \mathbf{a}) \times [\boldsymbol{\omega} \times (\mathbf{x} - \mathbf{a})] \; \rho \, dV \, ,$$

and the inertia tensor $I_{ij}(\mathbf{a})$ about $\mathbf{a}$ is defined by the relation

$$L_i = I_{ij}(\mathbf{a}) \, \omega_j \, .$$

(a) Given that $\mathbf{L}$ is a vector for any choice of the vector $\boldsymbol{\omega}$, show from first principles that $I_{ij}(\mathbf{a})$ is indeed a tensor, of rank 2.

Assuming that the centre of mass of B is located at the origin $\mathbf{0}$, so that

$$\int_B x_i \, dV \; = \; 0 \, ,$$

show that

$$I_{ij}(\mathbf{a}) \; = \; I_{ij}(\mathbf{0}) \; + \; M(\, a_k a_k \delta_{ij} - a_i a_j \,) \, ,$$

and find an explicit integral expression for $I_{ij}(\mathbf{0})$.

(b) Now suppose that B is a cube centred at $\mathbf{0}$ with edges of length ℓ parallel to the coordinate axes, *i.e.* B occupies the region $-\frac{1}{2}\ell \leqslant x_i \leqslant \frac{1}{2}\ell$. Using symmetry, explain in outline why $I_{ij}(\mathbf{0}) \; = \; \lambda \, \delta_{ij}$ for some constant λ.

Given that $\lambda = M\ell^2/6$, find $I_{ij}(\mathbf{a})$ when $\mathbf{a} = \frac{1}{2}\ell(1, 1, 0)$, writing the result in matrix form. Hence, or otherwise, show that if the cube is rotating relative to $\mathbf{a}$ with $|\boldsymbol{\omega}| = 1$ then, depending on the direction of the angular velocity, $|\mathbf{L}|$ has a maximum value that is four times larger than its minimum value.

END OF PAPER