UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Friday, 3 June, 2022 1.30pm to 4:30pm

PAPER 2

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. Of the Section II questions, no more than three may be on the same course.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet ***for each question*** *that you have attempted, and place it at the front of your answer to that question.*

Complete a green master cover sheet listing ***all the questions*** *that you have attempted.*

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into ***a single bundle****, with the master cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.*

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1A Differential Equations

Consider the integral

$$I(x) = \int_0^{\pi} e^{x\cos\theta} d\theta \ .$$

Show, by differentiating under the integral sign, that

$$\frac{dI}{dx} = \int_0^{\pi} x \sin^2\theta \, e^{x\cos\theta} d\theta \ .$$

Hence, or otherwise, show that

$$\frac{d^2 I}{dx^2} + \frac{1}{x}\frac{dI}{dx} - I = 0 \ .$$

2B Differential Equations

Solve the difference equation

$$x_{n+3} - 6x_{n+2} + 12x_{n+1} - 8x_n = 0 \,,$$

given initial conditions $x_0 = 0, x_1 = 4, x_2 = 24$.

3F Probability

What does it mean to say a function is *convex*? State Jensen's inequality for a convex function f and an integrable random variable X.

Let $x_1, \ldots, x_n$ be positive real numbers. Show that

$$\frac{\sum_{i=1}^n x_i \log x_i}{\sum_{i=1}^n x_i} \geqslant \log\left(\frac{\sum_{i=1}^n x_i}{n}\right) .$$

[You may use without proof a standard sufficient condition for convexity if it is stated carefully.]

4F Probability

Let X be a random variable with mean μ and variance σ^2. Let

$$G(a) = \mathbb{E}\big[(X-a)^2\big].$$

Show that $G(a) \geqslant \sigma^2$ for all a. For what value of a is there equality?

Let

$$H(a) = \mathbb{E}\big[|X-a|\big].$$

Supposing that X is a continuous random variable with probability density function f, express $H(a)$ in terms of f. Show that H is minimised for a such that $\int_{-\infty}^{a} f(x)dx = 1/2$.

SECTION II

5C Differential Equations

(a) What is meant by an ordinary point and a regular singular point of a linear second-order ordinary differential equation?

Consider

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + \lambda y = 0, \qquad (\dagger)$$

where λ is a real constant.

Find a solution to ($\dagger$) in the form of a series expansion around $x=0$. Obtain the general expression for the coefficients in the series.

For what values of λ do you obtain polynomial solutions?

(b) Determine the Wronskian of the equation ($\dagger$) as a function of x.

Let $\lambda = 1$. Verify that $y_1 = 1-x$ is a solution to ($\dagger$). Using the Wronskian, calculate a second solution y_2 in the form

$$y_2 = (1-x)\log x + b_1 x + b_2 x^2 + \ldots,$$

where b_1 and b_2 are constants you need to find.

6A Differential Equations

(a) Let $f(x,y)$ be a real-valued function depending smoothly on real variables x and y, and $g(t) = f(\,a + t\cos\gamma,\, b + t\sin\gamma\,)$, where a, b and γ are constants. Express $g'(t)$ and $g''(t)$ in terms of partial derivatives of f.

Write down sufficient conditions for g to have a local minimum at $t=0$ and deduce that a stationary point of f at $(x,y) = (a,b)$ is a local minimum if

$$\frac{\partial^2 f}{\partial y^2} > 0 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} > \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 .$$

(b) Now let

$$f(x,y) = x^4 - 3x^2 + 2xy + y^2 .$$

Find all stationary points of f and show that those at $(x,y) \neq (0,0)$ are local minima.

Show also that $g(t)$ with $a = b = 0$ has either (i) a local minimum or (ii) a local maximum at $t=0$, depending on the value of γ. Determine carefully the ranges of values of $\tan\gamma$ for which cases (i) and (ii) occur and sketch the typical behaviour of $g(t)$ in each of these cases.

7C Differential Equations

Consider the system of linear differential equations

$$\frac{d\mathbf{z}}{dt} - A\mathbf{z} = \mathbf{f}, \quad \text{where} \quad A = \begin{pmatrix} 3 & -6 \\ 1 & -2 \end{pmatrix}. \tag{$\dagger$}$$

(a) Suppose $\mathbf{f} = \mathbf{0}$. Show that the general solution to ($\dagger$) takes the form

$$\mathbf{z} = \alpha \mathbf{u}_1 e^{\lambda_1 t} + \beta \mathbf{u}_2 e^{\lambda_2 t}, \tag{$\star$}$$

where α and β are arbitrary constants. Calculate $\mathbf{u}_1$, $\mathbf{u}_2$, λ_1, and λ_2.

(b) Suppose now that $\mathbf{f} = (1,\, a)^T$, where a is a constant parameter.

By writing $\mathbf{f}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$, determine the value(s) of a for which the particular integral depends on time.

Using matrix methods, find the general solution to ($\dagger$).

(c) Consider

$$\frac{d^n\mathbf{z}}{dt^n} - A\mathbf{z} = \mathbf{0},$$

where $n > 1$ is an integer.

Show that ($\star$) is a solution to this system of equations. How many other linearly independent solutions must there be?

8B Differential Equations

(a) Consider the system

$$\dot{x} = 8x - 2x^2 - 2xy^2\,, \qquad \dot{y} = xy - y\,, \tag{$*$}$$

for $x(t) \geqslant 0\,, y(t) \geqslant 0$.

Find all the equilibrium points of ($*$) and determine their type. Explain how solutions close to each equilibrium point will evolve, sketching their trajectories. [You may quote general results without proof.]

(b) Consider the system

$$\dot{x} = x(1-y)\,, \qquad \dot{y} = 3y(x-1)\,, \tag{$**$}$$

defined for $x > 0, y > 0$.

Show that it has precisely one equilibrium point in the given range. Obtain an equation for dy/dx. Show that this equation is separable and hence obtain a solution in the form $E(x,y) = C$, where C is a constant and $E(x,y)$ is a nontrivial conserved quantity for solutions of ($**$). Show that $E(x,y)$ has a single stationary point in the quadrant $x > 0, y > 0$, and identify what type of stationary point it is. Hence show that solutions close to the equilibrium point at time $t = 0$ remain close at all times.

9F Probability

(a) Let U and V be two *bounded* random variables such that $\mathbb{E}[U^k] = \mathbb{E}[V^k]$ for all non-negative integers k. Show that U and V have the same moment generating function.

(b) Let X be a continuous random variable with probability density function

$$f(x) = Ae^{-x^2/2}$$

for all real x, where A is a normalising constant. Compute the moment generating function of X.

(c) Let Y be a discrete random variable with probability mass function

$$\mathbb{P}(Y = n) = Be^{-n^2/2}$$

for all integers n, where B is a normalising constant. Show that

$$\mathbb{E}[e^{kY}] = \mathbb{E}[e^{kX}]$$

for all integers k, where X is a standard normal random variable.

(d) Let U and V be *unbounded* random variables such that U^k and V^k are integrable and $\mathbb{E}[U^k] = \mathbb{E}[V^k]$ for all non-negative integers k. Does it follow that U and V have the same distribution?

10F Probability

(a) Let X be a random variable valued in $\{1, 2, \ldots\}$ and let G_X be its probability generating function. Show that

$$\mathbb{P}(X = n) = \frac{G_X^{(n)}(0)}{n!}$$

where $G_X^{(n)}$ denotes the nth derivative of G_X.

(b) Let Y be another random variable valued in $\{1, 2, \ldots\}$, independent of X. Prove that $G_{X+Y}(s) = G_X(s)G_Y(s)$ for all $0 \leqslant s \leqslant 1$.

(c) Compute G_X in the case where X is a geometric random variable taking values in $\{1, 2, \ldots\}$ with $\mathbb{P}(X = 1) = p$ for a given constant $0 < p \leqslant 1$.

(d) A jar contains n marbles. Initially, all of the marbles are red. Every minute, a marble is drawn at random from the jar, and then replaced with a blue marble. Let T be the number of minutes until the jar contains only blue marbles. Compute the probability generating function G_T.

11F Probability

Consider a coin that is biased such that when tossed the probability of heads is p and tails is $1-p$.

(a) Suppose that the coin was tossed n times. What is the probability that the coin came up heads exactly k times?

(b) Suppose that the coin was tossed n times. Given that the coin came up heads exactly k times, what is the probability that the coin came up heads k times in a row?

(c) Suppose that the coin was tossed repeatedly until heads came up k times. What is the probability that the total number of tosses was n?

(d) Suppose that the coin was tossed repeatedly until heads came up k times in a row. Find the expected number of tosses.

12F Probability

Let $A_1, A_2, \ldots$ be a collection of events. Let $N = \sum_{n\geqslant 1} \mathbf{1}_{A_n}$ be the random variable that counts how many of these events occur. Note that N takes values in $\{0,1,\ldots\}\cup\{\infty\}$.

(a) By considering the quantity $\mathbb{E}(N)$, show that if $\sum_{n\geqslant 1} \mathbb{P}(A_n) < \infty$ then $\mathbb{P}(\text{an infinite number of the events occur}) = 0$.

(b) Suppose now that the events are independent. Show the inequality $\mathbb{E}(2^{-N}) \leqslant e^{-\frac{1}{2}\mathbb{E}(N)}$, with the convention that $2^{-\infty} = 0$. [*Hint: use the inequality* $1 - x \leqslant e^{-x}$ *for all* x.]

(c) Again suppose that the events are independent. Show that if $\sum_{n\geqslant 1} \mathbb{P}(A_n) = \infty$ then $\mathbb{P}(\text{an infinite number of the events occur}) = 1$.

(d) A monkey types by randomly striking keys on a 26-letter keyboard, with each letter of the alphabet equally likely to be struck and the keystrokes independent. Show that with probability one, the word HELLO appears infinitely often.

END OF PAPER